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ON INJECTIVE NEAR–RING MODULES

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1. **Introduction.** Let N be a left near ring and let M be a right N-module. We recall [1] that M is called injective iff every diagram

$$\begin{array}{ccc} 0\longrightarrow A & \xrightarrow{f} & B \\ & & \downarrow g \\ & & M \end{array}$$

can be embedded into a commutative diagram

$$\begin{array}{ccc} 0\longrightarrow A & \xrightarrow{f} & B \\ g\downarrow & \swarrow h & \\ M & & \end{array}$$

where A and B are right N-modules with $0\longrightarrow A\xrightarrow{f} B$ exact.

The purpose of this note is to show that if N is a d.g. near-ring with identity, then M is injective iff for every right ideal u of N and every N-homomorphism $f:u\to N$, there exists an element m in M such that $f(a)=ma$ for all a in u.

For the definition of near-rings, near-ring modules etc., see [2].

2. **Main theorem.** We first prove the following

LEMMA. *Let N be a left d.g. near ring with identity, generated by a distributive semi group S. Let M be a right N-module and A any subset of M. Then the submodule of M generated by A is*

$$\bar{A} = \left\{\sum_{i=1}^{n} -m_i+a_is_i+m_i \mid m_i \in M,\ a_i \in A \text{ and either } s_i \in S \text{ or } -s_i \in S\right\}.$$

Proof. Clearly $\bar{A}$ is a right N-group. To show that $\bar{A}$ is normal in M, take $m \in M$ and $\sum_{i=1}^{n} -m_i+a_is_i+m_i \in \bar{A}$. Then

$$\begin{aligned}
&-m+\left\{\sum_{i=1}^{n}-m_i+a_is_i+m_i\right\}+m \\
&= -m+\{(-m_1+a_1s_1+m_1)+(-m_2+a_2s_2+m_2)+\cdots+(-m_n+a_ns_n+m_n)\}+m \\
&= \{(-m-m_1)+a_1s_1+(m_1+m)\}+\cdots+\{(-m-m_n)+a_ns_n+(m_n+m)\} \\
&= \sum_{i=1}^{n}\{-(m_i+m)+a_is_i+(m_i+m)\} \quad \text{is in } \bar{A}.
\end{aligned}$$

Let $m \in M$, $\bar{a}=\{\sum_{i=1}^{n} -m_i+a_is_i+m_i\} \in \bar{A}$ and $r \in N$. Then

$$(m+\bar{a})r-mr = (m+\bar{a})(t_1+t_2+\cdots+t_n)-m(t_1+t_2+\cdots+t_n)$$

where t_i or $-t_i \in S$.

To show that $(m+\bar{a})r-mr$ is in $\bar{A}$, we use induction on n. For $n=1$, it is easy to see that the result is true. Suppose it is true for $n-1$. Then

$$\begin{aligned}(m+\bar{a})r-mr &= (m+\bar{a})(t_1+t_2+\cdots+t_{n-1}+t_n)-m(t_1+t_2+\cdots+t_{n-1}+t_n)\\ &= (m+\bar{a})(t_1+t_2+\cdots+t_{n-1})+(m+\bar{a})t_n-m(t_1+t_2+\cdots+t_n)\\ &= (m+\bar{a})(t_1+t_2+\cdots+t_{n-1})-m(t_1+t_2+\cdots+t_{n-1})\\ &\quad +m(t_1+\cdots+t_{n-1})+(m+\bar{a})t_n-m(t_1+\cdots+t_n).\end{aligned}$$

Therefore if $t_n \in S$, then

$$\begin{aligned}(m+\bar{a})r-mr &= \{(m+\bar{a})(t_1+t_2+\cdots+t_{n-1})-m(t_1+\cdots+t_{n-1})\}\\ &\quad + \{m(t_1+\cdots+t_{n-1})+mt_n+\bar{a}t_n-m(t_1+\cdots+t_n)\}\\ &= a'+\{mr+\bar{a}t_n-mr\}\end{aligned}$$

where

$$a' = \{(m+\bar{a})(t_1+\cdots+t_{n-1})-m(t_1+\cdots+t_{n-1})\} \in \bar{A}$$

Thus, since $\bar{A}$ is normal in M, $(m+\bar{a})r-mr$ is in $\bar{A}$. If $-t_n \in S$ then

$$\begin{aligned}&(m+\bar{a})r-mr\\ &\qquad = a'+\{m(t_1+t_2+\cdots+t_{n-1})+\bar{a}t_n+mt_n-m(t_1+t_2+\cdots+t_n)\}\\ &\qquad = a'+\{m(t_1+\cdots+t_{n-1})+\bar{a}t_n+mt_n-mt_n-mt_{n-1}-\cdots-mt_2-mt_1\\ &\qquad = a'+\{m(t_1+\cdots+t_{n-1})+\bar{a}t_n-m(t_1+t_2+\cdots+t_{n-1})\}\end{aligned}$$

is in $\bar{A}$. Hence $\bar{A}$ is a submodule of M. Clearly $A \subseteq \bar{A}$. To show that $\bar{A}$ is the smallest submodule of M containing A, we take a submodule B of M containing A. Let $\bar{a}=\sum_{i=1}^{n}-m_i+a_is_i+m_i$ be an arbitrary element of $\bar{A}$. Since $a_i \in A \subseteq B$ and B is a submodule of M, we have that $-m_i+a_is_i+m_i \in B$ $(i=1, 2, \ldots, n)$. Hence $\bar{A} \subseteq B$.

As an immediate observation we have

COROLLARY. *If A is a subset of M such that $AN \subseteq A$, then the submodule of M generated by A is* $\left\{\sum_{i=1}^{n} -m_i+a_i+m_i \,\middle|\, m_i \in M,\ a_i \in A\right\}$.

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THEOREM. *Let N be a left d.g. near-ring, with identity, and M be a right N-module. Then the following are equivalent:*

(i) *M is injective*

(ii) *every diagram* $\begin{array}{ccccc} 0 & \longrightarrow & C & \xrightarrow{j} & D \\ & & {\scriptstyle g}\downarrow & & \\ & & M & & \end{array}$ *can be embedded into a commutative diagram*

$\begin{array}{ccccc} 0 & \longrightarrow & C & \xrightarrow{j} & D \\ & & {\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\ & & M & & \end{array}$, *where C is a submodule of the right N-module D and j is the injection map.*

(iii) *for every right ideal u of N and every N-homomorphism $f:u\to M$ there exists an element m in M such that $f(a)=ma$ for all a in u.*

Proof. (i)$\Rightarrow$(ii): Trivial.

(ii)$\Rightarrow$(iii): Let u be a right ideal of N and $f:u\to M$ be a N-homomorphism and consider $\begin{array}{ccccc} 0 & \longrightarrow & u & \xrightarrow{j} & N. \\ & & \downarrow{\scriptstyle f} & & \\ & & M & & \end{array}$

Since u_N is a submodule of N_N, there exists a N-homomorphism $h:N_N\to M$ such that $h/u=f$. Then if $h(1)=m$ we have that $f(a)=ma$ for all a in u.

(iii)$\Rightarrow$(i): Let us consider the diagram $\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B, \\ & & \downarrow{\scriptstyle g} & & \\ & & M & & \end{array}$ where A and B are right N-modules and $0\longrightarrow A\xrightarrow{f} B$ is exact. Let $f(A)=C$. Then $C\subseteq B$ is a right N-module and $h:C\to M$ given by $h(f(a))=g(a)$ is a N-homomorphism.

Let $\bar{C}$ be the submodule of B generated by C, then

$$\bar{C}=\left\{\sum_{i=1}^{n}-b_i+f(a_i)+b_i/b_i\in B,\, a_i\in A\right\}.$$

Define

$$h':\bar{C}\to M \quad \text{by} \quad h'\left[\sum_{i=1}^{n}-b_i+f(a_i)+b_i\right]=\sum_{i=1}^{n}h(f(a_i)).$$

We note that $h'/C=h$.

To show that h' is well defined we take

$$\sum_{i=1}^{n}[-b_i+f(a_i+)b_i]=\sum_{i=1}^{m}[-b_i'+f(a_i')+b_i'] \quad \text{in } \bar{C}.$$

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Then

$$f(a_1) = \sum_{i=1}^{m} [(b_1-b_i')+f(a_i')+(b_i'-b_1)]+[(b_1-b_n)+f(-a_n)+(b_n-b_1)]$$
$$+\cdots+[(b_1-b_2)+f(-a_2)+(b_2-b_1)] \quad \text{is in } C$$

$$\therefore \quad h(f(a_1)) = h(\text{R.H.S.}) = h'(\text{R.H.S.})$$
$$= \sum_{i=1}^{m} h(f(a_i')+[h(f(-a_n)+\cdots+h(f(-a_2))]$$
$$= \sum_{i=1}^{m} h(f(a_i'))+h\{f(a_n)(-1)\}+\cdots+h\{f(a_2)(-1)\}$$
$$= \sum_{i=1}^{m} h(f(a_i'))-h(f(a_n))-\cdots-h(f(a_2)),$$

or

$$\sum_{i=1}^{n} h(f(a_i)) = \sum_{i=1}^{m} h(f(a_i')).$$

It can be seen that h' is a N-homomorphism such that $h'\circ f=g$. Put $X=\{(A', g')/$ $A'\subseteq B$ is an N-submodule of B containing $f(A)$ and

$$g':A'\to M \text{ an } N\text{-homomorphism} \ni g'\circ f = g\}$$

X is nonempty since $(\bar{C}, h')$ is in X.

As usual we can check that X has a maximal element, say (C_0, g_0). We claim that $C_0=B$. If not let $x\in B$ such that $x\notin C_0$. Let $u=\{a\in N/xa\in C_0\}$. Then u is a right ideal of N. Define $\psi:u\to M$ by $\psi(a)=g_0(xa)$ for all a in u. One can check that ψ is an N-homomorphism. So there exists an m in M such that $\psi(a)=ma=g_0(xa)$ for all a in u. Let $g_0':xN+C_0+xN\to M$ be given by

$$g_0'(xr+c_0+xr') = mr+g_0(c_0)+mr', r \qquad r'\in N, \quad c_0\in C_0$$

We wish to show that g_0' is well defined. To this end we show that

(a) g_0' takes the zero element of $xN+C_0+xN$ to the zero element of M, and

(b) g_0' is additive.

So let $xr+c_0+xr'=0$ for some r, r' in N, c_0 in C_0.

Then $c_0=-xr-xr'=x(-r-r')\in C_0$ and so $-r-r'\in u$ and hence

$$g_0(c_0) = g_0\{x(-r-r')\} = \psi(-r-r') = m(-r-r') = -mr-mr'$$

and so $mr+g_0(c_0)+mr'=0$; thus $g_0'(xr+c_0+xr')=0$. Secondly let $xr+c_0+xr'$ and $xt+c_0'+xt'$ be any two elements of $xN+C_0+xN$.

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Then

$$\begin{aligned}
&g_0'\{(xr+c_0+xr')+(xt+c_0'+xt')\}\\
&\quad = g_0'[(xr+xr')+(-xr'+c_0+xr')+(xt+c_0'-xt)+(xt+xt')]\\
&\quad = g_0'[x(r+r')+b_0+x(t+t')] \text{ where } b_0 = (-xr'+c_0+xr')+(xt+c_0'-xt) \in C_0\\
&\quad = m(r+r')+g_0[(-xr'+c_0+xr')+(xt+c_0'-xt)]+m(t+t')\\
&\quad = m(r+r')+g_0\{(-xr'+c_0+xr')\}+g_0\{(xt+c_0'-xt)\}+m(t+t')\\
&\quad = g_0'[(xr+xr')-xr'+c_0+xr']+g_0'[(xt+c_0'-xt)+xt+xt']\\
&\quad = g_0'[xr+c_0+xr']+g_0'[xt+c_0'+xt'],
\end{aligned}$$

hence g_0' is well defined and additive.

Now since g_0' is additive and $(N, +)$ is generated by a set of distributive elements, it can be checked that g_0' is N-linear, which gives us that g_0' is a N-homomorphism.

Moreover

$$g_0'/C_0 = g_0 \quad \text{and} \quad g_0' \circ f = g.$$

Let P be the N-submodule of B generated by $xN+C_0+xN$.

Then

$$P = \left\{\sum_{i=1}^{n} [-b_i+(xr_i+c_{0i}+xr_i')+b_i]/b_i \in B,\ r_i,\ r_i' \in N,\ c_{0i} \in C_0,\ 1 \leq i \leq n\right\},$$

and clearly $C=f(A)\subseteq C_0\subseteq xN+C_0+xN\subseteq P\subseteq B$. Consider the mapping $g_0'':P\to M$ given by

$$g_0''\left[\sum_{i=1}^{n} -b_i+(xr_i+c_{0i}+xr_i')+b_i\right] = \sum_{i=1}^{n} g_0'(xr_i+c_{0i}+xr_i').$$

As before it can be seen that g_0'' is well defined and is a N-homomorphism and $g_0'' \circ f=g$.

Hence we have $(P, g_0'') \in X$ and $(C_0, g_0)<(P, g_0'')$ which contradicts the choice of (C_0, g_0). Hence $C_0=B$, which proves the result.

References

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