Compositio Mathematica **108:** 241–245, 1997.

A construction of representations of affine Weyl groups

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Received 1 May 1996; accepted in revised form 13 May 1996

Abstract. Let G be a complex, semisimple, simply connected algebraic group with Lie algebra $\mathfrak{g}$. We extend scalars to the power series field in one variable $\mathbf{C}((\pi))$, and consider the space of Iwahori subalgebras containing a fixed nil-elliptic element of $\mathfrak{g} \otimes \mathbf{C}((\pi))$, i.e. fixed point varieties on the full affine flag manifold. We define representations of the affine Weyl group in the homology of these varieties, generalizing Kazhdan and Lusztig's topological construction of Springer's representations to the affine context.

Mathematics Subject Classifications (1991): Primary: 14L30, 20G25; Secondary: 22E65, 14M15.

Key words: Affine Weyl groups, fixed point varieties on affine flag manifolds, Springer representations.

In [KL1], Kazhdan and Lusztig have given an elementary procedure for constructing Springer representations of the Weyl group W of a complex semisimple group G in the homology of the varieties $\mathcal{B}_u$, where $\mathcal{B}_u$ is the variety of Borel subgroups of G containing a fixed unipotent element u. Their method is a special case of a construction of W-actions on the homology of a larger class of varieties. They start with a complex algebraic variety X and consider a locally trivial principal G-bundle $E \to X$. They form the associate $\mathcal{B}$-bundle $\widehat{E} = E \times_G \mathcal{B} \to X$ and assume given an algebraic fiberwise **C**-action φ on $\widehat{E}$. They denote the fixed point set of the action by $\widehat{E}_\varphi$; it is a closed subvariety of $\widehat{E}$. They fix a set S of simple reflections of the Coxeter group W, and for each $s \in S$, construct a map $\alpha_s\colon \widehat{E}_\varphi \to \widehat{E}_\varphi$, defined up to proper homotopy. The map $s \mapsto (\alpha_s)_*$ then defines a group homomorphism $W \to \mathrm{Aut}(H_*(\widehat{E}_\varphi))$, where $H_*(\widehat{E}_\varphi)$ denotes the Borel–Moore homology groups with rational coefficients. Springer representations may be obtained using this procedure by taking X to be a point and E to be G, so that $\widehat{E} = \mathcal{B}$. Any unipotent element $u \in G$ induces a **C**-action on $\mathcal{B}$ via $(t, B) \mapsto u^t B u^{-t}$, where u^t is the image of t in the one parameter subgroup of G defined by u. Since the fixed point variety is $\mathcal{B}_u$, we get representations of W in $H_*(\mathcal{B}_u)$.

Because Kazhdan and Lusztig's construction of W-actions relies heavily on the existence of simple reflections in Weyl groups, it is natural to conjecture that their procedure generalizes to other Coxeter groups. It is indeed the case that representations of affine Weyl group may be defined in this way.

Let G be a semisimple, simply connected, complex algebraic group with Lie algebra $\mathfrak{g}$ and affine Weyl group $\widehat{W}$. Let $F = \mathbf{C}((\pi))$ be the field of formal power series in one variable with $A = \mathbf{C}[[\pi]]$ its ring of integers. We extend scalars to form $G(F), G(A)$, $\mathfrak{g}_F = \mathfrak{g} \otimes_{\mathbf{C}} F$, and $\mathfrak{g}_A = \mathfrak{g} \otimes_{\mathbf{C}} A$. Fix a Borel subalgebra $\mathfrak{b} \in \mathcal{B}$. An Iwahori subalgebra of $\mathfrak{g}_F$ is any $G(F)$-conjugate of the pullback $\widehat{\mathfrak{b}}$ of $\mathfrak{b}$ under the natural projection $\mathfrak{g}_A \to \mathfrak{g}$; we denote the space of Iwahori subalgebras by $\widehat{\mathcal{B}}$. If $B \subset G$ corresponds to $\mathfrak{b}$ with preimage $\widehat{B} \subset G(A)$ under the map $G(A) \to G$, then $\widehat{\mathcal{B}}$ may be identified with the quotient space $G(F)/\widehat{B}$. The set $\widehat{\mathcal{B}}$ has the structure of an infinite-dimensional projective variety over $\mathbf{C}$, i.e. it is an increasing union of finite-dimensional projective varieties [KL2].

Given $N \in \mathfrak{g}_F$, we define $\widehat{\mathcal{B}}_N$ to be the space of Iwahori subalgebras containing N. This is a closed subvariety of $\widehat{\mathcal{B}}$. We shall restrict our attention to the topologically nilpotent (or nil) elements N, those elements such that $(\operatorname{ad} N)^m \to 0$ in the power series topology on End $\mathfrak{g}_F$ as $m \to \infty$. More concretely, these are the elements of $\mathfrak{g}_F$ which may be conjugated into some $N' \in \mathfrak{g}_A$ whose image under projection to $\mathfrak{g}$ is nilpotent. Kazhdan and Lusztig have shown in [KL2] that for N nil, the closed subvariety $\widehat{\mathcal{B}}_N$ is finite-dimensional (although possibly having infinitely many irreducible components) precisely when N is regular semisimple. If we further assume that N is elliptic, i.e. the connected centralizer of N in $G(F)$ is an anisotropic maximal torus, then $\widehat{\mathcal{P}}_N$ is a projective variety in the usual sense.

Let N be a fixed nil-elliptic element of $\mathfrak{g}_F$. We define an algebraic $\mathbf{C}$-action on $\widehat{\mathcal{B}}$ via $\varphi(t, \mathfrak{b}) = \exp(tN)(\mathfrak{b})$; the exponential makes sense because tN is nil for every $t \in \mathbf{C}$. The fixed point set of this action is simply $\widehat{\mathcal{B}}_N$. We then have the following result on the existence of affine Springer representations:

THEOREM. *Let G be a complex, semisimple, simply connected algebraic group, $\mathfrak{g}$ its Lie algebra, and $F = \mathbf{C}((\pi))$ the field of formal power series. Then for any nil-elliptic element $N \in \mathfrak{g}_F$, the affine Weyl group $\widehat{W}$ acts naturally on the homology of $\widehat{B}_N$.*

Remarks. (1) Kazhdan and Lusztig have shown that $\widehat{W}$ acts on the top homology of $\widehat{B}_N$ by a different method in [KL2]. Lusztig also has another construction of affine Springer representations [L].

(2) Kato has shown that a modified version of the affine Weyl group acts on the top homologies of the classical fixed point varieties $\mathcal{B}_u$. [K].

Choose a set S of simple reflections for $\widehat{W}$. For each $s \in S$, we let $\widehat{\mathcal{P}}_s$ be the set of minimal parahoric subalgebras of type s. More concretely, if we fix an Iwahori subalgebra $\widehat{\mathfrak{b}}$ with corresponding Iwahori subgroup $\widehat{B}$, then $\widehat{\mathfrak{p}}_s = \widehat{\mathfrak{b}} + s \cdot \mathfrak{b}$ is a subalgebra of $\mathfrak{g}_F$ with stabilizer $\widehat{P}_s$. A minimal parahoric subalgebra of type s is just a conjugate of $\widehat{\mathfrak{p}}_s$, so $\widehat{\mathcal{P}}_s$ may be identified with $G(F)/\widehat{P}_s$. This quotient space may be endowed with the structure of an infinite-dimensional variety compatible with the variety structure on $\widehat{\mathcal{B}}$ and the natural map $\widehat{\mathcal{B}} \xrightarrow{\rho} \widehat{\mathcal{P}}_s$, which becomes a $\mathbf{P}^1$-fibration.

The $\mathbf{C}$-action on $\widehat{\mathcal{B}}$ induces a unique algebraic $\mathbf{C}$-action (also denoted φ) on $\widehat{\mathcal{P}}_s$ which makes the diagram

$$\begin{array}{ccc} \mathbf{C} \times \widehat{\mathcal{B}} & \xrightarrow{\varphi} & \widehat{\mathcal{B}} \\ \downarrow & & \downarrow \\ \mathbf{C} \times \widehat{\mathcal{P}}_s & \xrightarrow{\varphi} & \widehat{\mathcal{P}}_s \end{array}$$

commute. We consider the restriction $\mathcal{E}^s \to \widehat{\mathcal{P}}_{s,N}$ of the $\mathbf{P}^1$-bundle $\widehat{\mathcal{B}} \to \widehat{\mathcal{P}}_s$ to the fixed point variety $\widehat{\mathcal{P}}_{s,\varphi} = \widehat{\mathcal{P}}_{s,N}$. The map φ is a fibrewise $\mathbf{C}$-action on $\mathcal{E}^s$ with fixed point set $\widehat{\mathcal{B}}_N$. Since a $\mathbf{C}$-action on $\mathbf{P}^1$ is just a conjugate of a translation (viewing $\mathbf{P}^1(\mathbf{C})$ as the two-sphere), there is a fixed point in each fiber. (The translation may be the identity transformation, in which case the entire fiber is fixed.) The variety $\widehat{\mathcal{P}}_{s,N}$ is thus the image of $\widehat{\mathcal{B}}_N$ under the natural projection map and is itself a finite-dimensional algebraic variety.

We now construct maps $\widehat{\mathcal{B}}_N \xrightarrow{\alpha_s} \widehat{\mathcal{B}}_N$ (defined up to homotopy) following the procedure of Kazhdan and Lusztig. We choose Riemannian metrics on the fibers, depending continously on the base space, so that each fiber is isometric to S^2. We also take a closed neighborhood M of the fixed point variety such that the inclusion $\widehat{\mathcal{B}}_N \stackrel{i}{\hookrightarrow} M$ is a proper homotopy equivalence. The map α_s is then the composition of the antipodal map on the fibers and a flow via the action into M, followed by a proper homotopy inverse to i.

In order to show that the $(\alpha_s)_*$'s define a representation of $\widehat{W}$ in $H_*(\widehat{\mathcal{B}}_N)$, it suffices to show that they satisfy the relations of $\widehat{W}$ viewed as an abstract Coxeter group with simple reflections S. Moreover, since Coxeter group relations involve at most two simple reflections at a time, it will be enough to show that the above maps induce representations of W_s in homology for each s, where W_s is the finite Weyl group obtained by omitting the generator s and ignoring relations involving it. This is possible even in the case of $\mathrm{SL}_2(\mathbf{C})$ where the affine Weyl group (the infinite dihedral group) has only two simple reflections because there is no relation linking the two.

For each s, consider the space $\mathcal{K}_s$ of maximal parahoric subalgebras of type s, i.e. those parahoric subalgebras corresponding to the subset $S \setminus \{s\}$ of S. If we identify $\widehat{\mathcal{B}}$ with $G(F)/\widehat{B}$, where $\widehat{B}$ is the stabilizer of a fixed Iwahori subalgebra $\widehat{\mathfrak{b}}$, and let K_s and $\mathfrak{k}_s$ be the maximal parahorics of type s containing $\widehat{B}$ and $\widehat{\mathfrak{b}}$ respectively, then $\mathcal{K}_s \simeq G(F)/K_s$. Just as for $\widehat{\mathcal{P}}_s$, the set $\mathcal{K}_s$ has the structure of an infinite-dimensional variety compatible with the variety structure on $\widehat{\mathcal{B}}$ and the natural projection (and with the structures on $\widehat{\mathcal{P}}_t$ for $t \neq s$). It is known that K_s

https://doi.org/10.1023/A:1000167027904 Published online by Cambridge University Press

is a projective limit of algebraic groups $(\overline{K}_s^n)_{n\geqslant 1}$ defined over the residue field **C** where $\overline{K}_s^1$ is a complex semisimple group (denoted for convenience G_s) with Weyl group W_s [BT]. There is a natural surjective homomorphism $K_s \to G_s$; let K_s^1 be its kernel and $\mathcal{K}_s^1 = G(F)/K_s^1$.

The inclusions $K_s^1 \subset \widehat{B} \subset K_s$ give us a principal G_s-bundle $\mathcal{K}_s^1 \to \mathcal{K}_s$ with associate $\mathcal{B}_s$-bundle $\widehat{\mathcal{B}} \to \mathcal{K}_s$, where $\mathcal{B}_s$ is the flag variety of G_s. The **C**-action on $\widehat{\mathcal{B}}$ induces a unique algebraic **C**-action on $\mathcal{K}_s$ making a diagram analogous to the above diagram commute. Let $\mathcal{K}_{s,\varphi} = \mathcal{K}_{s,N}$ be the fixed point variety.

LEMMA. $\mathcal{K}_{s,N}$ *is finite-dimensional.*

Proof. It suffices to show that it is the image of $\widehat{\mathcal{B}}_N$ under the natural projection. Note that φ is a fiberwise **C**-action on the restriction of the bundle $\widehat{\mathcal{B}} \to \mathcal{K}_s$ to the fixed point set $\mathcal{K}_{s,N}$. Since the fibers are projective and **C** is a connected, solvable affine algebraic group, there is a fixed point, i.e. an element of $\widehat{\mathcal{B}}_N$, in each fiber.

Form a principal G_s-bundle by restricting the principal G_s-bundle $\mathcal{K}_s^1 \to \mathcal{K}_s$ to the fixed point variety $\mathcal{K}_{s,N}$. Taking the associate $\mathcal{B}_s$-bundle, we obtain the restriction of the natural fibration $\widehat{\mathcal{B}} \to \mathcal{K}_s$. The map φ furnishes this bundle with an algebraic fiberwise **C**-action whose fixed point set is precisely $\widehat{\mathcal{B}}_N$. We are in the precise context of Kazhdan and Lusztig's original construction and thus obtain a representation of W_s in $H_*(\widehat{\mathcal{B}}_N)$.

It only remains to show that the map $\widehat{\mathcal{B}}_N \xrightarrow{\gamma_s} \widehat{\mathcal{B}}_N$ is the same as α_s for each $s \in W_s$. However, in both procedures, the maps are constructed by passing to the $\mathbf{P}^1$-bundle obtained by omitting every simple reflection besides s. In other words, the actual construction takes place in the bundle $\mathcal{E}^s \to \widehat{P}_{s,N}$ in both cases, and it does not matter that in the second method, we remove the other simple reflections in two steps instead of in one. Therefore, the $(\alpha_s)_*$'s satisfy the Coxeter relations, so the map $s \mapsto (\alpha_s)_*$ defines a group homomorphism $\widehat{W} \to \mathrm{Aut}(H_*(\widehat{\mathcal{B}}_N))$. This proves the theorem.

Remarks. (1) It is possible to prove that $(\alpha_s)_*^2 = 1$ (and in particular, to prove the theorem for the special case $G = \mathrm{SL}_2(\mathbf{C})$) without considering the varieties of maximal parahoric subalgebras. Simply notice that for the algebraic $\mathbf{P}^1$-bundle $\mathcal{E}^s \to \widehat{P}_{s,N}$, the structure group is $\mathrm{PGL}_2(\mathbf{C})$ because these are the only algebraic automorphisms of $\mathbf{P}^1$. It is then a general fact that this bundle must be the associate $\mathbf{P}^1$-bundle to a principal $\mathrm{PGL}_2(\mathbf{C})$-bundle. Therefore, Kazhdan and Lusztig's result, applied to the group $\mathrm{PGL}_2(\mathbf{C})$, shows that $s \mapsto (\alpha_s)_*$ gives a representation of the Weyl group $\mathbf{Z}/2\mathbf{Z}$.

It seems possible to extend this construction to the following more general situation. Let X be a complex algebraic variety and $E \xrightarrow{\rho} X$ a 'principal $G(F)$-bundle', i.e. let E be a set on which $G(F)$ acts freely and transitively on the right with ρ defining a bijective correspondence between orbits in E and points of X. We can formally define the associated bundles $E \times_{G(F)} \widehat{\mathcal{P}} \to X$ for any variety $\widehat{\mathcal{P}}$ of

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parahoric subalgebras. Suppose that these varieties are endowed with the structures of infinite-dimensional complex varieties compatible with the natural projections. Now assume that we have an algebraic fiberwise **C**-action φ on $\widehat{E} = E \times_{G(F)} \widehat{\mathcal{B}}$, whose fixed point variety $\widehat{E}_\varphi$ is complex algebraic in the usual sense. The above argument, replacing $\widehat{\mathcal{P}}$ by the associated $\widehat{\mathcal{P}}$-bundle, then yields a representation of $\widehat{W}$ in $H_*(\widehat{E}_\varphi)$. Affine Springer representations are the result of this construction for the special case where X is a point and $E = G(F)$.

Acknowledgements

This paper is based on part of my doctoral thesis at the University of Chicago [S]. It is a great pleasure to thank my advisor, Robert Kottwitz, for many extremely helpful comments and observations. I would also like to thank George Lusztig for some helpful comments and Sam Evens for a very useful suggestion. Finally, I am happy to acknowledge the support of a National Science Foundation Graduate Fellowship and a University of Chicago McCormick Fellowship.

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