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Limit Sets of Typical Homeomorphisms

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Abstract. Given an integer $n \geq 3$, a metrizable compact topological n-manifold X with boundary, and a finite positive Borel measure μ on X, we prove that for the typical homeomorphism $f\colon X \to X$, it is true that for μ-almost every point x in X the limit set $\omega(f,x)$ is a Cantor set of Hausdorff dimension zero, each point of $\omega(f,x)$ has a dense orbit in $\omega(f,x)$, f is non-sensitive at each point of $\omega(f,x)$, and the function $a \to \omega(f,a)$ is continuous at x.

1 Introduction

Let X be a compact metric space with metric d. We denote by $H(X)$ (respectively $C(X)$) the set of all homeomorphisms from X onto X (respectively the set of all continuous functions from X into X) endowed with the supremum metric

$$\widetilde{d}(f,g) = \sup_{x \in X} d\big(f(x), g(x)\big).$$

Moreover, X^* denotes the set of all non-empty closed subsets of X endowed with the Hausdorff metric

$$d_H(A,B) = \max\big\{\max_{a \in A} d(a,B), \max_{b \in B} d(b,A)\big\}.$$

If M is a Baire space, we say that "the typical element of M" satisfies a certain property P if the set of all $x \in M$ that satisfy property P contains a residual subset of M (that is, a countable intersection of dense open sets). The term "generic" is often used instead of "typical".

Given $f\colon X \to X$ and $x \in X$, recall that the limit set $\omega(f,x)$ of f at x is the set of all limit points of the sequence $(f^j(x))_{j \geq 0}$.

Properties of limit sets $\omega(f,x)$ of the typical function $f \in H(X)$ (respectively $f \in C(X)$) that hold at the typical point $x \in X$ were studied by Akin, Hurley, and Kennedy in [3] (respectively by Lehning in [9]) for certain classes of spaces X including the case X is a closed manifold (respectively in the case X is an arbitrary metrizable compact topological manifold with boundary). The case of the space $C([0,1])$ was considered by Agronsky, Bruckner, and Laczkovich in [1]. Here we consider a different point of view. Our goal is to study properties of limit sets $\omega(f,x)$ of the typical homeomorphism $f \in H(X)$ that hold for almost every point x with respect to a given measure on X. More precisely, we shall establish the following.

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Theorem 1.1 *Let X be a metrizable, compact, topological manifold with (or without) boundary* [10], *and fix a metric d that is compatible with its topology. Assume X has dimension $n \geq 2$; if the boundary of X is nonempty, then assume $n \geq 3$. Let μ be a finite positive Borel measure on X. For the typical $f \in H(X)$, there exists a residual set G_f such that $\mu(G_f) = \mu(X)$ and the following properties hold for every point x in G_f:*

(a) *$\omega(f, x)$ is a Cantor set of Hausdorff dimension zero.*
(b) *Each point of $\omega(f, x)$ has a dense orbit in $\omega(f, x)$.*
(c) *f is non-sensitive at each point of $\omega(f, x)$.*
(d) *The function $a \in X \mapsto \omega(f, a) \in X^*$ is continuous at x.*

Recall that a Cantor set is a totally disconnected perfect set, and recall from [2, 4] that f is non-sensitive at a if for every $\epsilon > 0$ there is $\delta > 0$ such that for any choice of points $a_0 \in B(a; \delta)$, $a_1 \in B(f(a_0); \delta)$, $a_2 \in B(f(a_1); \delta), \dots$, we have that

$$d\big(a_m, f^m(a)\big) < \epsilon \text{ for every } m \geq 0.$$

A version of Theorem 1.1 for the space $C(X)$ was obtained by the author in [6]. Some results in this direction were obtained earlier by Agronsky, Bruckner, and Laczkovich in [1] in the case $X = [0, 1]$ and $\mu =$ Lebesgue measure.

Let us remark that if X is a perfect compact metric space and μ is a finite positive Borel measure on X, then the set

$$N_\mu = \{A \in X^* \, ; \, \mu(A) = 0\}$$

is residual in X^*. In fact, since $N_\mu = \bigcap_{k=1}^{\infty} \{A \in X^* \, ; \, \mu(A) < 1/k\}$, it follows from the regularity of μ that N_μ is a G_δ set. Moreover, if Z is the set of atoms of μ, then Z is countable because μ is a finite positive measure. Since X is perfect and compact, the set $D = X - Z$ is dense in X. Therefore, the set of finite subsets of D is dense in X^* and is contained in N_μ, which proves that N_μ is dense in X^*.

2 $\mathcal{C}$-trees

Let X be a compact metric space with metric d. Given $A \subset X$, we denote by $\overline{A}$, $\operatorname{Int} A$, $\operatorname{Bd} A$, and $\operatorname{diam} A$ the closure, the interior, the boundary, and the diameter of A in X, respectively. For each $\delta > 0$,

$$N_\delta(A) = \{x \in X \, ; \, d(x, A) < \delta\}$$

is the δ-neighborhood of A. Note that

$$d_H(A, B) = \inf\{\delta > 0 \, ; \, A \subset N_\delta(B) \text{ and } B \subset N_\delta(A)\}$$

for every $A, B \in X^*$. If A and B are subsets of X, we write $A \Subset B$ to mean that $\overline{A} \subset \operatorname{Int} B$.

Given a collection $\mathcal{C}$ of nonempty closed subsets of X, recall from [5] that a $\mathcal{C}$-tree is a pair (T, φ), where T is a finite rooted tree ([8]), and φ is a bijective correspondence between the set $V(T)$ of all vertices of T and a collection of pairwise disjoint sets in $\mathcal{C}$. If (T, φ) is a $\mathcal{C}$-tree, we usually omit the correspondence φ and speak just of the $\mathcal{C}$-tree T; moreover, we identify each vertex of T with its corresponding set of $\mathcal{C}$ (under φ). If T is a $\mathcal{C}$-tree and $A, B \in V(T)$, we write "$B < A$" or "$A > B$" to mean that B is the successor of A in the tree; that is, A and B are adjacent, and the unique path connecting B to the root of T passes through A. Moreover, we define

$$Q(T) = \bigcup\{A \,;\, A \in V(T)\} \quad \text{and} \quad \theta(T) = \max\{\operatorname{diam} A \,;\, A \in V(T)\}.$$

Two $\mathcal{C}$-trees, T_1 and T_2, are said to be disjoint if $A \cap B = \varnothing$ whenever $A \in V(T_1)$ and $B \in V(T_2)$.

Given a function $f\colon X \to X$, a *$\mathcal{C}$-tree for f* is a $\mathcal{C}$-tree T that satisfies the following conditions:

(α) If $A, B \in V(T)$ and $B < A$, then $f(B) \Subset A$.
(β) If R is the root of T, then there is an $S \in V(T)$ such that $f(R) \Subset S$.

Note that such an S is necessarily unique, since the sets in $V(T)$ are pairwise disjoint. The chain

$$\widetilde{T} = \{S = A_1 < \cdots < A_k = R\}$$

of successive elements of $V(T)$ connecting S to R is called the *special branch* of T. Moreover, a *thickening* of T is a $\mathcal{C}$-tree T' with the same number of vertices such that for each $A \in V(T)$ there is (a necessarily unique) $A' \in V(T')$ with $A \Subset A'$ so that the following properties hold:

(α') If $A, B \in V(T)$ and $B < A$, then $f(B') \Subset A$.
(β') $f(R') \Subset S$.

Note that T' is also a $\mathcal{C}$-tree for f.

Proposition 2.1 *If $f \in C(X)$ and T is a $\mathcal{C}$-tree for f, then there exists $\delta > 0$ such that*

$$f\big(N_\delta(B)\big) \Subset A$$

whenever $A, B \in V(T)$ and $B < A$, and such that

$$f\big(N_\delta(R)\big) \Subset S,$$

where R and S are as in (β). Moreover,

(A) *If $a \in Q(T)$ and we choose $a_0 \in B(a;\delta)$, $a_1 \in B(f(a_0);\delta)$, $a_2 \in B(f(a_1);\delta), \dots,$ then for every $j = 1, 2, \dots$ there exists a unique $A^j \in V(T)$ such that $f(a_{j-1}), f^j(a) \in A^j$; in particular,*

$$d\big(a_j, f^j(a)\big) < \theta(T) + \delta.$$

(B) *If $x \in N_\delta(Q(T))$, then*

$$\omega(f, x) \subset \bigcup\{f(A) \,;\, A \in V(\widetilde{T})\} \Subset Q(\widetilde{T})$$

and

$$\omega(f, x) \cap A \neq \varnothing \text{ for every } A \in V(\widetilde{T}).$$

In particular, if $y \in N_\delta(Q(T))$, then $d_H(\omega(f, x), \omega(f, y)) \le \theta(T)$.

Proof Suppose $C, D \in V(T)$ and $f(D) \Subset C$. Then we have a positive distance from the compact set $f(D)$ to the compact set $X - \operatorname{Int} C$. Therefore, by the uniform continuity of f, there exists $\eta > 0$ such that $f(N_\eta(D)) \Subset C$. By choosing one such η for each pair $A, B \in V(T)$ with $B < A$ and one such η for the pair S, R, we see that the smallest of these η's is the δ we are looking for. Property (A) follows easily by induction. If $x \in N_\delta(Q(T))$, then $f(x) \in Q(T)$, and so $f^k(x) \in R$ for a certain $k \geq 1$. It follows that $f^j(x)$ belongs to the compact set $\bigcup\{f(A) \,;\, A \in V(\widetilde{T})\}$ for every $j \geq k$, and therefore $\omega(f, x)$ is contained in this compact set. Moreover, since each $A \in V(\widetilde{T})$ contains a subsequence of the sequence $(f^j(x))_{j \geq 0}$, the compactness of X implies that $\omega(f, x) \cap A \neq \varnothing$. The other assertions in (B) are clear. ■

3 Proof of Theorem 1.1

Let X, d, and n be as in the statement of Theorem 1.1, and let $i(X)$ (respectively $b(X)$) be the interior (respectively the boundary) of the manifold X ([10]). We denote by B^n and D^n the closed unit ball and the open unit ball of $\mathbb{R}^n$ with respect to the Euclidean norm, respectively, and we define

$$H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,;\, x_n \geq 0\}.$$

Moreover, $\mathcal{U}_X$ (respectively $\mathcal{V}_X$) denotes the set of all subsets A of $i(X)$ (respectively A of X) for which there is a homeomorphism $\psi\colon A \to B^n$ (respectively $\psi\colon A \to B^n \cap H^n$) with $\psi(\operatorname{Int} A) = D^n$ (respectively $\psi(\operatorname{Int} A) = D^n \cap H^n$). Finally, $\mathcal{W}_X$ (respectively $\mathcal{Z}_X$) denotes the set of all $A \in \mathcal{U}_X$ (respectively $A \in \mathcal{V}_X$) such that A has a fundamental system of neighborhoods that belong to $\mathcal{U}_X$ (respectively to $\mathcal{V}_X$). Note that each point $a \in i(X)$ (respectively $a \in b(X)$) has a fundamental system of neighborhoods that belong to $\mathcal{W}_X$ (respectively to $\mathcal{Z}_X$).

Suppose that Theorem 1.1 is true if we replace "residual set" by "G_δ set" in its statement. Let $(O_k)_{k \geq 1}$ be a countable basis for the topology of X consisting of sets O_k for which there is a homeomorphism $h_k\colon \overline{O_k} \to B^n$ with $h_k(i(X) \cap O_k) = D^n$. For each $k \geq 1$ and each Borel set S of X, define

$$\lambda_k(S) = m_n\big(h_k(S \cap \overline{O_k})\big),$$

where m_n denotes n-dimensional Lebesgue measure. Then

$$\lambda = \sum_{k=1}^{\infty} \frac{1}{2^k} \lambda_k$$

is a finite positive Borel measure on X with the property that $\lambda(U) > 0$ for every nonempty open set U in X. By replacing μ by $\mu + \lambda$, it follows from our assumption that for the typical $f \in H(X)$, there is a G_δ set G_f such that $(\mu+\lambda)(G_f) = (\mu+\lambda)(X)$ and properties (a)–(d) hold for every $x \in G_f$. Clearly, $\mu(G_f) = \mu(X)$ and $\lambda(G_f) = \lambda(X)$. Since $\lambda(G_f) = \lambda(X)$ implies that G_f is dense in X, we conclude that G_f is necessarily a residual subset of X. This shows that it is enough to prove Theorem 1.1 with "G_δ set" instead of "residual set". Moreover, it is enough to consider the case

"$\mu(b(X)) = 0$" and the case "$b(X) \neq \varnothing$ and $\mu(i(X)) = 0$". So, we divide the proof into these two cases.

CASE I: $\mu(b(X)) = 0$.

For each integer $k \geq 1$, let $\mathcal{O}_k$ be the set of all $f \in H(X)$ for which there are finitely many pairwise disjoint $\mathcal{W}_X$-trees $T_1, \ldots, T_s$ for f such that:

(i) $\theta(T_i) < 1/k$ for all $1 \leq i \leq s$;

(ii) $\mu(X - (Q(T_1) \cup \cdots \cup Q(T_s))) < 1/k$.

(iii) For each $1 \leq i \leq s$, the special branch of T_i has the form

$$S_i = A_{i,1} < A_{i,2} < \cdots < A_{i,d_i} < B_{i,1} < B_{i,2} < \cdots < B_{i,d_i} = R_i,$$

where

$$\operatorname{diam}(A_{i,j} \cup B_{i,j}) < 1/k \text{ for } 1 \leq j \leq d_i.$$

(iv) $\sum_{i=1}^{s} \sum_{j=1}^{d_i} [(\operatorname{diam} f(A_{i,j}))^{1/k} + (\operatorname{diam} f(B_{i,j}))^{1/k}] < 1$.

Clearly, each $\mathcal{O}_k$ is open in $H(X)$. Let $f \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$. Then, for each $k \geq 1$, there are pairwise disjoint $\mathcal{W}_X$-trees $T_{k,1}, \ldots, T_{k,s_k}$ for f so that (i)–(iv) hold with $T_{k,1}, \ldots, T_{k,s_k}$ in place of $T_1, \ldots, T_s$. By replacing each $T_{k,i}$ by a thickening, if necessary, we may assume the following strengthening of property (ii):

(ii′) $\mu(X - (\operatorname{Int} Q(T_{k,1}) \cup \cdots \cup \operatorname{Int} Q(T_{k,s_k}))) < 1/k$.

Put

$$Q_k = Q(T_{k,1}) \cup \cdots \cup Q(T_{k,s_k}) \quad (k \geq 1) \quad \text{and} \quad G = \bigcap_{r=1}^{\infty} \bigcup_{k=r}^{\infty} \operatorname{Int} Q_k.$$

Then G is a G_δ set. By property (ii′), $\mu(X - \operatorname{Int} Q_k) < 1/k$ for every $k \geq 1$, and therefore $\mu(G) = \mu(X)$. Moreover, by Proposition 2.1, for each $k \geq 1$ there exists $0 < \delta_k < 1/k$ such that $f(N_{\delta_k}(B)) \Subset A$ whenever $A, B \in V(T_{k,i})$ and $B < A$ $(1 \leq i \leq s_k)$, and such that

$$f\big(N_{\delta_k}(R_{k,i})\big) \Subset S_{k,i},$$

where $R_{k,i}$ and $S_{k,i}$ are related to $T_{k,i}$ $(1 \leq i \leq s_k)$ as R and S are related to T in property (β). Property (iii) tells us that

$$S_{k,i} = A_{k,i,1} < \cdots < A_{k,i,d_{k,i}} < B_{k,i,1} < \cdots < B_{k,i,d_{k,i}} = R_{k,i},$$

where

$$\operatorname{diam}(A_{k,i,j} \cup B_{k,i,j}) < 1/k \ \text{ for } 1 \leq j \leq d_{k,i}. \tag{3.1}$$

By Proposition 2.1(B), if $x \in Q_k$, $y \in X$ and $d(y, x) < \delta_k$, then

$$\omega(f, y) \subset f(A_{k,i,1}) \cup \cdots \cup f(A_{k,i,d_{k,i}}) \cup f(B_{k,i,1}) \cup \cdots \cup f(B_{k,i,d_{k,i}}) \tag{3.2}$$

$$\subset A_{k,i,1} \cup \cdots \cup A_{k,i,d_{k,i}} \cup B_{k,i,1} \cup \cdots \cup B_{k,i,d_{k,i}}, \tag{3.3}$$

$$\omega(f, y) \cap A_{k,i,j} \neq \varnothing \text{ and } \omega(f, y) \cap B_{k,i,j} \neq \varnothing \text{ for } 1 \leq j \leq d_{k,i}, \tag{3.4}$$

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and because of (i),

$$d_H\big(\omega(f,y),\omega(f,x)\big) < 1/k, \tag{3.5}$$

where $1 \le i \le s_k$ is such that $x \in Q(T_{k,i})$. Thus, if $x \in G$, then the limit set $\omega(f,x)$ is totally disconnected (by (i) and (3.3)), perfect (by (3.1), (3.3), and (3.4)) and has Hausdorff dimension zero (by (i), (iv), and (3.2)), which gives property (a). Property (d) follows from (3.5), and property (b) follows from (i) and (3.3). Finally, (c) holds in view of (i), (3.3), and Proposition 2.1(A).

It remains to show that each $\mathcal{O}_k$ is dense in $H(X)$. Fix $k \ge 1$, $h \in H(X)$, and $\epsilon > 0$. In the proof of [7, Theorem 1 (case I)], we saw that the set of all $f \in H(X)$ for which there are finitely many pairwise disjoint $\mathcal{W}_X$-trees $T_1,\dots,T_s$ for f satisfying properties (i) and (ii) is dense in $H(X)$. Therefore, if we choose an integer $k' \ge k$ such that $1/k' < \epsilon/3$, then there are a function $g \in H(X)$ and pairwise disjoint $\mathcal{W}_X$-trees $T_1',\dots,T_s'$ for g so that $\widetilde{d}(g,h) < \epsilon/3$ and (i) and (ii) hold with $T_1',\dots,T_s'$ in place of $T_1,\dots,T_s$ and k' in place of k. Let R_i' denote the root of T_i', and let $S_i' \in V(T_i')$ be such that $g(R_i') \Subset S_i'$ $(1 \le i \le s)$. Now we will have to enlarge the trees $T_1',\dots,T_s'$ and make some small pertubations on g in order to obtain property (iii). Let

$$S_i' = A_{i,1} < \cdots < A_{i,d_i} = R_i'$$

be the special branch of T_i'. For each $1 \le i \le s$ and each $1 \le j \le d_i$, choose a neighborhood $U_{i,j}$ of $A_{i,j}$ that belongs to $\mathcal{U}_X$ and has diameter $< 1/k'$ so that the family $\{U_{i,j}\}_{1\le i\le s, 1\le j\le d_i}$ is pairwise disjoint,

$$g(U_{i,j}) \Subset A_{i,j+1} \ \text{ for } 1 \le j < d_i \quad \text{and} \quad g(U_{i,d_i}) \Subset A_{i,1}.$$

We may also assume that $U_{i,j}$ is disjoint from every $A \in (V(T_1')\cup\cdots\cup V(T_s')) - \{A_{i,j}\}$. For each $1 \le i \le s$ and each $1 \le j \le d_i$, choose $B_{i,j} \in \mathcal{W}_X$ such that $B_{i,j} \Subset U_{i,j}$ and $B_{i,j} \cap A_{i,j} = \varnothing$. We enlarge each T_i' by putting $B_{i,1},\dots,B_{i,d_i}$ as new vertices satisfying $A_{i,d_i} < B_{i,1} < \cdots < B_{i,d_i}$. In this way we obtain a new tree T_i whose root is $R_i = B_{i,d_i}$ $(1 \le i \le s)$. Clearly, the trees $T_1,\dots,T_s$ so constructed satisfy (i), (ii), and (iii). For each $1 \le i \le s$ and each $1 < j \le d_i$, let $\phi_{i,j} : U_{i,j} \to U_{i,j}$ be a homeomorphism such that

- $\phi_{i,j}\big(g(B_{i,j-1})\big) \Subset B_{i,j}$, and
- $\phi_{i,j}$ is the identity map on $\operatorname{Bd} U_{i,j}$ and on $g(A)$ whenever $A < A_{i,j}$.

For each $1 \le i \le s$, let $\phi_{i,1} : U_{i,1} \to U_{i,1}$ be a homeomorphism such that

- $\phi_{i,1}\big(g(A_{i,d_i})\big) \Subset B_{i,1}$, and
- $\phi_{i,1}$ is the identity map on $\operatorname{Bd} U_{i,1}$, on $g(B_{i,d_i})$ and on $g(A)$ whenever $A < A_{i,1}$.

The existence of the homeomorphisms $\phi_{i,j}$ follows from the following argument. We may think of $U_{i,j}$ as being B^n with $\operatorname{Int} U_{i,j}$ being D^n. There are a finite number of subsets of D^n, say $D_1,\dots,D_\alpha$, on which we want $\phi_{i,j}$ to coincide with the identity map. Since each of these sets has a fundamental system of neighborhoods whose boundaries are path connected, it follows that $D^n - (D_1 \cup \cdots \cup D_\alpha)$ is path connected. There are also two subsets of D^n, say D and E, such that we want $\phi_{i,j}$

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to satisfy $\phi_{i,j}(D) \subset \operatorname{Int} E$. Choose a point $a \in D$ and a point $b \in \operatorname{Int} E$, and let $\gamma\colon [0,1] \to D^n - (D_1 \cup \cdots \cup D_\alpha)$ be a path from a to b. Cover $\gamma([0,1])$ by a finite number of open balls $B_1, \ldots, B_\beta$ whose closures are contained in $D^n - (D_1 \cup \cdots \cup D_\alpha)$ so that

$$a \in B_1, B_i \cap B_{i+1} \neq \varnothing \ (1 \leq i < \beta) \quad \text{and} \quad b \in B_\beta \subset \operatorname{Int} E.$$

Let V_D be a neighborhood of D contained in $D^n - (D_1 \cup \cdots \cup D_\alpha)$ such that there is a homeomorphism from V_D onto B^n mapping $\operatorname{Int} V_D$ onto D^n. By working on $V_D \cup \overline{B_1} \cup \cdots \cup \overline{B_\beta}$, we can construct a homeomorphism $\phi_{i,j}\colon U_{i,j} \to U_{i,j}$ such that $\phi_{i,j}(D) \subset \operatorname{Int} E$ and $\phi_{i,j}(x) = x$ for all $x \in U_{i,j} - (V_D \cup \overline{B_1} \cup \cdots \cup \overline{B_\beta})$.

Now, we define $\phi\colon X \to X$ by $\phi = \phi_{i,j}$ on each $U_{i,j}$ and

$$\phi(x) = x \text{ for all } x \in X - \bigcup\{U_{i,j}\,;\, 1 \leq i \leq s, 1 \leq j \leq d_i\}.$$

Put $u = \phi \circ g$. Then $u \in H(X)$, $\widetilde{d}(u,g) < \epsilon/3$ and $T_1, \ldots, T_s$ are trees for u. Finally, in order to obtain property (iv), let us now denote the special branch of T_i by

$$S_i = C_{i,1} < \cdots < C_{i,2d_i} = R_i.$$

For each $1 \leq i \leq s$ and each $1 \leq j \leq 2d_i$, choose a neighborhood $V_{i,j}$ of $C_{i,j}$ that belongs to $\mathcal{U}_X$ and has diameter $< 1/k'$ so that the family $\{V_{i,j}\}_{1 \leq i \leq s, 1 \leq j \leq 2d_i}$ is pairwise disjoint and each $V_{i,j}$ is disjoint from every $A \in \big(V(T_1') \cup \cdots \cup V(T_s')\big) - \{C_{i,j}\}$. Let ψ be a homeomorphism of X onto X that coincides with the identity map outside the $V_{i,j}$'s, maps each $V_{i,j}$ onto itself, and maps each $C_{i,j}$ onto a subset of $\operatorname{Int} C_{i,j}$ of very small diameter. Then $f = \psi \circ u \in H(X)$, $\widetilde{d}(f,u) < \epsilon/3$ (so that $\widetilde{d}(f,h) < \epsilon$), and $T_1, \ldots, T_s$ are trees for f. Moreover, by choosing ψ so that the sets $\psi(C_{i,j})$ have sufficiently small diameters, we have that f also satisfies (iv).

CASE II: $b(X) \neq \varnothing$ and $\mu(i(X)) = 0$.

For each integer $k \geq 1$, let $\mathcal{O}_k$ be the set of all $f \in H(X)$ for which there are finitely many pairwise disjoint $\mathcal{Z}_X$-trees $T_1, \ldots, T_s$ for f satisfying properties (i)–(iv) as in CASE I. Then, each $\mathcal{O}_k$ is open in $H(X)$ and each $f \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$ has the desired properties. In order to prove that each $\mathcal{O}_k$ is dense in $H(X)$, we fix $k \geq 1$, $h \in H(X)$ and $\epsilon > 0$. In the proof of [7, Theorem 1 (Case II)], we saw that the set of all $f \in H(X)$ for which there are finitely many pairwise disjoint $\mathcal{Z}_X$-trees $T_1, \ldots, T_s$ for f satisfying properties (i) and (ii) is dense in $H(X)$. Therefore, if we choose an integer $k' \geq k$ such that $1/k' < \epsilon/3$, then there are a function $g \in H(X)$ and pairwise disjoint $\mathcal{Z}_X$-trees $T_1', \ldots, T_s'$ for g so that $\widetilde{d}(g,h) < \epsilon/3$ and (i) and (ii) hold with $T_1', \ldots, T_s'$ in place of $T_1, \ldots, T_s$ and k' in place of k. Now, it is enough to continue by arguing as in CASE I, but we need to consider the collection $\mathcal{Z}_X$ instead of $\mathcal{W}_X$ and the collection $\mathcal{V}_X$ instead of $\mathcal{U}_X$.

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