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TWO NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXTENSION OF MÖBIUS GROUPS

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Let $SL(2, \Gamma_n)$ be the n-dimensional Clifford matrix group and $G \subset SL(2, \Gamma_n)$ be a non-elementary subgroup. We show that G is the extension of a subgroup of $SL(2, \mathbb{C})$ if and only if G is conjugate in $SL(2, \Gamma_n)$ to a group G' which satisfies the following properties:

(1) there exist loxodromic elements g_0, $h \in G'$ such that $\operatorname{fix}(g_0) = \{0, \infty\}$, $\operatorname{fix}(g_0) \cap \operatorname{fix}(h) = \emptyset$ and $\operatorname{fix}(h) \cap \mathbb{C} \neq \emptyset$;

(2) $\operatorname{tr}(g) \in \mathbb{C}$ for each loxodromic element $g \in G'$.

Further G is the extension of a subgroup of $SL(2, \mathbb{R})$ if and only if G is conjugate in $SL(2, \Gamma_n)$ to a group G' which satisfies the following properties:

(1) there exists a loxodromic element $g_0 \in G'$ such that $\operatorname{fix}(g_0) \cap \{0, \infty\} \neq \emptyset$;

(2) $\operatorname{tr}(g) \in \mathbb{R}$ for each loxodromic element $g \in G'$.

The discreteness of subgroups of $SL(2, \Gamma_n)$ is also discussed.

1. INTRODUCTION AND MAIN RESULTS

As in [1] or [8], let $SL(2, \Gamma_n)$ denote the n-dimensional Clifford matrix group and $M(\overline{R}^n)$ the full group of n-dimensional sense-preserving Möbius transformations.

In the study of higher dimensional Möbius groups, the following two problems are fundamental and interesting.

PROBLEM 1. When is a subgroup $G \subset SL(2, \Gamma_n)$ the extension of a group of $SL(2, \mathbb{R})$?

PROBLEM 2. When is a subgroup $G \subset SL(2, \Gamma_n)$ the extension of a group of $SL(2, \mathbb{C})$?

Here G is called *the extension* of a subgroup of $SL(2, \mathbb{C})$ (or $SL(2, \mathbb{R})$) if G is conjugate in $SL(2, \Gamma_n)$ to a subgroup of $SL(2, \mathbb{C})$ (or $SL(2, \mathbb{R})$, respectively).

Many authors have discussed these two problems. For *Problem* 1, when $n = 2$, Maskit ([6]) proved

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THEOREM M. *Let $G \subset \mathrm{SL}(2,\mathbb{C})$ be a Kleinian group in which $\mathrm{tr}^2(g) \geqslant 0$ for all $g \in G$. Then G is Fuchsian.*

When $n \geqslant 2$, Apanasov ([**2**]) proved

THEOREM A. *If $G \subset \mathrm{SL}(2,\Gamma_n)$ is non-elementary and each nontrivial element of G is either hyperbolic or strictly parabolic or strictly elliptic, then G is the extension of a group of $\mathrm{SL}(2,\mathbb{R})$.*

Subsequently, we generalised Theorems M and A into the following form, (see [**7**]).

THEOREM WY. *Let $G \subset \mathrm{SL}(2,\Gamma_n)$ be non-elementary. If each loxodromic element of G is hyperbolic, then G is the extension of a group of $\mathrm{SL}(2,\mathbb{R})$.*

It is well-known that the trace of an element of $\mathrm{SL}(2,C)$ is conjugate invariant in $\mathrm{SL}(2,C)$. This property does not hold in $\mathrm{SL}(2,\Gamma_n)$ when $n \geqslant 3$. In order to overcome this difficulty, Theorems A and WY require that each loxodromic element of G is hyperbolic, since the trace of a hyperbolic element is conjugate invariant in $\mathrm{SL}(2,\Gamma_n)$. A natural problem is how to characterise subgroups of $\mathrm{SL}(2,\Gamma_n)$ without requiring each loxodromic element be hyperbolic. As the first aim of this paper, we shall consider this problem. By using different method, we shall prove

THEOREM 1. *Let $G \subset \mathrm{SL}(2,\Gamma_n)$ be non-elementary. Then G is the extension of a group of $\mathrm{SL}(2,\mathbb{R})$ if and only if G is conjugate in $\mathrm{SL}(2,\Gamma_n)$ to G' which satisfies the following properties:*

(1) *there exists a loxodromic element $g_0 \in G'$ such that $\mathrm{fix}(g_0) \cap \{0,\infty\} \neq \emptyset$; and*

(2) *$\mathrm{tr}(g) \in \mathbb{R}$ for each loxodromic element $g \in G'$.*

COROLLARY 1. *Let $G \subset \mathrm{SL}(2,\Gamma_n)$ be non-elementary. If each loxodromic element of G is hyperbolic and each elliptic element of G (if any) is of finite order, then G is discrete.*

REMARK 1. Obviously, Theorem 1 is a generalisation of Theorems M, A and WY. Example 1 shows the difference between Theorem 1 and Theorem WY.

Concerning Problem 2, recently Chen ([**4**]) proved

THEOREM C. *Let $G \subset \mathrm{SL}(2,\Gamma_n)$ be non-elementary. If G contains hyperbolic elements, then G is the extension of a group of $\mathrm{SL}(2,\mathbb{C})$ if and only if G is conjugate in $\mathrm{SL}(2,\Gamma_n)$ to G' which satisfies the following properties:*

(1) *there exist hyperbolic elements g_0, $h \in G'$ such that $\mathrm{fix}(g_0) = \{0,\infty\}$, $\mathrm{fix}(g_0) \cap \mathrm{fix}(h) = \emptyset$ and $\mathrm{fix}(h) \cap \mathbb{C} \neq \emptyset$; and*

(2) *$\mathrm{tr}(g) \in \mathbb{C}$ for each $g \in G'$.*

The following statement is obvious.

FACT. Each non-elementary subgroup of $\mathrm{SL}(2,\mathbb{C})$ (that is, $\mathrm{SL}(2,\Gamma_2)$) is the extension of a group of $\mathrm{SL}(2,\mathbb{C})$.

But when $n = 2$, Theorem C does not coincide with the above stated fact. This means that the condition "G containing hyperbolic elements" in Theorem C is too strict. We can see from [4] that this condition plays a key role in the proof. As the second aim of this paper, we shall study Theorem C further and prove the following.

THEOREM 2. *Let $G \subset \mathrm{SL}(2, \Gamma_n)$ be non-elementary. Then G is the extension of a group of $\mathrm{SL}(2, \mathbb{C})$ if and only if G is conjugate in $\mathrm{SL}(2, \Gamma_n)$ to G' which satisfies the following properties:*

(i) *there exist loxodromic elements g_0, $h \in G'$ such that $\mathrm{fix}(g_0) = \{0, \infty\}$, $\mathrm{fix}(g_0) \cap \mathrm{fix}(h) = \emptyset$ and $\mathrm{fix}(h) \cap \mathbb{C} \neq \emptyset$; and*

(ii) *$\mathrm{tr}(g) \in \mathbb{C}$ for each loxodromic element $g \in G'$.*

COROLLARY 2. *Let $G \subset \mathrm{SL}(2, \Gamma_n)$ be non-elementary. If G is conjugate in $\mathrm{SL}(2, \Gamma_n)$ to G' which satisfying properties* (i) *and* (ii) *as in Theorem 2, then G is discrete if and only if each non-elementary subgroup of G generated by two loxodromic elements is discrete.*

REMARK 2. Obviously, Theorem 2 is a generalisation of Theorem C. Also when $n = 2$, Theorem 2 completely coincides with the above stated fact, since the traces of elements of $\mathrm{SL}(2, \mathbb{C})$ are invariant under the conjugation in $\mathrm{SL}(2, \mathbb{C})$.

We shall prove Theorems 1, 2 and Corollaries 1, 2 in Section 3. In Section 2, we shall introduce some necessary material which is needed in Section 3.

2. PRELIMINARIES

We need the following preliminaries, see [1, 8] for more detail.

Let Γ_n denote the n-dimensional Clifford group, $\mathrm{SL}(2, \Gamma_n)$ the group of all n-dimensional Clifford matrices and

$$\mathrm{PSL}(2, \Gamma_n) = \mathrm{SL}(2, \Gamma_n)/\{\pm I\},$$

where I is the unit matrix.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \Gamma_n)$ correspond to the mapping in $\overline{R^n}$

$$x \mapsto Ax = (ax + b)(cx + d)^{-1}.$$

Then this is an isomorphism between $\mathrm{PSL}(2, \Gamma_n)$ and $M(\overline{R^n})$. We shall identify the element in $M(\overline{R^n})$ with its corresponding element in $\mathrm{PSL}(2, \Gamma_n)$.

In the following, we shall consider a more general case; that is, we shall consider subgroups in $\mathrm{SL}(2, \Gamma_n)$ instead of those in $\mathrm{PSL}(2, \Gamma_n)$.

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A nontrivial element $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\Gamma_n)$ is called is *loxodromic* if f is conjugate in $\mathrm{SL}(2,\Gamma_n)$ to $\begin{pmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix}$, where $r > 0$, $r \neq 1$, $\lambda \in \Gamma_n$ and $|\lambda| = 1$; in particular, we say that f is *hyperbolic* if $\lambda = \pm 1$.

Let

$$\mathrm{tr}(f) = a + d^* \quad \text{and} \quad \mathrm{fix}(f) = \{x \in \overline{\mathbb{R}}^n : f(x) = x\}.$$

We say that f is vectorial if $b, c \in \overline{\mathbb{R}}^n$. Then we have (see [1])

Lemma 1. *A nontrivial element f is hyperbolic if and only if f is vectorial and* $\mathrm{tr}^2(f) > 4$.

Corollary 3. *Let $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\Gamma_n)$ be loxodromic. Then f is hyperbolic if and only if $b^* = b$, $c^* = c$ and* $\mathrm{tr}(f) \in \mathbb{R}$.

For any loxodromic element $g \in \mathrm{SL}(2,\mathbb{C})$, g is hyperbolic if and only if $\mathrm{tr}(g) \in \mathbb{R}$. But the following example shows that when $n > 2$, this statement is not true.

Example 1. Let

$$g = \begin{pmatrix} 2e_1e_2 & 3e_1e_2 \\ e_1e_2 & 2e_1e_2 \end{pmatrix}.$$

Then g is loxodromic and $\mathrm{tr}(g) \in \mathbb{R}$, but g is not hyperbolic.

Let $\mathbb{H}^{n+1} = \{x : x = x_0 + x_1e_1 + \cdots + x_ne_n \in \overline{\mathbb{R}}^{n+1},\ x_n > 0\}$ and $\overline{\mathbb{H}}^{n+1} = \mathbb{H}^{n+1} \cup \overline{\mathbb{R}}^n$.

As in [3] we call, a subgroup $G \subset \mathrm{SL}(2,\Gamma_n)$, *elementary* if there exists some $x \in \overline{\mathbb{H}}^{n+1}$ such that the G-orbit $G(x) = \{g(x) : g \in G\}$ at x is finite. Otherwise G is called *non-elementary*. It follows from [**3, 7**] that if G is non-elementary, then G contains infinitely many loxodromic elements, no two of which have a common fixed points.

3. Proofs of the main results

Firstly, we introduce a lemma.

Lemma 2. *Let $G \subset \mathrm{SL}(2,\Gamma_n)$ be non-elementary and $g_0 \in G$ be loxodromic with $\mathrm{fix}(g_0) = \{0,\infty\}$. If $\mathrm{tr}(g) \in \mathbb{C}$ for any loxodromic element $g \in G$, then for any $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, $a, d \in \mathbb{C}$.*

Proof: If f interchanges the two fixed points of g_0 or $\mathrm{fix}(f) \cap \mathrm{fix}(g_0) \neq \emptyset$, then the result is obvious. Now we assume that g does not interchange 0 and ∞, and $\mathrm{fix}(g) \cap \{0,\infty\} = \emptyset$. Then $\max\{|a|,|d|\} > 0$ and $bc \neq 0$. To replace f by f^{-1} if needed, we may assume that $a \neq 0$. Then by [**7**, Lemma 3.3], we see that $g_0^m f$ are loxodromic for all large enough m. This completes the proof. ∎

PROOF OF THEOREM 1: The proof follows from [7, Theorem 4.1] and the following lemma.

LEMMA 3. *Let $G' \subset \mathrm{SL}(2,\Gamma_n)$ be non-elementary. If G' satisfies the following properties:*

(1) *there exists a loxodromic element $g_0 \in G'$ such that $\mathrm{fix}(g_0) \cap \{0,\infty\} \neq \emptyset$;*

(2) *$\mathrm{tr}(g) \in \mathbb{R}$ for each loxodromic element $g \in G'$,*

then each loxodromic element in G' is hyperbolic.

PROOF: Without loss of generality, we may assume that

$$g_0 = \begin{pmatrix} r & t \\ 0 & r^{-1} \end{pmatrix},$$

where $r \in \mathbb{R}$, $|r| > 1$ and $t \in \overline{\mathbb{R}^n}$.

By the similar reasoning as in the proof of [7, Theorem 4.1], we may assume further that $t \in \mathbb{R}$.

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any loxodromic element in G'.

If $c = 0$, then $g_0^m g$ are loxodromic for all large enough m. Condition (2) in Lemma 3 implies that $a,\ d \in \mathbb{R}$ and $b,\ c \in \overline{\mathbb{R}^n}$. By Corollary 3, we know that g is hyperbolic.

Now we assume that $c \neq 0$. To replace g by g^{-1} if needed, we may assume that $g(\infty) \notin \mathrm{fix}(g_0)$. Then $g_0^m g$ and $g g_0^m$ are loxodromic for all sufficiently large m. Condition (2) in Lemma 3 implies that

$$\left\{a + \frac{t}{r - r^{-1}}c,\ \ a + \frac{t}{r - r^{-1}}c^*\right\} \subset \mathbb{R}.$$

Hence $c^* = c$. It follows from $\Delta(g) = ad^* - bc^* = 1$ that $b^* = b$. Then Corollary 3 tells us that g is hyperbolic.

The proof of our lemma is completed. □

PROOF OF THEOREM 2: The necessity is obvious. In the following we prove the sufficiency.

By conditions (i) and (ii), we may assume that g_0 has the form:

$$g_0 = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix},$$

where $r \in \mathbb{C}$ and $|r| > 1$.

In the following, we shall prove that $h \in \mathrm{SL}(2, C)$.

Let

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $bc \neq 0$, $\max\{|a|, |d|\} > 0$ and $a + d^* \in \mathbb{C}$. Without loss of generality, we may assume that $a \neq 0$. Otherwise we replace h by h^{-1}. Then Lemma 2 implies that $a, d \in \mathbb{C}$.

It follows from ab^* and $a^*c \in \overline{\mathbb{R}^n}$ that h has the form:

$$h = \begin{pmatrix} a & as \\ a'q & d \end{pmatrix},$$

where $s = s_0 + \sum_{i=2}^{n-1} s_i e_i$ ($s_0 \in \mathbb{C}$, $s_i \in \mathbb{R}$), $q = q_0 + \sum_{i=2}^{n-1} q_i e_i$ ($q_0 \in \mathbb{C}$, $q_i \in \mathbb{R}$).

Now $\Delta(h) = ad^* - (as)(a'q)^* = 1$ implies that $sq \in \mathbb{C}$. Hence $s \in \mathbb{C}$ if and only if $q \in \mathbb{C}$, since $sq \neq 0$.

It follows from $das \in \overline{\mathbb{R}^n}$ that $ad \in \mathbb{R}$ or $s \in \mathbb{C}$. We claim that $s \in \mathbb{C}$. Suppose $s \notin \mathbb{C}$. Then $ad \in \mathbb{R}$. We may assume that

$$d = ka',$$

where $k \in \mathbb{R}$. Then we have

$$h = \begin{pmatrix} a & as \\ a'q & ka' \end{pmatrix}.$$

This implies that $sq \in \mathbb{R}$. Hence there exists $k_1 \in \mathbb{R}$ such that $q = k_1 s'$. Under the conjugation of a suitable element in $\mathrm{SL}(2, \mathbb{R})$, we may assume that

$$g = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} a & as \\ \varepsilon a's' & ka' \end{pmatrix},$$

where $r \in \mathbb{C}$, $|r| > 1$, $\varepsilon = \pm 1$, $s = s_0 + \sum_{i=2}^{n-1} s_i e_i$, $s_0 \in \mathbb{C}$ and $s_i,\ k \in \mathbb{R}$.

We see from $\mathrm{fix}(h) \cap \mathbb{C} \neq \emptyset$ and h being loxodromic that $s_0 \neq 0$ and $a' = -\varepsilon a$. Hence

$$h = \begin{pmatrix} a & as \\ -as' & -\varepsilon ka \end{pmatrix}.$$

Since h^2 is loxodromic, by Lemma 2, we know that $a^2 - asas' \in \mathbb{C}$, which implies that $sas' \in \mathbb{C}$. Then $\overline{a} = a$, that is, $a \in \mathbb{R}\backslash\{0\}$. By Corollary 3, h is hyperbolic. Then, by [**1**],

$$\mathrm{fix}(h) = \{t_1 s,\ \ t_2 s\},$$

where $t_{1,2} = -\left[(1 + \varepsilon k)a \pm \sqrt{(1 - k\varepsilon)^2 a^2 - 4}\right] / 2a^{-1}|s|^{-2} \in \mathbb{R}$.

Condition (i) implies that $s \in \mathbb{C}$. This contradiction shows that $s \in \mathbb{C}$.

Our claim implies that h has the following form:

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c, $d \in \mathbb{C}$ with $bc \neq 0$.

For any nontrivial element

$$p = \begin{pmatrix} u & v \\ \alpha & \beta \end{pmatrix} \in G',$$

by Lemma 2, we know that u, $\beta \in \mathbb{C}$. By considering pg, Lemma 2 implies that v, $\alpha \in \mathbb{C}$. This shows that $p \in \mathrm{SL}(2, \mathbb{C})$ which completes the proof. ∎

PROOF OF COROLLARY 1: If each loxodromic element of G is hyperbolic, then Theorem 1 yields that G is conjugate in $\mathrm{SL}(2, \Gamma_n)$ to a group G' of $\mathrm{SL}(2, \mathbb{R})$. Then [**5**, Theorem 2] or [**3**, Theorem 8.4.1] implies that G' is discrete. Hence G is discrete. ∎

PROOF OF COROLLARY 2: The proof follows from [**9**, Theorem 2]. ∎

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