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Coupled Vortex Equations on Complete Kähler Manifolds

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Abstract. In this paper, we first investigate the Dirichlet problem for coupled vortex equations. Secondly, we give existence results for solutions of the coupled vortex equations on a class of complete noncompact Kähler manifolds which include simply-connected strictly negative curved manifolds, Hermitian symmetric spaces of noncompact type and strictly pseudo-convex domains equipped with the Bergmann metric.

1 Introduction

The Hermitian–Einstein equation is of great importance in the study of holomorphic vector bundles over Kähler manifolds. The main result due to Donaldson [2] and Uhlenbeck–Yau [14] is the Hitchin–Kobayashi correspondence relating the stability of the underlying bundle over closed Kähler manifold to the existence of Hermitian–Einstein metric, *i.e.,* a Hermitian metric H solving the Hermitian–Einstein equation

$$\sqrt{-1}\Lambda F_H = \lambda \operatorname{Id}_E, \tag{1.1}$$

where F_H is the curvature of the Chern connection of the metric H, Λ is the contraction with the Kähler form of M, and λ is a real number determined by the topology of the underlying bundle. The classical Hitchin–Kobayashi correspondence has many important generalizations, for example: Higgs bundles by Hitchin [8] and Simpson [12], the vortex equation by Bradlow [1], and the coupled vortex equation by Garcia-Prada [4].

Let M be a Kähler manifold, E_1 and E_2 holomorphic vector bundles over M, and ϕ a holomorphic morphism from E_2 to E_1, $\tau = (\tau_1, \tau_2) \in R^2$. Then $H = (H_1, H_2)$ is a Hermitian metric on bundle $E = E_1 \oplus E_2$, where H_i is a Hermitian metric on E_i. We say H satisfies the coupled vortex equations if

$$\sqrt{-1}\Lambda F_{H_1} + \tfrac{1}{2}\phi \circ \phi^{*H} = \tau_1 \operatorname{Id}_{E_1} \qquad \sqrt{-1}\Lambda F_{H_2} - \tfrac{1}{2}\phi^{*H} \circ \phi = \tau_2 \operatorname{Id}_{E_2}, \tag{1.2}$$

where $i = 1, 2$ and ϕ^{*H} is the adjoint of ϕ taken with respect to metric H. The coupled vortex equations can be seen as a generalization of the Hermitian–Einstein equation and the vortex equation. For example, if there is only one holomorphic vector bundle and ϕ is trivial, then (1.2) is just the Hermitian–Einstein equation (1.1). If $E_2 = O_M$, *i.e.,* the canonical line bundle over M, then (1.2) is just the vortex equation which was discussed by Bradlow [1].

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In this paper, we want to consider the solution of coupled vortex equations over complete Kähler manifolds. We would like to point out that Ni [10], Ren [11], and Zhang [15] discussed the existence of Hermitian–Einstein metrics and vortices metrics over complete Kähler manifolds, and we will adapt the techniques used by them.

The corresponding Dirichlet boundary value problem for the Hermitian–Einstein equation was done by Donaldson [3]. Moreover, Zhang [15] solved the case for the vortex equation. In this paper, we first consider the Dirichlet boundary value problem for the coupled vortex equations. We obtain the following theorem.

Theorem 1.1 *Let $E = E_1 \oplus E_2$ be a holomorphic vector bundle over the compact Kähler manifold $\overline{M}$ with non-empty smooth boundary ∂M, where E_1, E_2 are two holomorphic bundles on M with holomorphic morphism $\phi\colon E_2 \to E_1$. For any Hermitian metric $\varphi = (\varphi_1, \varphi_2)$ on the restriction of $E = E_1 \oplus E_2$ to ∂M, there is a unique Hermitian metric $H = (H_1, H_2)$ on E such that H satisfies the coupled vortex equations and the Dirichlet boundary condition:*

$$\sqrt{-1}\Lambda F_{H_1} + \frac{1}{2}\phi \circ \phi^{*H} = \tau_1 \operatorname{Id}_{E_1} \quad \sqrt{-1}\Lambda F_{H_2} - \frac{1}{2}\phi^{*H} \circ \phi = \tau_2 \operatorname{Id}_{E_2} \quad H_i|_{\partial M} = \varphi_i$$

where $i = 1, 2$.

We will use the heat equation method to prove Theorem 1.1, and adapt the techniques which already appear in the literature on the Hermitian Yang–Mills flow [3, 12, 13]. Using the above solubility of Dirichlet problem, we can study the existence of the solution of coupled vortex equations over a class of complete noncompact Kähler manifolds, under some assumptions on the initial metric and the holomorphic morphism ϕ. We obtain the following.

Theorem 1.2 *Let M be an m-dimensional complete noncompact Kähler manifold without boundary, let $E = E_1 \oplus E_2$ be a holomorphic vector bundle over M with initial Hermitian metric $H_0 = (H_0^1, H_0^2)$, where H_0^1, H_0^2 are Hermitian metrics on E_1, E_2 respectively, and let $\phi\colon E_2 \to E_1$ be a holomorphic morphism . Let*

$$\Theta^2 = \Big|\sqrt{-1}\Lambda F_{H_0^1} + \frac{1}{2}\phi \circ \phi^{*H_0} - \tau_1 \operatorname{Id}_{E_1}\Big|^2_{H_0^1} + \Big|\sqrt{-1}\Lambda F_{H_0^2} - \frac{1}{2}\phi^{*H_0} \circ \phi - \tau_2 \operatorname{Id}_{E_2}\Big|^2_{H_0^2}$$

Assume that $\lambda_1(M) > 0$, where $\lambda_1(M)$ denotes the lower bound of the spectrum of the Laplacian operator, and that $\|\Theta\|_{L^p(M)} < \infty$ for some $p > 1$ and real numbers τ_1, τ_2. Then there exists a Hermitian metric $H = (H_1, H_2)$ on E such that H satisfies the coupled vortex equations:

$$\sqrt{-1}\Lambda F_{H_1} + \frac{1}{2}\phi \circ \phi^{*H} = \tau_1 \operatorname{Id}_{E_1}, \quad \sqrt{-1}\Lambda F_{H_2} - \frac{1}{2}\phi^{*H} \circ \phi = \tau_2 Id_{E_2}. \tag{1.3}$$

The examples satisfying the assumption of Theorem 1.2 include simply-connected strictly negative curved manifolds, Hermitian symmetric spaces of noncompact type, and strictly pseudo-convex domains equipped with the Bergmann metric. According to [6, Theorem 1.4.A], the universal cover of any Kähler hyperbolic manifold (in the sense of Gromov) satisfies the assumption of Theorem 1.2, too. Therefore Theorem 1.2 is applicable to a relatively broad class of Kähler manifolds.

2 Preliminary Results

Let M be a compact Kähler manifold and $E = E_1 \oplus E_2$ a rank r complex vector bundle over M. Denote by ω the Kähler form, and define the operator Λ as the contraction with ω, *i.e.*, if $\alpha \in \Omega^{1,1}(M, E)$, then $\Lambda\alpha = \langle \alpha, \omega \rangle$. Let $H = (H_1, H_2)$ be a Hermitian metric on a holomorphic vector bundle $E = E_1 \oplus E_2$, and denote the holomorphic structure by $\bar{\partial}_E$. Then there exists a canonical metric connection which is denoted by A_H. Taking a local holomorphic basis $e_\alpha (1 \leq \alpha \leq r)$, the Hermitian metric H is a positive Hermitian matrix $(H_{\alpha\bar{\beta}})_{1 \leq \alpha, \beta \leq r}$, which can also be denoted by H for simplicity; here $H_{\alpha\bar{\beta}} = H(e_\alpha, e_\beta)$. In fact, the complex metric connection can be written $A_H = H^{-1}\partial H$ and the curvature form as $F_H = \bar{\partial} A_H = \bar{\partial}(H^{-1}\partial H)$. In the literature, sometimes the connection is written as $(\partial H)H^{-1}$ because of the reversal of the roles of the row and column indices.

It is known that any two Hermitian metrics H and K on bundle E are related by $H = Kh$, where $h = K^{-1}H \in \Omega^0(M, \mathrm{End}(E))$ is positive and self-adjoint with respect to K. It is easy to check that

$$\begin{aligned} A_H - A_K &= h^{-1}\partial_K h \\ F_H - F_K &= \bar{\partial}(h^{-1}\partial_K h) \end{aligned} \tag{2.1}$$

Let $H(0) = K$ be a Hermitian metric on E. Consider a family of Hermitian metric $H(t) = (H_1(t), H_2(t))$ on E with initial metric $H(0) = K$. Denote by $A_{H(t)}$ and $F_{H(t)}$ the corresponding connections and curvature forms and let $h(t) = (h_1(t), h_2(t))$ be a 2-tuple of endomorphisms $h_i = K_i^{-1}H_i$, where $i = 1, 2$. When there is no confusion, we will omit the parameter t and simply write H, h for $H(t)$, $h(t)$. We consider the following heat equations of (1.2)

$$\begin{aligned} H_1^{-1}\frac{\partial H_1}{\partial t} &= -2(\sqrt{-1}\Lambda F_{H_1} + \frac{1}{2}\phi \circ \phi^{*H} - \tau_1 \,\mathrm{Id}_{E_1}), \\ H_2^{-1}\frac{\partial H_2}{\partial t} &= -2(\sqrt{-1}\Lambda F_{H_2} - \frac{1}{2}\phi^{*H} \circ \phi - \tau_2 \,\mathrm{Id}_{E_2}), \end{aligned} \tag{2.2}$$

It is completely equivalent to the following evolution equations:

$$\begin{aligned} \frac{\partial h_1}{\partial t} &= -2\sqrt{-1}\Lambda\bar{\partial}_{E_1}\partial_{K_1} h_1 + 2\sqrt{-1}\Lambda(\bar{\partial}_{E_1} h_1 h_1^{-1}\partial_{K_1} h_1) \\ &\quad - 2\sqrt{-1}h_1\Lambda F_{K_1} + 2\tau_1 h_1 - h_1\phi h_2^{-1}\phi^{*K} h_1, \\ \frac{\partial h_2}{\partial t} &= -2\sqrt{-1}\Lambda\bar{\partial}_{E_2}\partial_{K_2} h_2 + 2\sqrt{-1}\Lambda(\bar{\partial}_{E_2} h_2 h_2^{-1}\partial_{K_2} h_2) \\ &\quad - 2\sqrt{-1}h_2\Lambda F_{K_2} + 2\tau_2 h_2 + \phi^{*K} h_1 \phi, \end{aligned} \tag{2.3}$$

where we have used the formula (2.2) and the identity

$$\phi^{*H} = h_2^{-1}\phi^{*K} h_1. \tag{2.4}$$

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We know that the above equations are a nonlinear parabolic system; as in [2], $h_i(t)$ are self-adjoint with respect to H_i for $t > 0$ since $h_i(0) = \mathrm{Id}_{E_i}$. We denote

$$\Theta^2 = |\sqrt{-1}\Lambda F_{H_1} + \frac{1}{2}\phi \circ \phi^{*H} - \tau_1 \mathrm{Id}_{E_1}|^2_{H_1} + |\sqrt{-1}\Lambda F_{H_2} - \frac{1}{2}\phi^{*H} \circ \phi - \tau_2 \mathrm{Id}_{E_2}|^2_{H_2}.$$

Proposition 2.1 *Let* $\mathbf{H}(t) = (H_1(t), H_2(t))$ *be a solution of heat flow* (2.2), *then*

$$\Big(\triangle - \frac{\partial}{\partial t}\Big)\Theta^2 \geq 0, \tag{2.5}$$

$$\Big(\triangle - \frac{\partial}{\partial t}\Big)\operatorname{Tr}(\theta_1 + \theta_2) = 0, \tag{2.6}$$

where we denote

$$\theta_1 = \sqrt{-1}\Lambda F_{H_1} + \frac{1}{2}\phi \circ \phi^{*H} - \tau_1 \mathrm{Id}_{E_1}, \theta_2 = \sqrt{-1}\Lambda F_{H_2} - \frac{1}{2}\phi^{*H} \circ \phi - \tau_2 \mathrm{Id}_{E_2}. \tag{2.7}$$

Proof By calculating directly, we have

$$\begin{aligned} \frac{\partial}{\partial t}\theta_1 &= \sqrt{-1}\Lambda\bar{\partial}_{E_1}\Big(\partial_{H_1}\big(h_1^{-1}\frac{dh_1}{dt}\big)\Big) - \frac{1}{2}\phi h_2^{-1}\frac{dh_2}{dt}\phi^{*H} + \frac{1}{2}\phi\phi^{*H}h_1^{-1}\frac{dh_1}{dt}, \\ \frac{\partial}{\partial t}\theta_2 &= \sqrt{-1}\Lambda\bar{\partial}_{E_2}\Big(\partial_{H_2}\big(h_2^{-1}\frac{dh_2}{dt}\big)\Big) + \frac{1}{2}h_2^{-1}\frac{dh_2}{dt}\phi^{*H}\phi - \frac{1}{2}\phi^{*H}h_1^{-1}\frac{dh_1}{dt}\phi, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \triangle|\theta_i|^2_{H_i} = {}& 2Re\langle -2\sqrt{-1}\Lambda\bar{\partial}_{E_i}\partial_{H_i}\theta_i, \theta_i\rangle_{H_i} \\ & + Re\langle[2\sqrt{-1}\Lambda F^{1,1}_{H_i}, \theta_i], \theta_i\rangle_{H_i} + 2|\partial_{H_i}\theta_i|^2_{H_i} + 2|\bar{\partial}_{E_i}\theta_i|^2_{H_i}. \end{aligned}$$

Using the above formulas, we have

$$\begin{aligned} \Big(\triangle - \frac{\partial}{\partial t}\Big)\Theta^2 &= 2\sum_{i=1}^{2}|\nabla\theta_i|^2_{H_i} + \big(|\phi^{*H}\theta_1|^2 - 2\langle\theta_2\phi^{*H}, \phi^{*H}\theta_1\rangle + |\theta_2\phi^{*H}|^2\big) \\ &\quad + \big(|\phi\theta_2|^2 - 2\langle\theta_1\phi, \phi\theta_2\rangle + |\theta_1\phi|^2\big) \\ &\geq 0. \end{aligned}$$

The formula (2.6) can be deduced from (2.8) directly. ∎

Proposition 2.2 *Let* $\mathbf{H}(t) = (H_1(t), H_2(t))$ *be a solution of heat flow* (2.2). *Then there exists a positive constant* C_1 *such that*

$$\Big(\triangle - \frac{\partial}{\partial t}\Big)|\phi|^2_H \geq 2|\partial_H\phi|^2 + C_1|\phi|^4_H - |\tau_2 - \tau_1||\phi|^2_H.$$

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Proof By calculating directly, we have

$$\Big(\triangle - \frac{\partial}{\partial t}\Big)\,|\phi|_H^2 = 2|\partial_H\phi|_H^2 + 2|\phi\phi^{*H}|^2 + 2(\tau_2 - \tau_1)|\phi|^2,$$

where we have used $\bar\partial_{E_2^*\otimes E_1}\phi = 0$, and equations (2.3). On the other hand, one can easily check that

$$|\phi\phi^{*H}|_{H_1}^2 \geq \frac{1}{r_2}|\phi|_{H_2}^4.$$

From the above equalities we have

$$\Big(\triangle - \frac{\partial}{\partial t}\Big)\,|\phi|_H^2 \geq 2|\partial_H\phi|^2 + C_1|\phi|_H^4 - |\tau_2 - \tau_1||\phi|_H^2. \qquad \blacksquare$$

Next, we will introduce Donaldson's distance on the space of Hermitian metrics as follows.

Definition 2.3 For any two Hermitian metrics H, K on a vector bundle E set

$$\sigma(H,K) = \operatorname{Tr} H^{-1}K + \operatorname{Tr} K^{-1}H - 2\operatorname{rank} E.$$

It is obvious that $\sigma(H,K) \geq 0$ with equality if and only if $H = K$. The function σ is not quite a metric but it serves almost equally well in our problem. In particular, a sequence of metrics H_t converges to H in the usual C^0 topology if and only if $\sup_M \sigma(H_t, H) \to 0$.

Let $\mathbf{H} = (H_1, H_2)$ and $\mathbf{K} = (K_1, K_2)$ be 2-tuples of Hermitian metrics. We define Donaldson's distance of 2-tuples as:

$$\sigma(\mathbf{H},\mathbf{K}) = \sum_{i=1}^{2}\sigma(H_i, K_i).$$

Denoting $\mathbf{h} = (h_1, h_2)$, where $h_i = K_i^{-1}H_i$, applying $-\sqrt{-1}\Lambda$ to (2.1), and taking the trace in the bundle E_i, we have

$$\operatorname{Tr}(\sqrt{-1}h_i(\Lambda F_{H_i}^{1,1} - \Lambda F_{K_i}^{1,1})) = -\frac{1}{2}\triangle \operatorname{Tr} h_i + \operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_{E_i}h_ih_i^{-1}\partial_{K_i}h_i).$$

Let $\mathbf{H}(t)$, $\mathbf{K}(t)$ be two solutions of heat flow (2.2). Using the above formula, we have

$$\begin{aligned}\Big(\triangle - \frac{\partial}{\partial t}\Big)(\operatorname{Tr} h_1(t) + \operatorname{Tr} h_2(t)) &= 2\sum_{i=1}^{2}\operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_{E_i}h_ih_i^{-1}\partial_{K_i}h_i)\\ &\quad + \operatorname{Tr}(h_1\phi h_2^{-1}\phi^{*K}h_1 - h_1\phi\phi^{*K})\\ &\quad + \operatorname{Tr}(h_2\phi^{*K}\phi - \phi^{*K}h_1\phi)\\ &= 2\sum_{i=1}^{2}\operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_{E_i}h_ih_i^{-1}\partial_{K_i}h_i)\\ &\quad + \operatorname{Tr}(h_1\phi h_2^{-1}\phi^{*K}h_1 - 2h_1\phi\phi^{*K} + h_2\phi^{*K}\phi).\end{aligned}$$

In the similar way, we have

$$\Big(\triangle-\frac{\partial}{\partial t}\Big)(\operatorname{Tr}h_1^{-1}(t)+\operatorname{Tr}h_2^{-1}(t))=2\sum_{i=1}^{2}\operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_{E_i}h_i^{-1}h_i\partial_{H_i}h_i^{-1})$$
$$+\operatorname{Tr}(h_1^{-1}\phi\phi^{*K}-2h_2^{-1}\phi^{*K}\phi+h_2^{-1}h_2^{-1}\phi^{*K}h_1\phi).$$

On the other hand, it is not hard to check that

$$\operatorname{Tr}(h_1\phi h_2^{-1}\phi^{*K}h_1-2h_1\phi\phi^{*K}+h_2\phi^{*K}\phi)\geq 0,$$
$$\operatorname{Tr}(h_1^{-1}\phi\phi^{*K}-2h_2^{-1}\phi^{*K}\phi+h_2^{-1}h_2^{-1}\phi^{*K}h_1\phi)\geq 0.$$

Using the above formula and the facts [2, 13]

$$\operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_{E_i}h_ih_i^{-1}\partial_{K_i}h)\geq 0,\quad \operatorname{Tr}(-\sqrt{-1}\Lambda\bar{\partial}_{E_i}h_i^{-1}h_i\partial_{H_i}h^{-1})\geq 0,$$

we have proved the following proposition.

Proposition 2.4 *Let* $\mathbf{H}(t)$, $\mathbf{K}(t)$ *be two solutions of the heat flow* (2.2). *Then*

$$\Big(\triangle-\frac{\partial}{\partial t}\Big)\sigma(\mathbf{H}(t),\mathbf{K}(t))\geq 0.$$

Corollary 2.5 *Let* $\mathbf{H}$ *and* $\mathbf{K}$ *be* 2*-tuples of Hermitian metrics satisfying the coupled vortex equation* (1.2). *Then* $\triangle\sigma(\mathbf{H},\mathbf{K})\geq 0$.

3 The Dirichlet Boundary Problem for Coupled Vortex Equations

In this section we will consider the case when M is the interior of the compact Kähler manifold $\overline{M}$ with non-empty smooth boundary ∂M, and the Kähler metric is smooth and non-degenerate on the boundary. The holomorphic vector bundle $E=E_1\oplus E_2$ is defined over $\overline{M}$. We will discuss the Dirichlet boundary problem for the coupled vortex equations by using the heat equation method to deform an arbitrary initial metric to the desired solution.

For given data φ on ∂M, we consider the evolution equation:

$$\begin{aligned}&H_1^{-1}\frac{\partial H_1}{\partial t}=-2(\sqrt{-1}\Lambda F_{H_1}+\frac{1}{2}\phi\circ\phi^{*H}-\tau_1\operatorname{Id}_{E_1}),\\ &H_2^{-1}\frac{\partial H_2}{\partial t}=-2(\sqrt{-1}\Lambda F_{H_2}-\frac{1}{2}\phi^{*H}\circ\phi-\tau_2\operatorname{Id}_{E_2}),\\ &H_i(t)|_{t=0}=K_i,\quad H_i|_{\partial M}=\varphi_i.\end{aligned}\tag{3.1}$$

Here $\mathbf{K}=(K_1,K_2)$ is an arbitrary smooth initial Hermitian metric satisfying the boundary condition. Denote $h_i(t)=K_i^{-1}H_i(t)$. Then the evolution equation (3.1) is

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completely equivalent to the following equation:

$$\begin{aligned}
(3.2)\qquad \frac{\partial h_1}{\partial t} &= -2\sqrt{-1}\Lambda\bar{\partial}_{E_1}\partial_{K_1}h_1 + 2\sqrt{-1}\Lambda(\bar{\partial}_{E_1}h_1h_1^{-1}\partial_{K_1}h_1)\\
&\quad - 2\sqrt{-1}h_1\Lambda F_{K_1} + 2\tau_1 h_1 - h_1\phi h_2^{-1}\phi^{*K}h_1\\
\frac{\partial h_2}{\partial t} &= -2\sqrt{-1}\Lambda\bar{\partial}_{E_2}\partial_{K_2}h_2 + 2\sqrt{-1}\Lambda(\bar{\partial}_{E_2}h_2h_2^{-1}\partial_{K_2}h_2)\\
&\quad - 2\sqrt{-1}h_2\Lambda F_{K_2} + 2\tau_2 h_2 + \phi^{*K}h_1\phi\\
&\qquad h(0) = \mathrm{Id} \quad h|_{\partial M} = \mathrm{Id}\,.
\end{aligned}$$

We know that the above equation is a parabolic equation, so standard theory gives short-time existence.

Proposition 3.1 *For sufficiently small* $\epsilon > 0$*, equation* (3.2) *and so also equation* (3.1) *have a smooth solution,* $\mathbf{H}(t) = (H_1(t), H_2(t))$*, defined for* $0 \le t < \epsilon$ *.*

The main point of the proof is to show that the solution of equation (3.1) persists for all time and converges to a limit. First we want to prove the long-time existence of the evolution equation. Let $\mathbf{H}(t)$ be a solution of the evolution equation (3.1), and $h_i = K_i^{-1}H_i$, $i = 1, 2$. Then

$$(3.3)\qquad \Big|\frac{\partial}{\partial t}(\lg \operatorname{Tr} h_i)\Big| = \Big|\frac{\operatorname{Tr}(\frac{\partial h_i}{\partial t})}{\operatorname{Tr} h_i}\Big| = 2\Big|\frac{\operatorname{Tr} h_i\theta_i}{\operatorname{Tr} h_i}\Big| \le 2|\theta_i|_{H_i},$$

and similarly

$$(3.4)\qquad \Big|\frac{\partial}{\partial t}(\lg \operatorname{Tr} h_i^{-1})\Big| \le 2|\theta_i|_{H_i},$$

where θ_i is given in (2.7).

Theorem 3.2 *Suppose that a smooth solution* $\mathbf{H}(t) = (H_1(t), H_2(t))$ *to the evolution equation* (2.2) *is defined for* $0 \le t < T$*. Then* $\mathbf{H}(t)$ *converges in* C^0 *to some continuous non-degenerate metric* $\mathbf{H}(T)$ *as* $t \to T$*.*

Proof Given $\epsilon > 0$, by continuity at $t = 0$ we can find a δ such that

$$\sup_M \sigma(\mathbf{H}(t), \mathbf{H}(t')) < \epsilon,$$

for $0 < t, t' < \delta$. Then Proposition 2.4 and the maximum principle imply that

$$\sup_M \sigma(\mathbf{H}(t), \mathbf{H}(t')) < \epsilon,$$

for all $t, t' > T - \delta$. This implies that $H_i(t)$ is a uniformly Cauchy sequence and converges to a continuous limiting metric $H_i(T)$, $i = 1, 2$. On the other hand, by Proposition 2.1, we know that $|\theta_i|_{H_i}$ are bounded uniformly. Using formulas (3.3) and (3.4), one can conclude that $\sigma(H_i(t), K_i(t))$ are bounded uniformly, therefore $H_i(T)$ is a non-degenerate metric. ■

We prove the following proposition in the same way as [2, Lemma 19] and [9, Lemma 6.4].

Proposition 3.3 *Let $H(t)$, for $0 \le t < T$ (or ∞), be any one-parameter family of Hermitian metrics on a complex vector bundle E and that satisfy the Dirichlet boundary condition. If $H(t)$ converges in C^o to some continuous metric $H(T)$ as $t \to T$ (or ∞), and if $\sup_M |\Lambda F_H^{1,1}|$ is bounded uniformly in t, then $H(t)$ is bounded in $C^{1,\alpha}$ and also bounded in L_2^p (for any $1 < p < \infty$) uniformly in t.*

Theorem 3.4 *The evolution equation* (3.1) *has a unique solution* $\mathbf{H}(\mathbf{t})$ *which exists for* $0 \le t < \infty$.

Proof Proposition 3.1 guarantees that a solution exists for a short time. Suppose that the solution $\mathbf{H}(t)$ exists for $0 \le t < T$. By Theorem 3.2, $\mathbf{H}(t)$ converges in C^o to a non-degenerate continuous limit metric $H(T)$ as $t \to T$. From Proposition 2.1 and the maximum principle, we conclude that $|\theta_i|_{H_i}$ are bounded independently of t. Moreover, from Proposition 2.2, we have

$$\Big(\triangle - \frac{\partial}{\partial t}\Big)|\phi|_H^2 \ge 2|\partial_H \phi|^2 + C_1|\phi|_H^4 - |\tau_2 - \tau_1||\phi|_H^2.$$

Assume that $|\phi|_H^2$ attains its maximum on $M \times [0, T)$ at the point (x_0, t_0) with $0 < t_0 < T$, $x_0 \in M$. If $|\phi|^2(x_0, t_0) > \frac{|\tau_2-\tau_1|}{C_1}$, then $(\triangle - \frac{\partial}{\partial t})|\phi|^2 \ge 0$, This is contradicted by the maximum principle of the heat operator. Then we have

$$|\phi|^2 \le \max\Big\{ |\phi|_K^2, \frac{|\tau_2 - \tau_1|}{C_1} \Big\}.$$

So, $\sup_M |\Lambda F_{H_i}^{1,1}|_{K_i}^2$ are bounded independently of t, here $i = 1, 2$. Hence by Proposition 3.3, $H_i(t)$ are bounded in C^1 and also bounded in L_2^p (for any $(1 < p < \infty)$) uniformly in t. Since the evolution equations (3.1) and (3.2) are quadratic in the first derivative of h_i we can apply Hamilton's method [7] to deduce that $H_i(t) \to H_i(T)$ in C^∞, $i = 1, 2$, and the solution can be continued past T. Then the evolution equations (3.1) have a solution $\mathbf{H}(t)$ defined for all time.

By Proposition 2.4 and the maximum principle, it is easy to conclude the uniqueness of the solution. ∎

Proof of Theorem 1.1 For given data φ on ∂M we consider the evolution equation (3.1). By Theorem 3.4, we know that there exists a unique solution $\mathbf{H}(t)$ of equation (3.1). Next, we want to prove that $\mathbf{H}(t)$ will converge to the metric which satisfies the coupled vortex equation.

By direct calculation, one can check that $|\nabla_H \theta_i|_H^2 \ge |\nabla|\theta_i|_H|^2$ for any section θ_i in $\mathrm{End}(E_i)$. Then, using formula (2.5),

$$\Big(\triangle - \frac{\partial}{\partial t}\Big)\Theta^2 = 2\Theta\Big(\triangle - \frac{\partial}{\partial t}\Big)\Theta + 2|\nabla\Theta|^2 \ge 2\sum_{i=1}^{2}|\nabla\theta_i|_{H_i}^2.$$

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So

$$2\Theta\Big(\triangle - \frac{\partial}{\partial t}\Big)\Theta \geq 2\Big(\sum_{i=1}^{2}|\nabla\theta_i|_{H_i}^2 - |\nabla\Theta|^2\Big)$$

$$\geq 2\sum_{i=1}^{2}(|\nabla\theta_i|_{H_i}^2 - |\nabla|\theta_i|_{H_i}|^2) \geq 0,$$

and we have

$$\Big(\triangle - \frac{\partial}{\partial t}\Big)\Theta \geq 0. \tag{3.5}$$

We first solve the following Dirichlet problem on M:

$$\triangle v = -\Theta(x,0), \quad v|_{\partial M} = 0. \tag{3.6}$$

Set $\omega(x,t) = \int_0^t \Theta(x,s)\,ds - v(x)$. From (3.5), (3.6), and the boundary condition satisfied by H_i it follows that, for $t > 0$, $\Theta(x,t)$ vanishes on the boundary of M. Then it is easy to check that $\omega(x,t)$ satisfies

$$\Big(\triangle - \frac{\partial}{\partial t}\Big)\omega(x,t) \geq 0, \quad \omega(x,0) = -v(x), \quad \omega(x,t)|_{\partial M} = 0.$$

By the maximum principle, we have

$$\int_0^t \Theta(x,s)\,ds \leq \sup_{y\in M} v(y), \tag{3.7}$$

for any $x \in M$,and $0 < t < \infty$.

Let $t_1 \leq t \leq t_2$, and $\bar{h}_i(x,t) = H_i^{-1}(x,t_1)H_i(x,t)$. It is easy to check that

$$\bar{h}_i^{-1}\frac{\partial \bar{h}_i}{\partial t} = -2\theta_i.$$

Then we have $\frac{\partial}{\partial t}\log \operatorname{Tr}(\bar{h}_i) \leq 2|\theta_i|_{H_i}$.

From the above formula, we have

$$\operatorname{Tr}(H_i^{-1}(x,t_1)H_i(x,t)) \leq r\exp{(2\int_{t_1}^t |\theta_i|_{H_i}\,ds)}.$$

We have a similar estimate for $\operatorname{Tr}\big(H_i^{-1}(x,t)H_i(x,t_1)\big)$. Combining them we have

$$\sigma(H(x,t),H(x,t_1)) \leq 2r(\exp{(2\int_{t_1}^t |\theta_i|_{H_i}\,ds)} - 1). \tag{3.8}$$

From (3.7) and (3.8), we know that $H(t)$ converge in C^0 topological to some continuous metric H_∞ as $t \to \infty$. Using Proposition 3.3 again, we know that $H(t)$ are bounded in C^1 and also bounded in L_2^p (for any $1 < p < \infty$) uniformly in t. On the other hand, $|\theta_i|_{H_i}$ is bounded uniformly. Then the standard elliptic regularity implies that there exists a subsequence $H_t \to H_\infty$ in the C_∞ topology. From formula (3.7), we know that H_∞ is the desired Hermitian metric satisfying the boundary condition. The uniqueness can be easily deduced from Corollary 2.5 and the maximum principle. So we have proved Theorem 1.1. ■

4 Existence of the Solution of Coupled Vortex Equations over Complete Kähler Manifolds

In this section, we consider the existence of the solution of coupled vortex equations on a class of complete Kähler manifolds. As above, here complete means complete, noncompact, without boundary. In the following we will use the above solubility of Dirichlet problem and the exhaustion method to obtain Theorem 1.2.

Let $\{\Omega_i\}_{i=1}^{\infty}$ be an exhausting sequence of compact sub-domains of M, *i.e.,* they satisfy $\Omega_i \subset \Omega_{i+1}$ and $\bigcup_{i=1}^{\infty} \Omega_i = M$. By Theorem 1.1, we can find Hermitian metrics $\mathbf{H}_i = (H_i^1, H_i^2)$ on $\mathbf{E}|_{\Omega_i}$ for each i such that

$$\begin{aligned}
\sqrt{-1}\Lambda F_{H_i^1} + \frac{1}{2}\phi \circ \phi^{*H_i} - \tau_1 \operatorname{Id}_{E_1} &= 0;\\
\sqrt{-1}\Lambda F_{H_i^2} - \frac{1}{2}\phi^{*H_i} \circ \phi - \tau_2 \operatorname{Id}_{E_2} &= 0;\\
\mathbf{H}_i(x)|_{\partial\Omega_i} &= \mathbf{H}_0(x).
\end{aligned}$$

In order to prove that we can pass to limit and eventually obtain a solution on the whole manifold, we need to establish some estimates. The keys are the C^0 and C^1-estimates. Set $h_i^1 = (H_0^1)^{-1}H_i^1$, $h_i^2 = (H_0^2)^{-1}H_i^2$,

$$\tilde{\sigma}_i = \ln \sum_{\nu=1}^{2} (\operatorname{Tr} h_i^\nu + \operatorname{Tr}(h_i^\nu)^{-1}) - \ln 2 \operatorname{rank} \mathbf{E},$$

$$f = 2\Big|\sqrt{-1}\Lambda F_{H_0^1} + \frac{1}{2}\phi \circ \phi^{*H_0} - \tau_1 \operatorname{Id}_{E_1}\Big| + 2\Big|\sqrt{-1}\Lambda F_{H_0^2} - \frac{1}{2}\phi^{*H_0} \circ \phi - \tau_2 \operatorname{Id}_{E_2}\Big|.$$

Denote $A = \sum_{\nu=1}^{2} (\operatorname{Tr} h_i^\nu + \operatorname{Tr}(h_i^\nu)^{-1})$

$$\begin{aligned}
\triangle\tilde{\sigma}_i &= \triangle \ln A = A^{-1}\triangle A - A^{-2}|\nabla A|^2\\
&= (A)^{-1}\Big[\sum_{\nu=1}^{2} 2\operatorname{Tr}\big(-\sqrt{-1}\Lambda\bar{\partial}_{E_\nu} h_i^\nu (h_i^\nu)^{-1}\partial_{H_0^\nu} h_i^\nu\big) - 2\operatorname{Tr}(\sqrt{-1}h_i^\nu(\Lambda F_{H_i^\nu} - \Lambda F_{H_0^\nu}))\\
&\quad + \sum_{\nu=1}^{2} 2\operatorname{Tr}\big(-\sqrt{-1}\Lambda\bar{\partial}_{E_\nu}(h_i^\nu)^{-1} h_i^\nu \partial_{H_0^\nu}(h_i^\nu)^{-1}\big)\\
&\quad - 2\operatorname{Tr}\big(\sqrt{-1}(h_i^\nu)^{-1}(\Lambda F_{H_i^\nu} - \Lambda F_{H_0^\nu})\big)\Big] - A^{-2}|\nabla A|^2.
\end{aligned}$$

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Direct calculation shows that [13]

$$2(\operatorname{Tr} h_i^\nu)^{-1}\operatorname{Tr}\big(-\sqrt{-1}\Lambda\bar\partial_{E_\nu} h_i^\nu (h_i^\nu)^{-1}\partial_{H_0^\nu} h_i^\nu\big) - (\operatorname{Tr} h_i^\nu)^{-2}|\nabla \operatorname{Tr} h_i^\nu|^2 \ge 0,$$

$$2(\operatorname{Tr}(h_i^\nu)^{-1})^{-1}\operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_{E_\nu}(h_i^\nu)^{-1}(h_i^\nu)\partial_{H_0^\nu}(h_i^\nu)^{-1}) \\ - (\operatorname{Tr}(h_i^\nu)^{-1})^{-2}|\nabla \operatorname{Tr}(h_i^\nu)^{-1}|^2 \ge 0.$$

From the above two inequalities, it is easy to check

$$A^{-1}\Big[\sum_{\nu=1}^{2} 2\operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_{E_\nu} h_i^\nu (h_i^\nu)^{-1}\partial_{H_0^\nu} h_i^\nu) \\ + \sum_{\nu=1}^{2} 2\operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_{E_\nu}(h_i^\nu)^{-1} h_i^\nu \partial_{H_0^\nu}(h_i^\nu)^{-1})\Big] \ge A^{-2}|\nabla A|^2.$$

So,

$$\triangle\tilde\sigma_i \ge A^{-1}\big[-2\operatorname{Tr}(\sqrt{-1}h_i^\nu(\Lambda F_{H_i^\nu} - \Lambda F_{H_0^\nu})) - 2\operatorname{Tr}(\sqrt{-1}(h_i^\nu)^{-1}(\Lambda F_{H_i^\nu} - \Lambda F_{H_0^\nu}))\big]$$

By direct calculation, we have

$$\triangle\tilde\sigma_i \ge -f, \quad \tilde\sigma_i|_{\partial\Omega_i} = 0. \tag{4.1}$$

For further consideration, we need the following lemma [10, Lemma 2.1].

Lemma 4.1 *Let M be a complete Riemannian manifold. Assume that $\lambda_1(M) > 0$. Then for a non-negative function ψ the Poisson equation $\triangle u = -\psi$ has a non-negative solution $u \in W^{2,n}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ $(0 < \alpha < 1)$ if $\psi \in L^p(M)$ for some $p \ge 1$.*

From Lemma 4.1 and the conditions in Theorem 1.2, we can solve the above Poisson equation when $\psi = f$, *i.e.,* there exists a non-negative function $u \in W^{2,n}_{\mathrm{loc}}(M) \cap C^{1,\alpha}_{\mathrm{loc}}(M)$ such that $\triangle u = -f$. Using formula (4.1) and the maximum principle, we can conclude that $\tilde\sigma_i \le u$. So the Donaldson distance $\sigma_i = \sigma(\mathbf{H}_0, \mathbf{H}_i)$ between $\mathbf{H}_i$ and $\mathbf{H}_0$ must satisfy

$$\sigma_i \le 2\operatorname{rank}\mathbf{E}\cdot \exp u - 2\operatorname{rank}\mathbf{E}, \tag{4.2}$$

on Ω_i, *i.e.,* we have obtained a C^0-estimate for $\mathbf{H}_i$.

Next, we want to obtain a uniform C^1-estimate for $\mathbf{H}_i$. For any point $x \in M$, choose a small ball $B_x(r)$ such that the bundled E can be trivialized locally, and let $\{e_\alpha\}$ be the holomorphic frame of E. Choose a locally normal coordinate $\{Z^\alpha\}$ on $B_x(r)$ and centered at x.

It is easy to check that

$$
\begin{aligned}
(4.3)\qquad & \triangle|(H_i^1)^{-1}\nabla H_i^1|^2_{H_i^1} \\
&= 2\operatorname{Tr}(\partial_{\bar\delta}((H_i^1)^{-1}\partial_{\bar\alpha}H_i^1)(H_i^1)^{-1}\overline{\partial_{\bar\delta}((H_i^1)^{-1}\partial_{\bar\alpha}H_i^1)}^t H_i^1) \\
&\quad + 2\operatorname{Tr}(\partial_{\bar\delta}((H_i^1)^{-1}\partial_{\alpha}H_i^1)(H_i^1)^{-1}\overline{\partial_{\bar\delta}((H_i^1)^{-1}\partial_{\alpha}H_i^1)}^t H_i^1) \\
&\quad + 2g^{\alpha\bar\beta}_{,\delta\bar\delta}\operatorname{Tr}((H_i^1)^{-1}\partial_\alpha H_i^1(H_i^1)^{-1}\partial_{\bar\beta}H_i^1) \\
&\quad + 2g^{\alpha\bar\beta}_{,\delta}\operatorname{Tr}(\partial_{\bar\delta}((H_i^1)^{-1}\partial_\alpha H_i^1)(H_i^1)^{-1}\partial_{\bar\beta}H_i^1) \\
&\quad + 2g^{\alpha\bar\beta}_{,\delta}\operatorname{Tr}((H_i^1)^{-1}\partial_\alpha H_i^1\partial_{\bar\delta}((H_i^1)^{-1}\partial_{\bar\beta}H_i^1)) \\
&\quad + 2g^{\alpha\bar\beta}_{,\bar\delta}\operatorname{Tr}(\partial_{\delta}((H_i^1)^{-1}\partial_\alpha H_i^1)(H_i^1)^{-1}\partial_{\bar\beta}H_i^1) \\
&\quad + 2g^{\alpha\bar\beta}_{,\bar\delta}\operatorname{Tr}((H_i^1)^{-1}\partial_\alpha H_i^1\partial_{\delta}((H_i^1)^{-1}\partial_{\bar\beta}H_i^1)) \\
&\quad - 2g^{\delta\bar\gamma}_{,\alpha}\operatorname{Tr}(\bar\partial_{\gamma}((H_i^1)^{-1}\partial_\delta H_i^1)(H_i^1)^{-1}\partial_{\bar\alpha}H_i^1) \\
&\quad - 2g^{\delta\bar\gamma}_{,\bar\alpha}\operatorname{Tr}((H_i^1)^{-1}\partial_\alpha H_i^1\bar\partial_{\gamma}((H_i^1)^{-1}\partial_\delta H_i^1)) \\
&\quad + \operatorname{Tr}(\phi\partial_\alpha\phi^{*H}(H_i^1)^{-1}\partial_{\bar\alpha}H_i^1) + \operatorname{Tr}((H_i^1)^{-1}\partial_\alpha H_i^1\phi\partial_{\bar\alpha}\phi^{*H}).
\end{aligned}
$$

We can get a similar formula for $\triangle|(H_i^2)^{-1}\nabla H_i^2|^2_{H_i^2}$, since

$$|\mathbf{H}_i^{-1}\nabla\mathbf{H}_i|^2_{\mathbf{H}_i} = |(H_i^1)^{-1}\nabla H_i^1|^2_{H_i^1} + |(H_i^2)^{-1}\nabla H_i^2|^2_{H_i^2}.$$

Using (4.3), (2.4) and the Cauchy inequality, one can easily check that

$$\triangle|\mathbf{H}_i^{-1}\nabla\mathbf{H}_i|^2_{\mathbf{H}_i} \geq -C_2|\mathbf{H}_i^{-1}\nabla\mathbf{H}_i|^2_{H_i}, \tag{4.4}$$

where C_2 is a uniform constant independent of i.

Direct calculation as before shows that

$$
\begin{aligned}
\triangle(\operatorname{Tr}h_i^1 + \operatorname{Tr}h_i^2) &= 2\operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_{E_1}h_i^1(h_i^1)^{-1}\partial_{H_0^1}h_i^1) \\
&\quad + 2\operatorname{Tr}(-\sqrt{-1}h_i^1(\Lambda F_{H_i^1} - \Lambda F_{H_0^1})) \\
&\quad + 2\operatorname{Tr}(-\sqrt{-1}\Lambda\bar\partial_{E_2}h_i^2(h_i^2)^{-1}\partial_{H_0^2}h_i^2) \\
&\quad + 2\operatorname{Tr}(-\sqrt{-1}h_i^2(\Lambda F_{H_i^2} - \Lambda F_{H_0^2})) \\
&\geq -2f\cdot(\operatorname{Tr}h_i^1 + \operatorname{Tr}h_i^2) + 2e(h_i^1) + 2e(h_i^2),
\end{aligned}
$$

where

$$
\begin{aligned}
e(h_i^1) &= -\operatorname{Tr}(\sqrt{-1}\Lambda\bar\partial_{E_1}h_i^1(h_i^1)^{-1}\partial_{H_0^1}h_i^1) \geq 0, \\
e(h_i^2) &= -\operatorname{Tr}(\sqrt{-1}\Lambda\bar\partial_{E_2}h_i^2(h_i^2)^{-1}\partial_{H_0^2}h_i^2) \geq 0.
\end{aligned}
$$

Choosing i sufficiently large such that $B_o(4R) \subset \Omega_i$, let ψ be a cut-off function which equals 1 in $B_o(2R)$ and is supported in $B_o(4R)$. Now multiply the above inequality by $\mathrm{Tr}(h_i^1 + h_i^2)\psi^2$ and integrate it over M. Then

$$\int_M \mathrm{Tr}(h_i^1 + h_i^2)\psi^2 \triangle \mathrm{Tr}(h_i^1 + h_i^2) \geq -2\int_M f\,\mathrm{Tr}(h_i^1 + h_i^2)\psi^2 + 2\int_M \mathrm{Tr}(h_i^1 + h_i^2)\psi^2(e(h_i^1) + e(h_i^2)).$$

Integrating by parts, we have

$$\int_M \mathrm{Tr}(h_i^1 + h_i^2)\psi^2(e(h_i^1) + e(h_i^2)) \leq \int_M f \cdot \mathrm{Tr}(h_i^1 + h_i^2)\psi^2 + \int_M |\nabla\psi|^2(\mathrm{Tr}(h_i^1 + h_i^2))^2.$$

Using the above C^0-estimate (4.2), we obtain the following estimate

$$\int_{B_o(2R)} e(h_i^1) + e(h_i^2) \leq C_3, \tag{4.5}$$

where C_3 is a constant independent of i. Because $e(h_i^1)+e(h_i^2)$ contains all the squares of the first order derivatives of h_i^1 and h_i^2, the above inequality implies that h_i^1 and h_i^2, (*i.e.*, $\mathbf{H}_i$) are uniformly bounded in $L_1^2(B_0(2R))$. Using (4.4), (4.5), and the mean value inequality, we conclude that there exists a uniform constant C_4 independent of i such that $\sup_{B_o(R)} |\mathbf{H}_i^{-1}\nabla\mathbf{H}_i|^2_{\mathbf{H}_i} \leq C_4$.

So we have obtained that the C^1-norm of H_i is bounded uniformly on any $B_o(R)$. From the above C^0, C^1-estimates of $\mathbf{H}_i$, then the standard elliptic theory shows that by passing to a subsequence, $\mathbf{H}_i$ converges uniformly over any compact sub-domain of M to a smooth Hermitian metric $\mathbf{H} = (H_1, H_2)$ satisfying the coupled vortex equations (1.3). So we have proved Theorem 1.2.

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