RINGS IN WHICH CERTAIN RIGHT IDEALS ARE DIRECT SUMMANDS OF ANNIHILATORS

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Abstract

This paper is a continuation of the study of the rings for which every principal right ideal (respectively, every right ideal) is a direct summand of a right annihilator initiated by Stanley S. Page and the author in [20, 21].

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Introduction

In this paper, we continue the study of left AP-injective and left AGP-injective rings which were introduced and discussed in [20]. Following [20], a ring $R$ is called left AP-injective if every principal right ideal is a direct summand of a right annihilator, and the ring $R$ is called left AGP-injective if, for any $0 \neq a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and $a^nR$ is a direct summand of $\text{rI}(a^n)$. Recall that a ring $R$ is left principally injective (P-injective) if every principal right ideal is a right annihilator, and the ring $R$ is left generalized principally injective (GP-injective) if, for any $0 \neq a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and $a^nR$ is a right annihilator. The detailed discussion of left P-injective and left GP-injective rings can be found in [3, 7, 12, 15, 16, 17, 22, 23, 24, 26]. Clearly, every left AP-injective ring is left P-injective and every left AGP-injective ring is left GP-injective. But there exist left AP-injective rings which are not left GP-injective [20]. In fact, a left AP-injective ring is not necessarily a left mininjective ring. (The ring $R$ is left mininjective if,
for any minimal left ideal $Ra$, $aR$ is a right annihilator [18], and every left GP-injective ring is left mininjective.) In [20], several results which are known for left P-injective (respectively, left GP-injective) rings were shown to hold for left AP-injective (respectively, left AGP-injective) rings. It has been noted that it is unknown whether there exists a left GP-injective ring that is not left P-injective (see [6, 24]). This may put a bit more weight on our excuse for carrying on the study of the left AGP-injective rings. In this paper, we discuss left AGP-injective rings with various chain conditions.

It is well known that a ring $R$ is quasi-Frobenius (QF) if and only if $R$ is left self-injective and left (or right) noetherian. In [9], Faith proved that any left self-injective ring satisfying the ACC on left annihilators is QF. Björk [2] extended this result from a left self-injective ring to a left $f$-injective ring, and then Rutter [23] further proved that, if $R$ satisfies the ACC on left annihilators, then $R$ is QF if and only if $R$ is left 2-injective, where the ring $R$ is called left $f$-injective (respectively, left 2-injective) if, for any finitely generated (respectively, 2-generated) left ideal $I$ of $R$, every $R$-homomorphism from $I$ to $R$ extends to an $R$-homomorphism from $R$ to $R$. Note that a left $f$-injective rings need not be left self-injective, and a left P-injective ring need not be left 2-injective. It was also proved in [23] that any left P-injective ring satisfying the ACC on left annihilators is right artinian. The latter result was extended from a left P-injective ring to a left GP-injective ring in Chen and Ding [7]. It is clear, by Rutter's example in [23], that a left P-injective ring satisfying the ACC on left annihilators need not be left artinian, and hence not be QF. The main result in Section 2 states that a left AGP-injective ring with the ACC on left annihilators is always semiprimary, but is not necessarily right artinian.

A ring is called a right dual ring if every right ideal is a right annihilator. The study of right noetherian, right dual rings was initiated by Johns [14], and continued by Faith and Menal in [10, 11] where they gave a counterexample to Johns' result that every right noetherian, right dual ring is right artinian. Recently, Gómez Pardo and Guil Asensio [12] proved that if $R$ is right noetherian and left P-injective, then $J(R)$ is nilpotent and $I(J(R))$ is essential both as a left and a right ideal of $R$, and this result allows them to show that every left Kasch, right noetherian and left P-injective ring is right artinian. In Section 2, among other things, we prove that, for a right noetherian and left AGP-injective ring $R$, $J(R)$ is nilpotent and $I(J(R))$ is essential both as a left and a right ideal of $R$. As a corollary of this, we show that every right noetherian, left AGP-injective ring with right (GC2) is right artinian.

In Section 3, we consider right quasi-dual rings. A ring $R$ is called right quasi-dual if every right ideal of $R$ is a direct summand of a right annihilator [21]. The right quasi-dual rings form an interesting class of left AP-injective rings. In Section 3, it is proved that, for a right quasi-dual ring, $J(R) = r(S_r)$, $S_r = r(Z_r)$ and $I(J(R))$ is essential in $rR$. Consequently, for a two-sided quasi-dual ring $R$, the left socle
coincides with the right socle and is essential both as a left and a right ideal of $R$. We also improve a result of [21] by showing that a ring $R$ is a two-sided PF-ring if and only if every right Goldie torsion $R$-module is cogenerated by $R_R$ and every left Goldie torsion $R$-module is cogenerated by $r_R$.

Throughout, $R$ is an associative ring with identity and modules are unitary. We use $M_R$ (respectively, $rM$) to indicate that $M$ is a right (respectively, left) module over $R$. For a subset $X$ of $R$, $l(X)$ (respectively, $r(X)$) is the left (respectively, right) annihilator of $X$ in $R$, and we write $l(x)$ (respectively, $r(x)$) for $l(\{x\})$ (respectively, $r(\{x\})$) when $x \in R$. The left socle, right socle, left singular ideal, right singular ideal and Jacobson radical of $R$ are denoted by $S_l$, $S_r$, $Z_l$, $Z_r$ and $J(R)$, respectively. For a submodule $N$ of $M$, we use $N \leq M$ to mean that $N$ is essential in $M$.

1. Left AGP-injective rings with left chain conditions

Following [20], the ring $R$ is left $AP$-injective if, for any $a \in R$, $aR$ is a direct summand of $rR(a)$, and $R$ is left AGP-injective if, for any $0 \neq a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and $a^nR$ is a direct summand of $rR(a^n)$. Every left $P$-injective ring is left AP-injective and every left GP-injective ring is left AGP-injective. The rings $R$ in [21, Examples 2.3, 2.4] are commutative AP-injective rings, but not mininjective and hence not GP-injective.

In this section, we prove several results of left AGP-injective rings with some chain conditions on left ideals.

A module $M$ is said to satisfy the generalized C2-condition (or (GC2)) if, for any $N \subseteq M$ and $N = M$, $N$ is a summand of $M$. Note that the GC2-condition is the same as the (*)-condition in [20, page 713].

**Lemma 1.1.** Let $M$ satisfy (GC2). If $M$ is finitely dimensional, then $\text{End}(M)$ is semilocal.

**Proof.** Let $\sigma : M \to M$ be a monomorphism. Then $M = \sigma(M) \oplus N$ for some $N \subseteq M$. It must be that $N = 0$ since $M$ is finitely dimensional. So, $\sigma$ is an isomorphism. Therefore, $M$ satisfies the assumptions in Camps-Dicks [5, Theorem 5], and so $\text{End}(M)$ is semilocal.

The next corollary extends [21, Proposition 2.12].

**Corollary 1.2.** Let $R$ be a left AGP-injective ring.

1. If $R_R$ is of finite Goldie dimension, then $R$ is semilocal.
2. $R$ is left noetherian if and only if $R$ is left artinian.
PROOF. (1). By [20, Proposition 2.13], $R$ satisfies (GC2). Since $R$ has finite Goldie dimension, $R$ is semilocal by Lemma 1.1.

(2). If $R$ is left noetherian, then $R$ is semilocal by (1). By [20, Corollary 2.11], $J(R)$ is nilpotent. So, $R$ is left artinian. 

LEMMA 1.3 ([20]). If $R$ is a left AGP-injective ring, then $J(R) = Z_t$.

LEMMA 1.4. Let $R$ be a left AGP-injective ring and $a \in R$. If $a \notin J(R)$ then there exists $r \in R$ such that the inclusion $l(a) \subseteq l(a - ara)$ is proper.

PROOF. Let $a \in R$ but $a \notin J(R)$. By Lemma 1.3, $a \notin Z_t$ and hence $l(a)$ is not essential in $R$. So, we have $l(a) \cap I = 0$ for some $0 \neq I \subseteq R$. Take $0 \neq b \in I$. Then $ba \neq 0$. By the hypothesis, there exists $n > 0$ such that $(ba)^n \neq 0$ and $rl((ba)^n) = (ab)^n R \oplus X$ where $X$ is a right ideal of $R$. Since $l(a) \cap I = 0$, $l((ba)^n) = l((ba)^n - 1)b)$. It follows that $(ba)^n - 1b \in rl((ba)^n - 1b) = rl((ba)^n) = (ba)^n R \oplus X$. Thus, there exists $r \in R$ such that $(ba)^n - 1b = (ba)^n r + x$ where $r \in R$ and $x \in X$. This gives that $(ba)^n - 1b(1 - ar) = x$ and hence $(ba)^n - 1b(a - ara) = xa \in (ba)^n R \cap X$. It follows that $(ba)^n - 1b(a - ara) = 0$. Let $c = a - ara$. Then $l(a) \subseteq l(c)$. Since $(ba)^n - 1b$ is in $l(c)$ but not in $l(a)$, the inclusion $l(a) \subseteq l(c)$ is proper. 

The next result extends [7, Theorem 3.4, Corollary 3.6]. Following [1], a module $M$ is called finitely projective (respectively, singly projective) if, for each epimorphism $f : N \to M$ and each finitely generated (respectively, cyclic) submodule $M_0$ of $M$, there exists $g \in \text{Hom}_R(M_0, N)$ such that the restriction of $g \circ f$ to $M_0$ is the identity on $M_0$.

THEOREM 1.5. The following are equivalent for a left AGP-injective ring $R$:

1. $R$ is a left Perfect ring.
2. Every flat left $R$-module is finitely projective.
3. Every flat left $R$-module is singly projective.
4. For any infinite sequence $x_1, x_2, x_3, \ldots$ of elements in $R$, the chain $l(x_1) \subseteq l(x_1x_2) \subseteq l(x_1x_2x_3) \subseteq \cdots$ terminates.

PROOF. (1) implies (2) and (2) implies (3) are obvious. (3) implies (4) is by [1, Corollary 25].

(4) implies (1). Firstly, we prove $R/J(R)$ is a von Neumann regular ring. For any $x \in R$, let $\bar{x} = x + J(R)$. Let $a_1 \in R$ but $a_1 \notin J(R)$. We want to show that $\bar{a}_1 = \bar{a}_1 \bar{x} \bar{a}_1$ for some $x \in R$. By Lemma 1.4, there exists $r_1 \in R$ such that $l(a_1) \subseteq l(a_2)$ where $a_2 = a_1 - a_1 r_1 a_1$. If $a_2 \in J(R)$, then $\bar{a}_1 = \bar{a}_1 r_1 \bar{a}_1$ and we are done. If $a_2 \notin J(R)$, then, by Lemma 1.4, there exists $r_2 \in R$ such that $l(a_2) \subseteq l(a_3)$ where $a_3 = a_2 - a_2 r_2 a_2$. The induction principle and the hypothesis ensure the existence of a positive integer
n and two sequences \(\{a_i : i = 1, \ldots, n + 1\}\) and \(\{r_i : i = 1, \ldots, n\}\) of elements in \(R\) such that \(a_{n+1} \in J(R)\) and \(a_{i+1} = a_i - a_i r_i a_i\) for \(i = 1, \ldots, n\). Thus, \(\tilde{a}_n = \tilde{a}_n r_n \tilde{a}_n\). It follows that

\[
\tilde{a}_{n-1} = \tilde{a}_n + \tilde{a}_{n-1} r_{n-1} \tilde{a}_{n-1}
\]

\[
= (\tilde{a}_{n-1} - \tilde{a}_{n-1} r_{n-1} \tilde{a}_{n-1}) r_n (\tilde{a}_{n-1} - \tilde{a}_{n-1} r_{n-1} \tilde{a}_{n-1}) + \tilde{a}_{n-1} r_{n-1} \tilde{a}_{n-1}
\]

\[
= \tilde{a}_{n-1} [r_n (I - \tilde{a}_{n-1} r_{n-1}) + r_{n-1} \tilde{a}_{n-1}],
\]

so \(\tilde{a}_{n-1}\) is also a regular element. Continuing this process, we see that \(\tilde{a}_1\) is a regular element.

Secondly, we prove that \(Z_i\) is left T-nilpotent. Let \(a_i \in Z_i\) for \(i = 1, 2, \ldots\). We have a chain \(I(a_i) \subseteq I(a_i a_2) \subseteq \cdots\). By our assumption, there exists \(n > 0\) such that \(I(a_1 \cdots a_n) = I(a_1 \cdots a_n a_{n+1})\). Thus, \(I(a_{n+1}) \cap Ra_1 \cdots a_n = 0\). Since \(I(a_{n+1})\) is essential in \(R\), we have \(a_1 \cdots a_n = 0\), so \(Z_i\) is left T-nilpotent. Therefore, by Lemma 1.3, we have proved that \(R/J(R)\) is a von Neumann regular ring and \(J(R)\) is left T-nilpotent. So, it suffices to show that \(R/J(R)\) is an artinian semisimple ring. By [13, Corollary 2.16], we only need to show that \(R/J(R)\) contains no infinite sets of nonzero orthogonal idempotents. This can be proved by arguing as in [7, page 2107].

**COROLLARY 1.6.** If \(R\) is a left AGP-injective ring with ACC on left annihilators, then \(R\) is semiprimary.

**PROOF.** It is well known that \(Z_i\) is nilpotent for any ring \(R\) with ACC on left annihilators. By Lemma 1.3 and Theorem 1.5, \(R\) is semiprimary. \(\square\)

**COROLLARY 1.7.** Let \(R\) be a left AGP-injective ring with ACC on left annihilators and \(S_r \subseteq S_i\). Then \(R\) is right artinian if and only if \(S_r\) is a finitely generated right ideal of \(R\).

**PROOF.** By Corollary 1.6, \(R\) is semiprimary. By [20, Corollary 2.7], \(S_i \subseteq S_r\), and so \(S = S_i = S_r\) by the hypothesis. Now the result follows from [4, Lemma 6]. \(\square\)

A left GP-injective ring with the ACC on left annihilators is always right artinian [7, Theorem 3.7]. The ring \(R\) [21, Example 2.4] is a commutative AP-injective ring with the ACC on annihilators, but \(R\) is not artinian.

Recall that a ring \(R\) is called left Kasch if \(r(K) \neq 0\) for every maximal left ideal \(K\) of \(R\).

**COROLLARY 1.8.** Let \(R\) be a left AGP-injective ring with ACC on left annihilators. If every minimal right ideal is a right annihilator, then \(R\) is right artinian. Moreover, \(R\) is left artinian if and only if \(S_i\) is finitely generated as a left ideal of \(R\).
PROOF. By Corollary 1.6, \( R \) is semiprimary. By [18, Corollary 3.15], \( R \) is right finite dimensional with \( S_r = S_i \). Now, by [4, Lemma 6], \( R \) is right artinian. The last assertion follows from [4, Lemma 6] again.

Now the following result, [7, Theorem 3.7], is an immediate corollary of the above:

**Corollary 1.9 ([7]).** Every left GP-injective ring with ACC on left annihilators is right artinian.

**Proof.** If \( R \) is a left GP-injective ring, then every minimal right ideal is a right annihilator. For, if \( I \) is a minimal right ideal of \( R \), then \( I = eR \) where \( e^2 = e \in R \) or \( I^2 = 0 \). If \( I = eR \), clearly \( I \) is an annihilator. On the other hand, if \( I = xR \) for some \( x \in R \) with \( I^2 = 0 \), it follows from the definition of left GP-injectivity that \( I = xR = rI(I) \). Now the result follows from Corollary 1.8.

2. Left AGP-injective rings with right chain conditions

In this section, we first consider right noetherian, left AGP-injective rings. We prove that, for a right noetherian, left AGP-injective ring \( R \), \( J(R) \) is nilpotent and \( l(J(R)) \) is essential as a left and as a right ideal of \( R \). As a corollary of this, we prove that every right noetherian, left AGP-injective ring \( R \) such that \( R_R \) satisfies (GC2) is right artinian. We next prove that every maximal left (respectively, right) annihilator of a semiprime left AGP-injective ring is a maximal left (respectively, right) ideal generated by an idempotent.

The next result extends [12, Theorem 2.7] from a left P-injective ring to a left AGP-injective ring.

**Theorem 2.1.** Let \( R \) be a right noetherian, and left AGP-injective ring. Then \( J(R) \) is nilpotent and \( l(J(R)) \) is essential both as a left and as a right ideal of \( R \).

**Proof.** Let \( J = J(R) \). First we prove that \( l(J) \leq e \) \( R \). Let \( 0 \neq x \in R \). Since \( R \) is right noetherian, the non-empty set \( \mathcal{F} = \{ r((ax)^k) : a \in R, k > 0 \text{ such that } (ax)^k \neq 0 \} \) has a maximal element, say \( r((yx)^n) \).

We claim that \( (yx)^n J = 0 \). If not, then there exists \( t \in J \) such that \( (yx)^n t \neq 0 \). Since \( R \) is left AGP-injective, there exists \( m > 0 \) such that \( ((yx)^n t)^m \neq 0 \) and \( ((yx)^n t)^m R \) is a direct summand of \( rI((yx)^n t)^m) \). Write \( ((yx)^n t)^m = (yx)^n s \) where \( s = t((yx)^n t)^{m-1} \in J \). Then \( rI((yx)^n s) = (yx)^n s R \oplus X \) for some right ideal \( X \) of \( R \). We proceed with the following two cases.

Case 1. \( rI((yx)^n) = rI((yx)^n s) \). Then \( (yx)^n \in rI((yx)^n) = (yx)^n s R \oplus X \). Write \( (yx)^n = (yx)^n sv + z, \) where \( v \in R \) and \( z \in X \). Then \( (yx)^n s = (yx)^n sv + zs \) and...
so $zs \in (yx)^n s R \cap X$. Thus, $zs = 0$ and hence $(yx)^n s = (yx)^n s vs$. It follows that $(yx)^n s (1 - vs) = 0$. Since $s \in J$, $1 - vs$ is a unit in $R$. So, we have $(yx)^n s = 0$. This is a contradiction.

Case 2. $rl((yx)^n) \neq rl((yx)^n s)$. Then $l((yx)^n) \neq l((yx)^n s)$. It follows that there exists $u \in l((yx)^n s)$ but $u \notin l((yx)^n)$. Thus, $u(yx)^n s = 0$ and $u(yx)^n \neq 0$. This gives that $s \in r(u(yx)^n)$ and $s \notin r((yx)^n)$. So, the inclusion $r((yx)^n) \subseteq r(u(yx)^n)$ is proper. This is a contradiction because $0 \neq u(yx)^n = (u(yx)^n)^{-1}yx$ and $r(u(yx)^n) \subsetneq \mathcal{F}$.

We have proved that $(yx)^n J = 0$, and so $Rx \cap I(J) = 0$. Therefore, $I(J)$ is an essential left ideal of $R$.

Next we prove that $J$ is nilpotent. Since $R$ is right noetherian, there exists $k > 0$ such that $I(J^k) = I(J^{k+n})$ for all $n > 0$. Suppose $J$ is not nilpotent. Then $J^k \neq 0$ and so $M_R = R/I(J^k)$ is a nonzero $R$-module. Since $R$ is right noetherian, the set \{r_R(m) : 0 \neq m \in M\} has a maximal element, $r_R(m_1)$ say. Write $m_1 = x + I(J^k)$ where $x \in R$. Then $x J^k \neq 0$. Since $I(J^{2k}) = I(J^k)$, we see $x J^k \nsubseteq I(J^k)$. So, there exists $b \in J^k$ such that $xb \notin I(J^k)$. Since $I(J) \leq x R$, $Rx b \cap I(J^k) \neq 0$. So, we have $0 \neq ax b \in I(J^k)$ for some $a \in R$. Let $m_2 = ax + I(J^k) \in M$. Then $m_2 \neq 0$ and $b \notin r_R(m_2)$. But, $b \notin r_R(m_1)$. So, the inclusion $r_R(m_1) \subsetneq r_R(m_2)$ is proper. This contradicts the choice of $m_1$.

Finally, for any $0 \neq x \in R$, $x J = 0$, or $x J^n \neq 0$ and $x J^{n+1} = 0$ for some $n > 0$. It follows that $x R \cap I(J) \neq 0$. So, $I(J)$ is an essential right ideal of $R$.

The next result extends [12, Corollary 2.9]. (Note that, if $R$ is left Kasch, then $R_R$ satisfies (C2) (see [25]) and hence satisfies (GC2)).

**Corollary 2.2.** Every right noetherian, left AGP-injective ring $R$ such that $R_R$ satisfies (GC2) is right artinian.

**Proof.** Since $R$ is right finitely dimensional and $R_R$ satisfies (GC2), $R$ is semilocal by Lemma 1.1. By Theorem 2.1, $J(R)$ is nilpotent. So, $R$ is semiprimary. Since $R$ is right noetherian, $R$ is right artinian. 

Next, we consider semiprime left AGP-injective rings.

**Lemma 2.3.** Let $R$ be an arbitrary ring and $a \in R$ such that $l(a)$ is a maximal left annihilator or $r(a)$ is a maximal right annihilator. Then $l(at) = l(a)$ for any $t \notin r(a)$ and $Z_l \subseteq r(a)$, and $r(ta) = r(a)$ for any $t \notin l(a)$ and $Z_r \subseteq l(a)$.

**Proof.** Let $x \in Z_l$. Then $l(x)$ is essential in $R$. So, $l(x) \cap Rr \neq 0$ for any $0 \neq r \in R$. Thus, there exists $y \in R$ such that $0 \neq yr$ and $yrx = 0$. So, the inclusion $l(r) \subseteq l(rx)$ is proper.
Case 1. Let $l(a)$ be a maximal left annihilator. As above, $l(a) \subseteq l(ax)$ for all $x \in Z_l$. It must be that $ax = 0$. This shows that $a \in l(Z_l)$. Clearly, in this case $l(at) = l(a)$ for any $t \notin r(a)$.

Case 2. Let $r(a)$ be a maximal right annihilator. If $t \notin r(a)$, then $at \neq 0$. For $x \in l(at)$, $t \notin r(xa)$ and so the inclusion $r(a) \subseteq r(xa)$ is proper. By the maximality of $r(a)$, $xa = 0$. Thus, $l(at) = l(a)$. It follows that $Ra \cap l(t) = 0$. Thus, $t \notin Z_l$. Therefore, $Z_l \subseteq r(a)$.

The remaining part is by the left-right symmetry of the hypothesis. \[\square\]

The next theorem extends [7, Theorem 3.1].

**THEOREM 2.4.** Let $R$ be a semiprime left AGP-injective ring. Then every maximal left (respectively, right) annihilator is a maximal left (respectively, right) ideal of $R$ which is generated by an idempotent.

**PROOF.** Let $L$ be a maximal left (respectively, right) annihilator. Then $L = l(a)$ (respectively, $r(a)$) for some $0 \neq a \in R$. Since $R$ is semiprime, $Z_l \cap l(Z_l) = 0$. Claim: $a \notin Z_l$. Otherwise, $a \notin l(Z_l)$, that is, $aZ_l \neq 0$. Take $x \in Z_l$ such that $ax \neq 0$. Since $x \notin r(a)$, $l(ax) = l(a)$ by Lemma 2.3. Thus, $l(x) \cap Ra = 0$, a contradiction, since $x \in Z_l$. Therefore, $a \notin Z_l$. By Lemma 1.3 and Lemma 1.4, the inclusion $l(a) \subseteq l(a - ara) = l[a(1 - ra)]$ is proper for some $r \in R$. It follows from Lemma 2.3 that $a - ara = 0$. Therefore, $L = l(ar)$ (respectively, $L = r(ra)$) with $ar$ (respectively, $ra$) an idempotent. So we can assume that $a = e$ is an idempotent. To see $L$ is a maximal left (respectively, right) ideal, we show that $Re$ (respectively, $eR$) is a minimal left (respectively, right) ideal of $R$. Since $R$ is semiprime, it suffices to show that $eRe$ is a division ring. Let $0 \neq d \in eRe$. Since $R$ is left AGP-injective, there exists $n > 0$ such that $d^n \neq 0$ and $d^nR$ is a direct summand of $rl(d^n)$. By Lemma 2.3, $l(d^n) = l(e)$ and so $rl(d^n) = r(l(e)) = eR$. Thus, $d^nR$ is a direct summand of $eR$ and hence of $R_R$. It follows that $d^nR = rl(d^n) = eR$. Write $e = d^n b$ where $b \in R$. Then $e = d(d^{n-1}be)$ with $d^{n-1}be \in eRe$. So, $eRe$ is a division ring. \[\square\]

A ring $R$ is a left PP ring if every principal left ideal of $R$ is projective. The next result extends [6, Theorem 2.9] from a left GP-injective ring to a left AGP-injective ring.

**PROPOSITION 2.5.** The ring $R$ is a von Neumann regular ring if and only if $R$ is left PP and left AGP-injective.

**PROOF.** One direction is obvious. Suppose that $R$ is left PP and left AGP-injective. For any nonzero element $a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and $rl(a^n) = a^nR \oplus X$ where $X$ is a right ideal of $R$. Since $R$ is left PP, $Ra^n$ is projective, and
hence $0 \to I(a^n) \to R \to Ra^n \to 0$ splits. Thus, $I(a^n) = Re$ where $e^2 = e \in R$. It follows that $rl(a^n) = r(Re) = (1 - e)R$. Thus, $a^n R$ is a direct summand of $(1 - e)R$, and hence a direct summand of $R_R$. This implies that $a^n$ is a regular element of $R$. If $a \neq 0$ but $a^2 = 0$, the argument above shows that $a$ is a regular element. So, by [6, Theorem 2.9], $R$ is a regular ring.

3. Right quasi-dual rings

Following [21], a ring $R$ is called right quasi-dual if every right ideal of $R$ is a direct summand of a right annihilator. As shown in [21], the ring $R$ is right quasi-dual if and only if every essential right ideal of $R$ is a right annihilator if and only if every singular cyclic right $R$-module is cogenerated by $R$. Every right dual ring is certainly right quasi-dual, and every right quasi-dual ring is left AP-injective.

**Lemma 3.1.** Let $R$ be a right quasi-dual ring. For any right ideal $I$ of $R$ and $a \in R$, $rl(Ra \cap I) = I + (X_{aI} : a)$, with $(X_{aI} : a) \cap I \subseteq rl(a)$ and $(X_{aI} : a)_r = \{ x \in R : ax \in X_{aI} \}$, where $X_{aI}$ is a right ideal of $R$ such that $rl(aI) = aI \oplus X_{aI}$.

**Proof.** Let $x \in rl(Ra \cap I)$. Then $rl(aI) \subseteq l(ax)$, and so $ax \in rl(ax) \subseteq rl(aI) = aI \oplus X_{aI}$. Write $ax = at + y$ where $t \in I$ and $y \in X_{aI}$. Then $a(x - t) = y \in X_{aI}$ and thus $x - t \in (X_{aI} : a)$. Therefore, $x \in I + (X_{aI} : a)$, and $rl(Ra \cap I) \subseteq I + (X_{aI} : a)$. It is easy to see that $(X_{aI} : a) \cap I \subseteq rl(a)$ and that $I \subseteq rl(Ra \cap I)$. Let $y \in (X_{aI} : a)$. Then $ay \in X_{aI} \subseteq rl(aI)$. For any $ra \in Ra \cap I$, $ral = 0$. This gives that $r \in I(aI)$. Since $ay \in rl(aI)$, it follows that $ray = 0$. Thus, $y \in rl(Ra \cap I)$ and $(X_{aI} : a)_r \subseteq rl(Ra \cap I)$.

**Theorem 3.2.** Let $R$ be a right quasi-dual ring and $J = J(R)$. Then

1. $J = Z_t = rl(S_r)$, $S_r = l(Z_r)$, and $R$ is right Kasch.
2. $l(J)$ is essential in $R_R$.

**Proof.** (1). Clearly, $S_r \subseteq rl(Z_r)$. Let $K$ be any essential right ideal of $R$. Then $l(K) \subseteq Z_t$ and so $K = rl(K) \supseteq rl(Z_r)$. It follows that $S_r \supseteq rl(Z_r)$ since $S_r$ is the intersection of all essential right ideals. Thus, $S_r = rl(Z_r)$. By [21, Lemma 2.5 and Lemma 2.6], $J = Z_t$ and $R$ is right Kasch. Since $R$ is right Kasch, $J = rl(S_r)$.

(2). Let $0 \neq a \in R$ and assume that $Ra \cap l(J) = 0$. Then, by Lemma 3.1, $R = rl(Ra \cap I(J)) = J + (X_{aJ} : a)$, where $X_{aJ}$ is a right ideal of $R$ such that $rl(aJ) = aJ \oplus X_{aJ}$. Since $J$ is small in $R_R$, $R = (X_{aJ} : a)$. It follows that $aR \subseteq X_{aJ}$ and so $aJ \subseteq aJ \cap X_{aJ} = 0$. Thus, $a \in Ra \cap l(J) = 0$, a contradiction.

**Corollary 3.3.** Let $R$ be a quasi-dual ring. Then $S = S_r = S_l$ is essential as a left and a right ideal of $R$. 


PROOF. By [21, Theorem 2.8] and Theorem 3.2.

It was proved in [21] that, for a two-sided quasi-dual ring $R$, every Goldie torsion right $R$-module is cogenerated by $R_R$ if and only if the second singular right ideal $Z_2(R_R)$ of $R$ is injective. This result can be improved as follows.

**Theorem 3.4.** Consider the following conditions on a ring $R$:

1. Every Goldie torsion right $R$-module is cogenerated by $R_R$.
2. $Z_2(R_R)$ is injective and $R$ is right Kasch.
3. $R$ is right self-injective and right Kasch.

Then (3) implies (2) and (2) implies (1). In addition (1) implies (3) if $R$ is left quasi-dual.

**Proof.** (3) implies (2) is obvious, and (2) implies (1) is by the proof of [21, Theorem 4.1].

Suppose $R$ is left quasi-dual and (1) holds. By [21, Theorem 4.1], $Z_2(R_R)$ is injective. Write $R_R = Z_2(R_R) \oplus K$ where $K$ is right ideal of $R$. It suffices to show that $K_R$ is injective. Note that $R$ is a two-sided quasi-dual ring, so $Z_l = Z_l$ and $S_r = I(Z_l)$ by [21, Theorem 2.8]. It follows that $K \subseteq I(Z_2(R_R)) \subseteq I(I(Z_l)) = S_r$. So, $K_R$ is semisimple. Thus, to show that $K_R$ is injective, it suffices to show that $K$ is $Z_2(R_R)$-injective. But, this is clear because $K$ is non-singular and $Z_2(R_R)$ is Goldie torsion.

A ring $R$ is right PF if $R$ is an injective cogenerator for Mod-$R$. It is known that $R$ is right PF if and only if $R$ is right self-injective and right Kasch. The next corollary improved [21, Corollaries 4.4–4.6].

**Corollary 3.5.** $R$ is a two-sided PF-ring if and only if every Goldie torsion right $R$-module is cogenerated by $R_R$ and every Goldie torsion left $R$-module is cogenerated by $R_R$.

Dischinger and Müller [8] constructed a left PF-ring that is not right PF. By Corollary 3.5, the left PF-ring in [8] does not cogenerate every Goldie torsion right $R$-module. Ososky [19] constructed a non-injective cogenerator for Mod-$R$. We note that Ososky’s ring $R$ has the property that $Z_2(R_R) = R$ (since $J(R)^2 = 0$ and $J(R)_R \leq R_R$). This shows the conditions (1) and (2) in Theorem 3.4 are not equivalent.

**Proposition 3.6.** The following are equivalent for a ring $R$:

1. $R$ is right PF.
2. $Z_2(R_R)$ is injective, $R$ is right Kasch and $R = Z_2(R_R) + S_r$.
**Proof.** (2) implies (1). It suffices to show that $R$ is right self-injective. Since $R = E(R) + S$, $R = E(R) \oplus K$ where $K$ is a non-singular semisimple right ideal of $R$. Clearly, $K_R$ is $E(R)$-injective and $K_R$-injective. So, $K_R$ is injective. Thus, $R_R$ is injective.

(1) implies (2). We only need to show that $R = E(R) + S$. Since $E(R)$ is injective, write $R = E(R) \oplus K$ where $K$ is a right ideal of $R$. Since $R$ is right PF, $J(R) = Z \subseteq E(R)$ and $S$ is a finitely generated essential right ideal of $R$. Thus $\text{Soc}(K_R)$ is finitely generated and essential in $K$. Since every minimal right ideal contained in $K$ is idempotent, $\text{Soc}(K_R)$ is a summand of $R_R$ and hence of $K_R$. Thus, $K = \text{Soc}(K_R)$ is semisimple. 

We do not know if the condition that $R = E(R) + S$, in Proposition 3.6 can be removed.

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**References**


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