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RINGS IN WHICH CERTAIN RIGHT IDEALS ARE DIRECT SUMMANDS OF ANNIHILATORS

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Abstract

This paper is a continuation of the study of the rings for which every principal right ideal (respectively, every right ideal) is a direct summand of a right annihilator initiated by Stanley S. Page and the author in [20, 21].

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Introduction

In this paper, we continue the study of left AP-injective and left AGP-injective rings which were introduced and discussed in [20]. Following [20], a ring R is called *left AP-injective* if every principal right ideal is a direct summand of a right annihilator, and the ring R is called *left AGP-injective* if, for any $0 \neq a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and $a^n R$ is a direct summand of $\mathbf{rl}(a^n)$. Recall that a ring R is *left principally injective* (*P-injective*) if every principal right ideal is a right annihilator, and the ring R is *left generalized principally injective* (*GP-injective*) if, for any $0 \neq a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and $a^n R$ is a right annihilator. The detailed discussion of left P-injective and left GP-injective rings can be found in [3, 7, 12, 15, 16, 17, 22, 23, 24, 26]. Clearly, every left AP-injective ring is left P-injective and every left AGP-injective ring is left GP-injective. But there exist left AP-injective rings which are not left GP-injective [20]. In fact, a left AP-injective ring is not necessarily a left mininjective ring. (The ring R is *left mininjective* if,

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for any minimal left ideal Ra, aR is a right annihilator [18], and every left GP-injective ring is left mininjective.) In [20], several results which are known for left P-injective (respectively, left GP-injective) rings were shown to hold for left AP-injective (respectively, left AGP-injective) rings. It has been noted that it is unknown whether there exists a left GP-injective ring that is not left P-injective (see [6, 24]). This may put a bit more weight on our excuse for carrying on the study of the left AGP-injective rings. In this paper, we discuss left AGP-injective rings with various chain conditions.

It is well known that a ring R is quasi-Frobenius (QF) if and only if R is left self-injective and left (or right) noetherian. In [9], Faith proved that any left self-injective ring satisfying the ACC on left annihilators is QF. Björk [2] extended this result from a left self-injective ring to a left f-injective ring, and then Rutter [23] further proved that, if R satisfies the ACC on left annihilators, then R is QF if and only if R is left 2-injective, where the ring R is called *left f-injective* (respectively, *left 2-injective*) if, for any finitely generated (respectively, 2-generated) left ideal I of R, every R-homomorphism from I to R extends to an R-homomorphism from R to R. Note that a left f-injective rings need not be left self-injective, and a left P-injective ring need not be left 2-injective. It was also proved in [23] that any left P-injective ring satisfying the ACC on left annihilators is right artinian. The latter result was extended from a left P-injective ring to a left GP-injective ring in Chen and Ding [7]. It is clear, by Rutter's example in [23], that a left P-injective ring satisfying the ACC on left annihilators need not be left artinian, and hence not be QF. The main result in Section 2 states that a left AGP-injective ring with the ACC on left annihilators is always semiprimary, but is not necessarily right artinian.

A ring is called a *right dual ring* if every right ideal is a right annihilator. The study of right noetherian, right dual rings was initiated by Johns [14], and continued by Faith and Menal in [10, 11] where they gave a counterexample to Johns' result that every right noetherian, right dual ring is right artinian. Recently, Gómez Pardo and Guil Asensio [12] proved that if R is right noetherian and left P-injective, then $J(R)$ is nilpotent and $\mathbf{l}(J(R))$ is essential both as a left and a right ideal of R, and this result allows them to show that every left Kasch, right noetherian and left P-injective ring is right artinian. In Section 2, among other things, we prove that, for a right noetherian and left AGP-injective ring R, $J(R)$ is nilpotent and $\mathbf{l}(J(R))$ is essential both as a left and a right ideal of R. As a corollary of this, we show that every right noetherian, left AGP-injective ring with right (GC2) is right artinian.

In Section 3, we consider right quasi-dual rings. A ring R is called *right quasi-dual* if every right ideal of R is a direct summand of a right annihilator [21]. The right quasi-dual rings form an interesting class of left AP-injective rings. In Section 3, it is proved that, for a right quasi-dual ring, $J(R) = \mathbf{r}(S_r)$, $S_r = \mathbf{r}(Z_r)$ and $\mathbf{l}(J(R))$ is essential in ${}_RR$. Consequently, for a two-sided quasi-dual ring R, the left socle coincides with the right socle and is essential both as a left and a right ideal of R. We also improve a result of [21] by showing that a ring R is a two-sided PF-ring if and only if every right Goldie torsion R-module is cogenerated by R_R and every left Goldie torsion R-module is cogenerated by ${}_RR$.

Throughout, R is an associative ring with identity and modules are unitary. We use M_R (respectively, ${}_RM$) to indicate that M is a right (respectively, left) module over R. For a subset X of R, $\mathbf{l}(X)$ (respectively, $\mathbf{r}(X)$) is the left (respectively, right) annihilator of X in R, and we write $\mathbf{l}(x)$ (respectively, $\mathbf{r}(x)$) for $\mathbf{l}(\{x\})$ (respectively, $\mathbf{r}(\{x\})$) when $x \in R$. The left socle, right socle, left singular ideal, right singular ideal and Jacobson radical of R are denoted by S_l, S_r, Z_l, Z_r and $J(R)$, respectively. For a submodule N of M, we use $N \leq_e M$ to mean that N is essential in M.

1. Left AGP-injective rings with left chain conditions

Following [20], the ring R is *left AP-injective* if, for any $a \in R$, aR is a direct summand of $\mathbf{rl}(a)$, and R is *left AGP-injective* if, for any $0 \neq a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and a^nR is a direct summand of $\mathbf{rl}(a^n)$. Every left P-injective ring is left AP-injective and every left GP-injective ring is left AGP-injective. The rings R in [21, Examples 2.3, 2.4] are commutative AP-injective rings, but not mininjective and hence not GP-injective.

In this section, we prove several results of left AGP-injective rings with some chain conditions on left ideals.

A module M is said to satisfy the *generalized* C2-*condition* (or (GC2)) if, for any $N \subseteq M$ and $N \cong M$, N is a summand of M. Note that the GC2-condition is the same as the $(*)$-condition in [20, page 713].

LEMMA 1.1. *Let* ${}_RM$ *satisfy* (GC2). *If* M *is finitely dimensional, then* End(M) *is semilocal.*

PROOF. Let $\sigma : M \to M$ be a monomorphism. Then $M = \sigma(M) \oplus N$ for some $N \subseteq M$. It must be that $N = 0$ since M is finitely dimensional. So, σ is an isomorphism. Therefore, M satisfies the assumptions in Camps-Dicks [5, Theorem 5], and so End(M) is semilocal. □

The next corollary extends [21, Proposition 2.12].

COROLLARY 1.2. *Let R be a left AGP-injective ring.*

(1) *If* ${}_RR$ *is of finite Goldie dimension, then R is semilocal.*
(2) *R is left noetherian if and only if R is left artinian.*

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PROOF. (1). By [20, Proposition 2.13], ${}_RR$ satisfies (GC2). Since ${}_RR$ has finite Goldie dimension, R is semilocal by Lemma 1.1.

(2). If R is left noetherian, then R is semilocal by (1). By [20, Corollary 2.11], $J(R)$ is nilpotent. So, R is left artinian. □

LEMMA 1.3 ([20]). *If R is a left AGP-injective ring, then $J(R) = Z_l$.*

LEMMA 1.4. *Let R be a left AGP-injective ring and $a \in R$. If $a \notin J(R)$ then there exists $r \in R$ such that the inclusion $\mathbf{l}(a) \subset \mathbf{l}(a - ara)$ is proper.*

PROOF. Let $a \in R$ but $a \notin J(R)$. By Lemma 1.3, $a \notin Z_l$ and hence $\mathbf{l}(a)$ is not essential in ${}_RR$. So, we have $\mathbf{l}(a) \cap I = 0$ for some $0 \neq I \subseteq {}_RR$. Take $0 \neq b \in I$. Then $ba \neq 0$. By the hypothesis, there exists $n > 0$ such that $(ba)^n \neq 0$ and $\mathbf{rl}((ba)^n) = (ab)^n R \oplus X$ where X is a right ideal of R. Since $\mathbf{l}(a) \cap I = 0, \mathbf{l}((ba)^n) = \mathbf{l}((ba)^{n-1}b)$. It follows that $(ba)^{n-1}b \in \mathbf{rl}((ba)^{n-1}b) = \mathbf{rl}((ba)^n) = (ba)^n R \oplus X$. Thus, there exists $r \in R$ such that $(ba)^{n-1}b = (ba)^n r + x$ where $r \in R$ and $x \in X$. This gives that $(ba)^{n-1}b(1 - ar) = x$ and hence $(ba)^{n-1}b(a - ara) = xa \in (ba)^n R \cap X$. It follows that $(ba)^{n-1}b(a - ara) = 0$. Let $c = a - ara$. Then $\mathbf{l}(a) \subseteq \mathbf{l}(c)$. Since $(ba)^{n-1}b$ is in $\mathbf{l}(c)$ but not in $\mathbf{l}(a)$, the inclusion $\mathbf{l}(a) \subset \mathbf{l}(c)$ is proper. □

The next result extends [7, Theorem 3.4, Corollary 3.6]. Following [1], a module M is called *finitely projective* (respectively, *singly projective*) if, for each epimorphism $f : N \to M$ and each finitely generated (respectively, cyclic) submodule M_0 of M, there exists $g \in \mathrm{Hom}_R(M_0, N)$ such that the restriction of $g \circ f$ to M_0 is the identity on M_0.

THEOREM 1.5. *The following are equivalent for a left AGP-injective ring R:*

(1) *R is a left Perfect ring.*

(2) *Every flat left R-module is finitely projective.*

(3) *Every flat left R-module is singly projective.*

(4) *For any infinite sequence $x_1, x_2, x_3, \ldots$ of elements in R, the chain $\mathbf{l}(x_1) \subseteq \mathbf{l}(x_1x_2) \subseteq \mathbf{l}(x_1x_2x_3) \subseteq \cdots$ terminates.*

PROOF. (1) implies (2) and (2) implies (3) are obvious. (3) implies (4) is by [1, Corollary 25].

(4) implies (1). Firstly, we prove $R/J(R)$ is a von Neumann regular ring. For any $x \in R$, let $\bar{x} = x + J(R)$. Let $a_1 \in R$ but $a_1 \notin J(R)$. We want to show that $\bar{a}_1 = \bar{a}_1\bar{x}\bar{a}_1$ for some $x \in R$. By Lemma 1.4, there exists $r_1 \in R$ such that $\mathbf{l}(a_1) \subset \mathbf{l}(a_2)$ where $a_2 = a_1 - a_1r_1a_1$. If $a_2 \in J(R)$, then $\bar{a}_1 = \bar{a}_1\bar{r}_1\bar{a}_1$ and we are done. If $a_2 \notin J(R)$, then, by Lemma 1.4, there exists $r_2 \in R$ such that $\mathbf{l}(a_2) \subset \mathbf{l}(a_3)$ where $a_3 = a_2 - a_2r_2a_2$. The induction principle and the hypothesis ensure the existence of a positive integer

n and two sequences $\{a_i : i = 1, \dots, n+1\}$ and $\{r_i : i = 1, \dots, n\}$ of elements in R such that $a_{n+1} \in J(R)$ and $a_{i+1} = a_i - a_i r_i a_i$ for $i = 1, \dots, n$. Thus, $\bar{a}_n = \bar{a}_n \bar{r}_n \bar{a}_n$. It follows that

$$\begin{aligned}\bar{a}_{n-1} &= \bar{a}_n + \bar{a}_{n-1}\bar{r}_{n-1}\bar{a}_{n-1} \\ &= (\bar{a}_{n-1} - \bar{a}_{n-1}\bar{r}_{n-1}\bar{a}_{n-1})\bar{r}_n(\bar{a}_{n-1} - \bar{a}_{n-1}\bar{r}_{n-1}\bar{a}_{n-1}) + \bar{a}_{n-1}\bar{r}_{n-1}\bar{a}_{n-1} \\ &= \bar{a}_{n-1}[(\bar{1} - \bar{r}_{n-1}\bar{a}_{n-1})\bar{r}_n(\bar{1} - \bar{a}_{n-1}\bar{r}_{n-1}) + \bar{r}_{n-1}]\bar{a}_{n-1},\end{aligned}$$

so $\bar{a}_{n-1}$ is also a regular element. Continuing this process, we see that $\bar{a}_1$ is a regular element.

Secondly, we prove that Z_l is left T-nilpotent. Let $a_i \in Z_l$ for $i = 1, 2, \dots$. We have a chain $\mathbf{l}(a_1) \subseteq \mathbf{l}(a_1 a_2) \subseteq \cdots$. By our assumption, there exists $n > 0$ such that $\mathbf{l}(a_1 \cdots a_n) = \mathbf{l}(a_1 \cdots a_n a_{n+1})$. Thus, $\mathbf{l}(a_{n+1}) \cap R a_1 \cdots a_n = 0$. Since $\mathbf{l}(a_{n+1})$ is essential in ${}_R R$, we have $a_1 \cdots a_n = 0$, so Z_l is left T-nilpotent. Therefore, by Lemma 1.3, we have proved that $R/J(R)$ is a von Neumann regular ring and $J(R)$ is left T-nilpotent. So, it suffices to show that $R/J(R)$ is an artinian semisimple ring. By [13, Corollary 2.16], we only need to show that $R/J(R)$ contains no infinite sets of nonzero orthogonal idempotents. This can be proved by arguing as in [7, page 2107]. □

COROLLARY 1.6. *If R is a left AGP-injective ring with ACC on left annihilators, then R is semiprimary.*

PROOF. It is well known that Z_l is nilpotent for any ring R with ACC on left annihilators. By Lemma 1.3 and Theorem 1.5, R is semiprimary. □

COROLLARY 1.7. *Let R be a left AGP-injective ring with ACC on left annihilators and $S_r \subseteq S_l$. Then R is right artinian if and only if S_r is a finitely generated right ideal of R.*

PROOF. By Corollary 1.6, R is semiprimary. By [20, Corollary 2.7], $S_l \subseteq S_r$, and so $S = S_l = S_r$ by the hypothesis. Now the result follows from [4, Lemma 6]. □

A left GP-injective ring with the ACC on left annihilators is always right artinian [7, Theorem 3.7]. The ring R [21, Example 2.4] is a commutative AP-injective ring with the ACC on annihilators, but R is not artinian.

Recall that a ring R is called left Kasch if $\mathbf{r}(K) \neq 0$ for every maximal left ideal K of R.

COROLLARY 1.8. *Let R be a left AGP-injective ring with ACC on left annihilators. If every minimal right ideal is a right annihilator, then R is right artinian. Moreover, R is left artinian if and only if S_l is finitely generated as a left ideal of R.*

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PROOF. By Corollary 1.6, R is semiprimary. By [18, Corollary 3.15], R is right finite dimensional with $S_r = S_l$. Now, by [4, Lemma 6], R is right artinian. The last assertion follows from [4, Lemma 6] again. □

Now the following result, [7, Theorem 3.7], is an immediate corollary of the above:

COROLLARY 1.9 ([7]). *Every left GP-injective ring with ACC on left annihilators is right artinian.*

PROOF. If R is a left GP-injective ring, then every minimal right ideal is a right annihilator. For, if I is a minimal right ideal of R, then $I = eR$ where $e^2 = e \in R$ or $I^2 = 0$. If $I = eR$, clearly I is an annihilator. On the other hand, if $I = xR$ for some $x \in R$ with $I^2 = 0$, it follows from the definition of left GP-injectivity that $I = xR = \mathbf{rl}(I)$. Now the result follows from Corollary 1.8. □

2. Left AGP-injective rings with right chain conditions

In this section, we first consider right noetherian, left AGP-injective rings. We prove that, for a right noetherian, left AGP-injective ring R, $J(R)$ is nilpotent and $\mathbf{l}(J(R))$ is essential as a left and as a right ideal of R. As a corollary of this, we prove that every right noetherian, left AGP-injective ring R such that R_R satisfies (GC2) is right artinian. We next prove that every maximal left (respectively, right) annihilator of a semiprime left AGP-injective ring is a maximal left (respectively, right) ideal generated by an idempotent.

The next result extends [12, Theorem 2.7] from a left P-injective ring to a left AGP-injective ring.

THEOREM 2.1. *Let R be a right noetherian, and left AGP-injective ring. Then $J(R)$ is nilpotent and $\mathbf{l}(J(R))$ is essential both as a left and as a right ideal of R.*

PROOF. Let $J = J(R)$. First we prove that $\mathbf{l}(J) \leq_e {}_R R$. Let $0 \neq x \in R$. Since R is right noetherian, the non-empty set $\mathscr{F} = \{\mathbf{r}((ax)^k) : a \in R, k > 0$ such that $(ax)^k \neq 0\}$ has a maximal element, say $\mathbf{r}((yx)^n)$.

We claim that $(yx)^n J = 0$. If not, then there exists $t \in J$ such that $(yx)^n t \neq 0$. Since R is left AGP-injective, there exists $m > 0$ such that $((yx)^n t)^m \neq 0$ and $((yx)^n t)^m R$ is a direct summand of $\mathbf{rl}(((yx)^n t)^m)$. Write $((yx)^n t)^m = (yx)^n s$ where $s = t((yx)^n t)^{m-1} \in J$. Then $\mathbf{rl}((yx)^n s) = (yx)^n sR \oplus X$ for some right ideal X of R. We proceed with the following two cases.

Case 1. $\mathbf{rl}((yx)^n) = \mathbf{rl}((yx)^n s)$. Then $(yx)^n \in \mathbf{rl}((yx)^n) = (yx)^n sR \oplus X$. Write $(yx)^n = (yx)^n sv + z$, where $v \in R$ and $z \in X$. Then $(yx)^n s = (yx)^n svs + zs$ and

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so $zs \in (yx)^n sR \cap X$. Thus, $zs = 0$ and hence $(yx)^n s = (yx)^n svs$. It follows that $(yx)^n s(1 - vs) = 0$. Since $s \in J$, $1 - vs$ is a unit in R. So, we have $(yx)^n s = 0$. This is a contradiction.

Case 2. $\mathbf{rl}((yx)^n) \neq \mathbf{rl}((yx)^n s)$. Then $\mathbf{l}((yx)^n) \neq \mathbf{l}((yx)^n s)$. It follows that there exists $u \in \mathbf{l}((yx)^n s)$ but $u \notin \mathbf{l}((yx)^n)$. Thus, $u(yx)^n s = 0$ and $u(yx)^n \neq 0$. This gives that $s \in \mathbf{r}(u(yx)^n)$ and $s \notin \mathbf{r}((yx)^n)$. So, the inclusion $\mathbf{r}((yx)^n) \subset \mathbf{r}(u(yx)^n)$ is proper. This is a contradiction because $0 \neq u(yx)^n = (u(yx)^{n-1}y)x$ and $\mathbf{r}(u(yx)^n) \in \mathscr{F}$.

We have proved that $(yx)^n J = 0$, and so $Rx \cap \mathbf{l}(J) \neq 0$. Therefore, $\mathbf{l}(J)$ is an essential left ideal of R.

Next we prove that J is nilpotent. Since R is right noetherian, there exists $k > 0$ such that $\mathbf{l}(J^k) = \mathbf{l}(J^{k+n})$ for all $n > 0$. Suppose J is not nilpotent. Then $J^k \neq 0$ and so $M_R = R/\mathbf{l}(J^k)$ is a nonzero R-module. Since R is right noetherian, the set $\{\mathbf{r}_R(m) : 0 \neq m \in M\}$ has a maximal element, $\mathbf{r}_R(m_1)$ say. Write $m_1 = x + \mathbf{l}(J^k)$ where $x \in R$. Then $xJ^k \neq 0$. Since $\mathbf{l}(J^{2k}) = \mathbf{l}(J^k)$, we see $xJ^k \nsubseteq \mathbf{l}(J^k)$. So, there exists $b \in J^k$ such that $xb \notin \mathbf{l}(J^k)$. Since $\mathbf{l}(J) \leq_e {}_R R$, $Rxb \cap \mathbf{l}(J^k) \neq 0$. So, we have $0 \neq axb \in \mathbf{l}(J^k)$ for some $a \in R$. Let $m_2 = ax + \mathbf{l}(J^k) \in M$. Then $m_2 \neq 0$ and $b \in \mathbf{r}_R(m_2)$. But, $b \notin \mathbf{r}_R(m_1)$. So, the inclusion $\mathbf{r}_R(m_1) \subset \mathbf{r}_R(m_2)$ is proper. This contradicts the choice of m_1.

Finally, for any $0 \neq x \in R$, $xJ = 0$, or $xJ^n \neq 0$ and $xJ^{n+1} = 0$ for some $n > 0$. It follows that $xR \cap \mathbf{l}(J) \neq 0$. So, $\mathbf{l}(J)$ is an essential right ideal of R. □

The next result extends [12, Corollary 2.9]. (Note that, if R is left Kasch, then R_R satisfies (C2) (see [25]) and hence satisfies (GC2)).

COROLLARY 2.2. *Every right noetherian, left AGP-injective ring R such that R_R satisfies* (GC2) *is right artinian.*

PROOF. Since R is right finitely dimensional and R_R satisfies (GC2), R is semilocal by Lemma 1.1. By Theorem 2.1, $J(R)$ is nilpotent. So, R is semiprimary. Since R is right noetherian, R is right artinian. □

Next, we consider semiprime left AGP-injective rings.

LEMMA 2.3. *Let R be an arbitrary ring and $a \in R$ such that $\mathbf{l}(a)$ is a maximal left annihilator or $\mathbf{r}(a)$ is a maximal right annihilator. Then $\mathbf{l}(at) = \mathbf{l}(a)$ for any $t \notin \mathbf{r}(a)$ and $Z_l \subseteq \mathbf{r}(a)$, and $\mathbf{r}(ta) = \mathbf{r}(a)$ for any $t \notin \mathbf{l}(a)$ and $Z_r \subseteq \mathbf{l}(a)$.*

PROOF. Let $x \in Z_l$. Then $\mathbf{l}(x)$ is essential in ${}_R R$. So, $\mathbf{l}(x) \cap Rr \neq 0$ for any $0 \neq r \in R$. Thus, there exists $y \in R$ such that $0 \neq yr$ and $yrx = 0$. So, the inclusion $\mathbf{l}(r) \subset \mathbf{l}(rx)$ is proper.

Case 1. Let $\mathbf{l}(a)$ be a maximal left annihilator. As above, $\mathbf{l}(a) \subset \mathbf{l}(ax)$ for all $x \in Z_l$. It must be that $ax = 0$. This shows that $a \in \mathbf{l}(Z_l)$. Clearly, in this case $\mathbf{l}(at) = \mathbf{l}(a)$ for any $t \notin \mathbf{r}(a)$.

Case 2. Let $\mathbf{r}(a)$ be a maximal right annihilator. If $t \notin \mathbf{r}(a)$, then $at \neq 0$. For $x \in \mathbf{l}(at)$, $t \in \mathbf{r}(xa)$ and so the inclusion $\mathbf{r}(a) \subset \mathbf{r}(xa)$ is proper. By the maximality of $\mathbf{r}(a)$, $xa = 0$. Thus, $\mathbf{l}(at) = \mathbf{l}(a)$. It follows that $Ra \cap \mathbf{l}(t) = 0$. Thus, $t \notin Z_l$. Therefore, $Z_l \subseteq \mathbf{r}(a)$.

The remaining part is by the left-right symmetry of the hypothesis. □

The next theorem extends [7, Theorem 3.1].

THEOREM 2.4. *Let R be a semiprime left AGP-injective ring. Then every maximal left (respectively, right) annihilator is a maximal left (respectively, right) ideal of R which is generated by an idempotent.*

PROOF. Let L be a maximal left (respectively, right) annihilator. Then $L = \mathbf{l}(a)$ (respectively, $\mathbf{r}(a)$) for some $0 \neq a \in R$. Since R is semiprime, $Z_l \cap \mathbf{l}(Z_l) = 0$. Claim: $a \notin Z_l$. Otherwise, $a \notin \mathbf{l}(Z_l)$, that is, $aZ_l \neq 0$. Take $x \in Z_l$ such that $ax \neq 0$. Since $x \notin \mathbf{r}(a)$, $\mathbf{l}(ax) = \mathbf{l}(a)$ by Lemma 2.3. Thus, $\mathbf{l}(x) \cap Ra = 0$, a contradiction, since $x \in Z_l$. Therefore, $a \notin Z_l$. By Lemma 1.3 and Lemma 1.4, the inclusion $\mathbf{l}(a) \subset \mathbf{l}(a - ara) = \mathbf{l}[a(1 - ra)]$ is proper for some $r \in R$. It follows from Lemma 2.3 that $a - ara = 0$. Therefore, $L = \mathbf{l}(ar)$ (respectively, $L = \mathbf{r}(ra)$) with ar (respectively, ra) an idempotent. So we can assume that $a = e$ is an idempotent. To see L is a maximal left (respectively, right) ideal, we show that Re (respectively, eR) is a minimal left (respectively, right) ideal of R. Since R is semiprime, it suffices to show that eRe is a division ring. Let $0 \neq d \in eRe$. Since R is left AGP-injective, there exists $n > 0$ such that $d^n \neq 0$ and d^nR is a direct summand of $\mathbf{rl}(d^n)$. By Lemma 2.3, $\mathbf{l}(d^n) = \mathbf{l}(e)$ and so $\mathbf{rl}(d^n) = \mathbf{rl}(e) = eR$. Thus, d^nR is a direct summand of eR and hence of R_R. It follows that $d^nR = \mathbf{rl}(d^n) = eR$. Write $e = d^nb$ where $b \in R$. Then $e = d(d^{n-1}be)$ with $d^{n-1}be \in eRe$. So, eRe is a division ring. □

A ring R is a left PP ring if every principal left ideal of R is projective. The next result extends [6, Theorem 2.9] from a left GP-injective ring to a left AGP-injective ring.

PROPOSITION 2.5. *The ring R is a von Neumann regular ring if and only if R is left PP and left AGP-injective.*

PROOF. One direction is obvious. Suppose that R is left PP and left AGP-injective. For any nonzero element $a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and $\mathbf{rl}(a^n) = a^nR \oplus X$ where X is a right ideal of R. Since R is left PP, Ra^n is projective, and hence $0 \to \mathbf{l}(a^n) \to R \to Ra^n \to 0$ splits. Thus, $\mathbf{l}(a^n) = Re$ where $e^2 = e \in R$. It follows that $\mathbf{rl}(a^n) = \mathbf{r}(Re) = (1-e)R$. Thus, a^nR is a direct summand of $(1-e)R$, and hence a direct summand of R_R. This implies that a^n is a regular element of R. If $a \neq 0$ but $a^2 = 0$, the argument above shows that a is a regular element. So, by [6, Theorem 2.9], R is a regular ring. □

3. Right quasi-dual rings

Following [21], a ring R is called right quasi-dual if every right ideal of R is a direct summand of a right annihilator. As shown in [21], the ring R is right quasi-dual if and only if every essential right ideal of R is a right annihilator if and only if every singular cyclic right R-module is cogenerated by R. Every right dual ring is certainly right quasi-dual, and every right quasi-dual ring is left AP-injective.

LEMMA 3.1. *Let R be a right quasi-dual ring. For any right ideal I of R and $a \in R$, $\mathbf{r}[Ra \cap \mathbf{l}(I)] = I + (X_{aI} : a)_r$ with $(X_{aI} : a)_r \cap I \subseteq \mathbf{r}(a)$ and $(X_{aI} : a)_r = \{x \in R : ax \in X_{aI}\}$, where X_{aI} is a right ideal of R such that $\mathbf{rl}(aI) = aI \oplus X_{aI}$.*

PROOF. Let $x \in \mathbf{r}[Ra \cap \mathbf{l}(I)]$. Then $\mathbf{l}(aI) \subseteq \mathbf{l}(ax)$, and so $ax \in \mathbf{rl}(ax) \subseteq \mathbf{rl}(aI) = aI \oplus X_{aI}$. Write $ax = at + y$ where $t \in I$ and $y \in X_{aI}$. Then $a(x-t) = y \in X_{aI}$ and thus $x - t \in (X_{aI} : a)_r$. Therefore, $x \in I + (X_{aI} : a)_r$ and $\mathbf{r}[Ra \cap \mathbf{l}(I)] \subseteq I + (X_{aI} : a)_r$. It is easy to see that $(X_{aI} : a)_r \cap I \subseteq \mathbf{r}(a)$ and that $I \subseteq \mathbf{r}[Ra \cap \mathbf{l}(I)]$. Let $y \in (X_{aI} : a)_r$. Then $ay \in X_{aI} \subseteq \mathbf{rl}(aI)$. For any $ra \in Ra \cap \mathbf{l}(I)$, $raI = 0$. This gives that $r \in \mathbf{l}(aI)$. Since $ay \in \mathbf{rl}(aI)$, it follows that $ray = 0$. Thus, $y \in \mathbf{r}[Ra \cap \mathbf{l}(I)]$ and $(X_{aI} : a)_r \subseteq \mathbf{r}[Ra \cap \mathbf{l}(I)]$. □

THEOREM 3.2. *Let R be a right quasi-dual ring and $J = J(R)$. Then*

(1) *$J = Z_l = \mathbf{r}(S_r)$, $S_r = \mathbf{r}(Z_r)$, and R is right Kasch.*
(2) *$\mathbf{l}(J)$ is essential in ${}_RR$.*

PROOF. (1). Clearly, $S_r \subseteq \mathbf{r}(Z_r)$. Let K be any essential right ideal of R. Then $\mathbf{l}(K) \subseteq Z_r$ and so $K = \mathbf{rl}(K) \supseteq \mathbf{r}(Z_r)$. It follows that $S_r \supseteq \mathbf{r}(Z_r)$ since S_r is the intersection of all essential right ideals. Thus, $S_r = \mathbf{r}(Z_r)$. By [21, Lemma 2.5 and Lemma 2.6], $J = Z_l$ and R is right Kasch. Since R is right Kasch, $J = \mathbf{r}(S_r)$.

(2). Let $0 \neq a \in R$ and assume that $Ra \cap \mathbf{l}(J) = 0$. Then, by Lemma 3.1, $R = \mathbf{r}[Ra \cap \mathbf{l}(J)] = J + (X_{aJ} : a)_r$ where X_{aJ} is a right ideal of R such that $\mathbf{rl}(aJ) = aJ \oplus X_{aJ}$. Since J is small in R_R, $R = (X_{aJ} : a)_r$. It follows that $aR \subseteq X_{aJ}$ and so $aJ \subseteq aJ \cap X_{aJ} = 0$. Thus, $a \in Ra \cap \mathbf{l}(J) = 0$, a contradiction. □

COROLLARY 3.3. *Let R be a quasi-dual ring. Then $S = S_r = S_l$ is essential as a left and a right ideal of R.*

PROOF. By [21, Theorem 2.8] and Theorem 3.2. □

It was proved in [21] that, for a two-sided quasi-dual ring R, every Goldie torsion right R-module is cogenerated by R_R if and only if the second singular right ideal $Z_2(R_R)$ of R is injective. This result can be improved as follows.

THEOREM 3.4. *Consider the following conditions on a ring R:*

(1) *Every Goldie torsion right R-module is cogenerated by R_R.*
(2) *$Z_2(R_R)$ is injective and R is right Kasch.*
(3) *R is right self-injective and right Kasch.*

Then (3) *implies* (2) *and* (2) *implies* (1). *In addition* (1) *implies* (3) *if R is left quasi-dual.*

PROOF. (3) implies (2) is obvious, and (2) implies (1) is by the proof of [21, Theorem 4.1].

Suppose R is left quasi-dual and (1) holds. By [21, Theorem 4.1], $Z_2(R_R)$ is injective. Write $R_R = Z_2(R_R) \oplus K$ where K is right ideal of R. It suffices to show that K_R is injective. Note that R is a two-sided quasi-dual ring, so $Z_l = Z_r$ and $S_r = \mathbf{l}(Z_l)$ by [21, Theorem 2.8]. It follows that $K \subseteq \mathbf{l}((Z_2(R_R)) \subseteq \mathbf{l}(Z_l) = S_r$. So, K_R is semisimple. Thus, to show that K_R is injective, it suffices to show that K is $Z_2(R_R)$-injective. But, this is clear because K is non-singular and $Z_2(R_R)$ is Goldie torsion. □

A ring R is right PF if R is an injective cogenerator for Mod-R. It is known that R is right PF if and only if R is right self-injective and right Kasch. The next corollary improved [21, Corollaries 4.4–4.6].

COROLLARY 3.5. *R is a two-sided PF-ring if and only if every Goldie torsion right R-module is cogenerated by R_R and every Goldie torsion left R-module is cogenerated by $_RR$.*

Dischinger and Müller [8] constructed a left PF-ring that is not right PF. By Corollary 3.5, the left PF-ring in [8] does not cogenerate every Goldie torsion right R-module. Osofsky [19] constructed a non-injective cogenerator for Mod-R. We note that Osofsky's ring R has the property that $Z_2(R_R) = R$ (since $J(R)^2 = 0$ and $J(R)_R \leq_e R_R$). This shows the conditions (1) and (2) in Theorem 3.4 are not equivalent.

PROPOSITION 3.6. *The following are equivalent for a ring R:*

(1) *R is right PF.*
(2) *$Z_2(R_R)$ is injective, R is right Kasch and $R = Z_2(R_R) + S_r$.*

PROOF. (2) implies (1). It suffices to show that R is right self-injective. Since $R = Z_2(R_R) + S_r$, $R = Z_2(R_R) \oplus K$ where K is a non-singular semisimple right ideal of R. Clearly, K_R is $Z_2(R_R)$-injective and K_R-injective. So, K_R is injective. Thus, R_R is injective.

(1) implies (2). We only need to show that $R = Z_2(R_R) + S_r$. Since $Z_2(R_R)$ is injective, write $R = Z_2(R_R) \oplus K$ where K is a right ideal of R. Since R is right PF, $J(R) = Z_r \subseteq Z_2(R_R)$ and S_r is a finitely generated essential right ideal of R. Thus $\mathrm{Soc}(K_R)$ is finitely generated and essential in K_R. Since every minimal right ideal contained in K is idempotent, $\mathrm{Soc}(K_R)$ is a summand of R_R and hence of K_R. Thus, $K = \mathrm{Soc}(K_R)$ is semisimple. □

We do not know if the condition that $R = Z_2(R_R) + S_r$ in Proposition 3.6 can be removed.

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