Generalisation of the "Orthopole" and Allied Theorems.

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The greater part of this paper consists of generalisations of well-known theorems regarding perpendiculars to the sides of a triangle, or other base-lines, the perpendiculars being replaced by isoclinals.

To save confusion and secure generality, the following conventions will be observed:

(i) The angle which a line falling on a base-line makes with the latter will be taken as the angle, acute or obtuse as the case may be, obtained by measuring the shortest way round from the base-line to the first line (produced if necessary) in the *positive* or counter-clock sense.

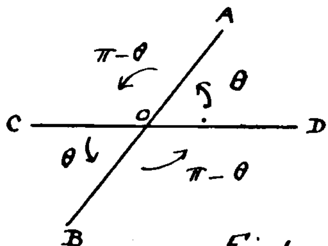

Fig. 1

Thus, in the figure, AO and BO make an angle θ with CD, but CO and DO make an angle $\pi - \theta$ with AB.

(ii) The areas of triangles will be regarded as positive when their vertices are taken counter-clockwise, as negative when their vertices are taken clockwise.

The following two propositions, which are so simple that their proofs may be omitted, supply the foundation for most of the subsequent theory.

(1). If from a point P a line PL be drawn to a base-line making

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with it an angle θ, and Q and R be any two points on the base-line,

$$PQ^2 - PR^2 = LQ^2 - LR^2 + 4\triangle PQR\cot\theta,$$

or $$LQ^2 - LR^2 = PQ^2 - PR^2 - 4\triangle PQR\cot\theta.$$

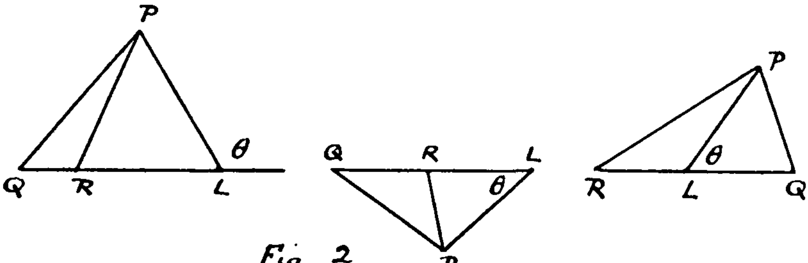

Fig. 2

This proposition as stated is universally true if (and not unless) we apply the two foregoing conventions and take the vertices of the triangle PQR in the following order—first, the vertex not lying on the base-line; and second, that vertex on the base-line which occurs first in the statement of the relation.

(2). If from any point O in the plane, lines OL, OM, ON be drawn to meet the sides BC, CA, AB respectively of a triangle ABC at the same angle θ,

$$LB^2 - LC^2 + MC^2 - MA^2 + NA^2 - NB^2 = -4\triangle ABC\cot\theta,$$

which may be written

$$[LMN] = -4\triangle ABC\cot\theta.$$

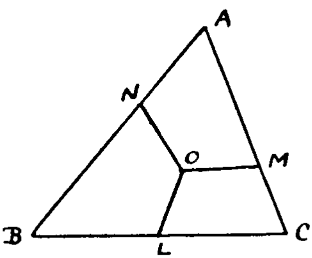

Fig. 3

This is an easy inference from (1), and is a generalisation of the standard theorem that if OL, OM, ON be perpendicular to the sides,

$$[LMN] = 0.$$

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If L, M, N be *any* points on BC, CA, AB, a value of θ can always be found to satisfy the general relation, and by a *reductio ad absurdum* we can prove that the lines from these points making angles having this value with the sides, meet in a point. Thus we obtain a criterion of concurrency for isoclinals of which use will be made in what follows.

The theorem can be extended to the polygon, but the converse in this case is not necessarily true, and hence cannot be used as a criterion of concurrency.

"*Isological*" *Triangles.*

(3). It is well known that if the three perpendiculars from the vertices of one triangle to the sides of another are concurrent, the three corresponding perpendiculars from the vertices of the latter to the sides of the former are also concurrent. Two such triangles are said to be reciprocally *orthological.**

This theorem may be generalised as follows: If ABC, A′B′C′ be two triangles such that lines AL′, BM′, CN′ meeting B′C′, C′A′, A′B′ respectively at an angle θ, meet in a point O, then lines A′L, B′M, C′N meeting BC, CA, AB at the angle $\pi - \theta$, also meet in a point O′.

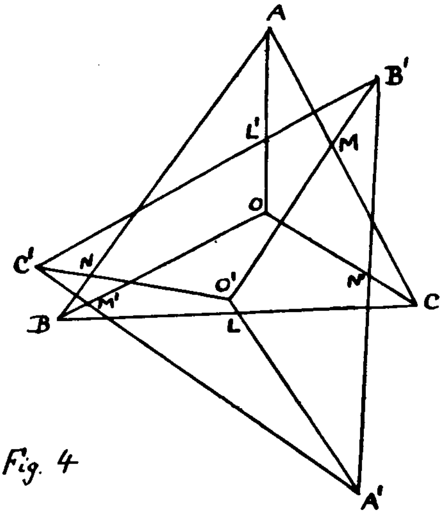

Fig. 4

* See a paper by Prof. Neuberg in *Mathesis*, 3rd series, vol. 1, p. 157.

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Proof. Join A′ to B and C; B′ to C and A; C′ to A and B.

Then, using (1) and remembering the convention regarding the areas of triangles, we have

$$\begin{aligned}
[\mathrm{LMN}] &= \mathrm{A'B^2 - A'C^2} - 4\Delta\mathrm{A'BC}\cot(\pi-\theta) \\
&\quad + \mathrm{B'C^2 - B'A^2} - 4\Delta\mathrm{B'CA}\cot(\pi-\theta) \\
&\quad + \mathrm{C'A^2 - C'B^2} - 4\Delta\mathrm{C'AB}\cot(\pi-\theta) \\
&= \mathrm{AC'^2 - AB'^2 + BA'^2 - BC'^2 + CB'^2 - CA'^2} \\
&\quad - 4(\Delta\mathrm{A'BC} + \Delta\mathrm{B'CA} + \Delta\mathrm{C'AB})\cot(\pi-\theta) \\
&= \mathrm{L'C'^2 - L'B'^2} + 4\Delta\mathrm{AC'B'}\cot\theta \\
&\quad + \mathrm{M'A'^2 - M'C'^2} + 4\Delta\mathrm{BA'C'}\cot\theta \\
&\quad + \mathrm{N'B'^2 - N'A'^2} + 4\Delta\mathrm{CB'A'}\cot\theta \\
&\quad - 4(\Delta\mathrm{A'BC} + \Delta\mathrm{B'CA} + \Delta\mathrm{C'AB})\cot(\pi-\theta) \\
&= -(\mathrm{L'B'^2 - L'C'^2 + M'C'^2 - M'A'^2 + N'A'^2 - N'B'^2}) \\
&\quad - 4(\Delta\mathrm{AC'B'} + \Delta\mathrm{BA'C'} + \Delta\mathrm{CB'A'} + \Delta\mathrm{A'BC} + \Delta\mathrm{B'CA} \\
&\qquad\qquad + \Delta\mathrm{C'AB})\cot(\pi-\theta) \\
&=, \text{ by (2)}, \ 4\Delta\mathrm{A'B'C'}\cot\theta \\
&\quad - 4(\Delta\mathrm{AC'B'} + \Delta\mathrm{BA'C'} + \Delta\mathrm{CB'A'} + \Delta\mathrm{A'BC} + \Delta\mathrm{B'CA} \\
&\qquad\qquad + \Delta\mathrm{C'AB})\cot(\pi-\theta) \\
&= -4(\Delta\mathrm{A'B'C'} + \Delta\mathrm{AC'B'} + \Delta\mathrm{BA'C'} + \Delta\mathrm{CB'A'} \\
&\qquad\qquad + \Delta\mathrm{A'BC} + \Delta\mathrm{B'CA} + \Delta\mathrm{C'AB})\cot(\pi-\theta) \\
&= -4\Delta\mathrm{ABC}\cot(\pi-\theta).
\end{aligned}$$

Therefore, by (2), A′L, B′M, C′N meet in a point O′.

For convenience of reference a pair of triangles related as ABC and A′B′C′ are, will be said to be reciprocally *isological* with respect to the angle which the isoclinals from the vertices of the first-mentioned make with the sides of the second.

(4). If A′B′C′ be isological with ABC and BCA with respect to the same angle θ, it is also isological with CAB with respect to that angle.

θ being the angle of inclination throughout, let

	A′L, B′M, C′N	be isoclinals to		BC, CA, AB	respectively,
	A′L′, B′M′, C′N′	,,	,, ,,	CA, AB, BC	,,
and	A′L″, B′M″, C′N″	,,	,, ,,	AB, BC, CA	,,

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If the first set meet in a point O, and the second in a point O′, we have to prove that the third set also meet in a point O″.

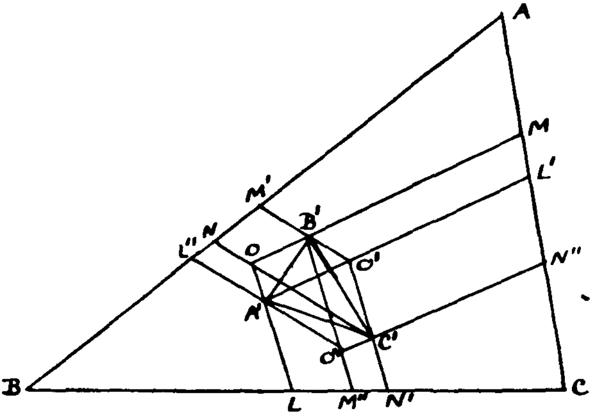

Fig. 5

We have, by (2) and (1),

$$
\begin{aligned}
-4\triangle ABC\cot\theta &= [LMN]\\
&= A'B^2 - A'C^2 - 4\triangle A'BC\cot\theta\\
&\quad + B'C^2 - B'A^2 - 4\triangle B'CA\cot\theta\\
&\quad + C'A^2 - C'B^2 - 4\triangle C'AB\cot\theta.
\end{aligned}
$$

Also

$$
\begin{aligned}
-4\triangle ABC\cot\theta &= [N'L'M']\\
&= C'B^2 - C'C^2 - 4\triangle C'BC\cot\theta\\
&\quad + A'C^2 - A'A^2 - 4\triangle A'CA\cot\theta\\
&\quad + B'A^2 - B'B^2 - 4\triangle B'AB\cot\theta.
\end{aligned}
$$

Adding and using (1) we have

$$
\begin{aligned}
-8\triangle ABC\cot\theta &= B'C^2 - B'B^2 + C'A^2 - C'C^2 + A'B^2 - A'A^2\\
&\qquad - 4(\triangle A'BC + \triangle A'CA + \triangle B'CA + \triangle B'AB\\
&\qquad + \triangle C'AB + \triangle C'BC)\cot\theta\\
&= M''C^2 - M''B^2 + 4\triangle B'CB\cot\theta\\
&\quad + N''A^2 - N''C^2 + 4\triangle C'AC\cot\theta\\
&\quad + L''B^2 - L''A^2 + 4\triangle A'BA\cot\theta\\
&\quad - 4(3\triangle ABC + \triangle A\ BA + \triangle B'CB + \triangle C'AC)\cot\theta\\
&= -[M''N''L''] - 12\triangle ABC\cot\theta.
\end{aligned}
$$

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Therefore $[M''N''L''] = -4\triangle ABC\cot\theta$.

Therefore, by (2), $A'L''$, $B'M''$, $C'N''$ meet in a point.

We can therefore assert that if two triangles be doubly isological, in the manner stated, with respect to a certain angle, they are triply isological with respect to that angle.

(5). Any two triangles ABC, A′B′C′ are, in general, reciprocally isological in six ways.

Suppose that the vertices as named lie the same way round; we can then make the following statements:

(i) If through the extremities of BC, CA, AB three circles be drawn such that their segments on the *same* sides of these lines *as* A, B, C respectively, contain angles respectively equal to $\pi - A'$, $\pi - B'$, $\pi - C'$, the circles meet in a point O_1, such that AO_1, BO_1, CO_1 make with $B'C'$, $C'A'$, $A'B'$ respectively the same angle θ_1, *i.e.* ABC is isological with A′B′C′ with respect to θ_1.

(ii) The same statement as the foregoing, with cyclical interchange of A′, B′, C′ and with O_2 for O_1 and θ_2 for θ_1.

(iii) The same as the foregoing, with further cyclical interchange of A′, B′, C′ and with O_3 for O_2 and θ_3 for θ_2.

(iv) If through the extremities of BC, CA, AB three circles be drawn such that their segments on the *opposite* sides of these lines *from* A, B, C respectively, contain angles respectively equal to $\pi - A'$, $\pi - C'$, $\pi - B'$, the circles meet in a point O_4 such that AO_4, BO_4, CO_4 make with $C'B'$, $B'A'$, $A'C'$ respectively the same angle θ_4, *i.e.* ABC is isological with A′C′B′ with respect to θ_4. (Note that here the vertices of A′C′B′ as named lie the *opposite* way round *from* those of ABC.)

(v) The same statement as the foregoing, with cyclical interchange in reverse order of A′, B′, C′ and with O_5 for O_4 and θ_5 for θ_4.

(vi) The same as the foregoing, with further cyclical interchange, still in reverse order, of A′, B′, C′ and with O_6 for O_5 and θ_6 for θ_5.

If the vertices of ABC and A′B′C′ instead of lying the same way round lie in opposite ways round, the above statements must be modified by the interchange of the expressions ***same as*** and ***opposite from*** wherever they occur.

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The truth of the statements, which need not be demonstrated in detail, will be manifest from an examination of the cyclic quadrilaterals in the figures of the different cases.

It will be noticed that O_1, O_2, O_3, O_4, O_5, O_6 are dependent only on the position of ABC and the shape of A′B′C′, and therefore remain invariable when ABC is fixed and A′B′C′ retains a constant shape, however much the latter triangle may vary in position or size. The respective conjugate points of concurrence of the lines from the vertices of A′B′C′ making angles $\pi - \theta_1$, $\pi - \theta_2$, etc., with the sides of ABC, are related to A′B′C′ in the same way as their conjugates are related to ABC, and consequently vary in position with A′B′C′.

If the triangles ABC, A′B′C′ are *directly* similar, O_1, O_2, O_3, O_4, O_5, O_6 become respectively the orthocentre, the negative Brocard point, the positive Brocard point, and the vertices A, B, C of ABC.

If the triangles ABC, A′B′C′ are *inversely* similar, O_1 is *any* point on the circumcircle of ABC, since the three circles giving rise to it coalesce in that circle. Hence we have the theorem which will be used later on, that any two inversely similar triangles are reciprocally isological with respect to every angle lying between 0 and π, *i.e.* in an infinite number of ways, and that, as the angle varies, the point of concurrence of the isoclinals from the vertices of either to the corresponding sides of the other has as locus the circumcircle of the former.

The positions assumed by the O-points in the other cases of inverse similarity corresponding to (ii), (iii), (iv), (v) and (vi) above, do not offer any special features of interest.

Orthopole and "Isopoles."

(6). It is well known that if perpendiculars AP, BQ, CR be drawn from the vertices of a triangle ABC to a base-line XY, and perpendiculars PL, QM, CN be drawn from P, Q, R to BC, CA, AB respectively, PL, QM, CN meet in a point O, which is called the *orthopole* of XY.*

* See a paper on the Orthopole by Mr William Gallatly (Glasgow University Press).

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This theorem may be generalised as follows: If isoclinals AP, BQ, CR be drawn from the vertices of a triangle ABC to a base-line XY making with it any angle θ, and if isoclinals PL, QM, CN be drawn from P, Q, R to BC, CA, AB respectively, making with them an angle $\pi-\theta$, PL, QM, CN meet in a point O, which may be called the *isopole* of XY with respect to the angle θ adopted.

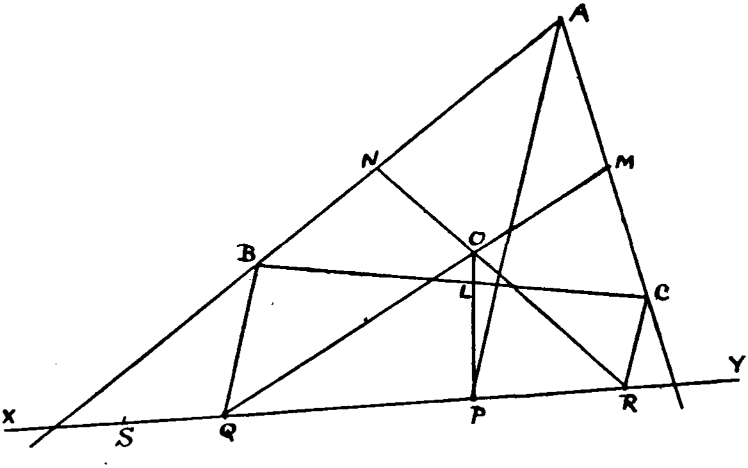

Fig. 6

Proof. Using (1), and remembering the convention regarding the areas of triangles, we have

$$\begin{aligned}
[LMN] &= PB^2 - PC^2 - 4\triangle PBC\cot(\pi-\theta) \\
&+ QC^2 - QA^2 - 4\triangle QCA\cot(\pi-\theta) \\
&+ RA^2 - RB^2 - 4\triangle RAB\cot(\pi-\theta) \\
&= AR^2 - AQ^2 + BP^2 - BR^2 + CQ^2 - CP^2 \\
&- 4(\triangle PBC + \triangle QCA + \triangle RAB)\cot(\pi-\theta) \\
&= PR^2 - PQ^2 + 4\triangle ARQ\cot\theta \\
&+ QP^2 - QR^2 + 4\triangle BPR\cot\theta \\
&+ RQ^2 - RP^2 + 4\triangle CQP\cot\theta \\
&- 4(\triangle PBC + \triangle QCA + \triangle RAB)\cot(\pi-\theta) \\
&= -4(\triangle ARQ + \triangle BPR + \triangle CQP + \triangle PBC \\
&\qquad + \triangle QCA + \triangle RAB)\cot(\pi-\theta) \\
&= -4\triangle ABC\cot(\pi-\theta).
\end{aligned}$$

Therefore, by (2), PL, QM, RN are concurrent.

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(7). The theory of isopoles could, if it were worth while, be considerably developed. I have had time only for a very cursory and superficial examination of the subject. The present section contains a selection of the results obtained, which are either loosely general in character or refer to the more obvious particular cases.

(i) If the base-line XY (Fig. 6) move parallel to itself, the locus of its isopole O with respect to a constant angle θ is a line making an angle θ with the fixed direction of XY.*

This follows from the obvious fact that, in the case supposed, the triangle QOR is always congruent to itself.

It follows also from the constant self congruence of QOR that the distance of O from XY is constant, and thus that the distance between a line and its isopole with respect to a given angle is unaltered by the movement of the line parallel to itself.

(ii) If XY (Fig. 6) vary in position while AP, BQ, CR remain fixed in direction thus making a variable angle θ with XY, and O be the isopole of XY in any position with respect to the value of θ for that position, the triangles QOR, ROP, POQ remain constant in shape; *e.g.* if ϕ be the constant angle made by AP with BC, then according as ϕ is less or greater than $\pi - C$, the angles at Q, R and O of the triangle QOR are respectively $\pi - C - \phi$, $\pi - B + \phi$ and $\pi - A$, or $-\pi + C + \phi$, $\pi + B - \phi$ and A. Thus if XY rotate round a fixed point S, since QR passes through this point and Q and R move on parallel straight lines, O will also move on a straight line, viz. the line joining the fixed second points of intersection of the circles SQO and SRO with BQ and CR respectively.

(iii) If we take O as the origin (Fig. 6) and $y = \mu x + \nu$ as the Cartesian equation of the line QR whose isopole with respect to a particular angle θ is O, and then form the equations of OQ and OR and hence those of BQ and CR, we find from the condition that BQ and CR make an angle θ with QR, taking β and γ as the angles made with the x-axis by AB and AC and (b_1, b_2), (c_1, c_2) as the coordinates of B and C, that

$$\nu\{\tan\theta - \tan\overline{\gamma - \theta} + \mu(1 + \tan\theta\,.\,\tan\overline{\gamma - \theta})\}$$

$$= \mu^2(b_1 + b_2\tan\theta) + \mu\{b_1(\tan\theta - \tan\overline{\gamma - \theta}) - b_2(1 + \tan\theta\,.\,\tan\overline{\gamma - \theta})\}$$

$$+ (b_2 - b_1\tan\theta)\,.\,\tan\overline{\gamma - \theta},$$

with a similar equation having c for b and β for γ.

* The corresponding statement for the orthopole is given in Mr Gallatly's paper, p. 4.

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Hence we obtain a cubic for the determination of μ; while for every value of μ there is only one value of ν.

Thus we conclude that, while every base-line has only one isopole for a particular value of θ, every point is in general the isopole, with respect to any given value of θ, of three different lines.

For example, the orthocentre H of the triangle ABC is obviously the orthopole of each of the three sides. Combining this statement with (i) above, we have the result that the locus of the orthopole of a line which moves parallel to one of the sides is the altitude corresponding to that side, and that the distance of the orthopole from the line is always equal to HD, where D is the foot of the altitude on the side.

One other example may be given. D is the orthopole of AD, of the line parallel to BC through the image of H with respect to BC, and, as might easily be shown, of the line joining A to the circumcentre.

(iv) If a line make an angle θ with a side of the triangle, it is obvious from a figure that the isopole of the line with respect to $\pi - \theta$ is the point where the line cuts that side.

(v) The locus of the isopoles, with respect to a given angle θ, of the lines passing through a fixed point S is in general a conic.

Take S as the origin and $y = x \,.\, \tan\phi$ as the equation of the variable base-line. Let also (b_1, b_2), (c_1, c_2) be the coordinates of B and C, and β, γ the angles made by AB and AC with the x-axis.

Then, since BQ and CR (Fig. 6) make an angle $\theta + \phi$, and QM and RN angles $\gamma - \theta$ and $\beta - \theta$ with the x-axis, we have as the equation of QM,

$$\begin{aligned}&\tan^2\phi(x\tan\theta \,.\, \tan\overline{\gamma - \theta} - y\tan\theta + b_1 + b_2\tan\theta)\\&+ \tan\phi\{b_1(\tan\theta - \tan\overline{\gamma - \theta}) - b_2(1 + \tan\theta \,.\, \tan\overline{\gamma - \theta})\}\\&+ x\tan\theta \,.\, \tan\overline{\gamma - \theta} - y\tan\theta - (b_1\tan\theta - b_2)\tan\overline{\gamma - \theta} = 0,\end{aligned}$$

with a similar equation for RN having β for γ and c for b.

Eliminating $\tan\phi$ between these equations, we have as the locus of O a conic in the form,

$$(ax + by + c)^2 - (dx + ey + f)(dx + ey + g) = 0.$$

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It obviously touches the parallel lines (def) and (deg) at the points where these are met by (abc), and its centre lies on (abc) mid-way between the former two lines.

If from S lines SX, SY, SZ be drawn to the sides BC, CA, AB respectively of the triangle, making with them an angle $\pi - \theta$, it follows from (iv) that the conic passes through X, Y, Z. If S be the negative (or positive) Brocard point, and θ the Brocard angle ω (or $\pi - \omega$), X, Y, Z coincide with C, A, B respectively (or B, C, A), and the conic becomes a circumconic.

If S be the circumcentre and $\theta = \dfrac{\pi}{2}$, so that for *isopole* we have *orthopole*, the conic becomes the nine-point circle.* In this case the orthopoles of the perpendiculars from S to the sides are, by (iv), the mid-points of the sides; the orthopoles of the parallels through S to the sides are, by (iii), the mid-points of the lines joining the orthocentre to the vertices; and the orthopoles of the lines joining S to the vertices are the feet of the altitudes.

When S coincides with a vertex of the triangle, it is evident from a figure that the locus of the isopole of a variable line through S with respect to a given angle θ, degenerates into the line through the vertex making with the opposite side the angle $\pi - \theta$.

(vi) The locus of the isopole of a given line with respect to an angle θ which varies from zero to π is, in general, a cubic.

The equation of this cubic, if we take the given base-line as the axis of x, and (a_1, a_2), (b_1, b_2), (c_1, c_2) as the coordinates of A, B, C, is found to be

$$\begin{aligned}&[x\,.\,\Sigma a_1 a_2(b_2 - c_2) - y\,.\,\Pi(b_2 - c_2) + \Sigma a_1 a_2(b_1 c_2 - b_2 c_1)]^2\\ &- [x\,.\,\Pi(b_2 - c_2) + y\,.\,\Sigma a_1 a_2(b_2 - c_2) + \Sigma a_1 b_2 c_2(b_2 - c_2)]\\ &\times [(x^2 + y^2)\,.\,\Sigma a_1(b_2 - c_2) - x\,.\,\Sigma a_1^2(b_2 - c_2)\\ &\qquad + y\{\Sigma a_1(b_2^2 - c_2^2) - \Sigma a_1 a_2(b_2 - c_2) + \Pi(b_1 - c_1)\}\\ &\qquad - \Sigma a_2 b_1 c_1(b_1 - c_1)] = 0.\end{aligned}$$

It is evident from (iv) that the cubic passes through the three points where the base-line cuts the sides of the triangle.

In the particular case when the base-line coincides with a side

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* Proved in Mr Gallatly's paper, p. 5.

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of the triangle, the cubic degenerates into the circle passing through the extremities of that side and the orthocentre H.

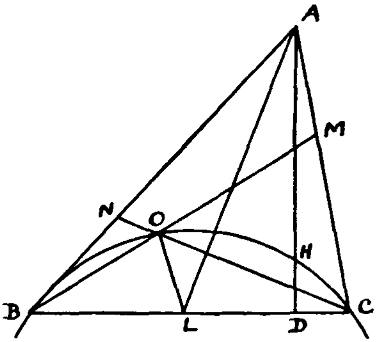

Fig 7

Let O, for example, be the isopole of BC with respect to any angle θ, and let BO, CO meet CA, AB in M and N respectively. Then BM, CN make an angle $\pi - \theta$ with CA, AB. Therefore AMON is a cyclic quadrilateral. Hence according as O lies on the same side of BC as A or on the opposite side, angle BOC $= \pi - A$ or A, so that O lies on the circle BHC. It is obvious that O is the image with respect to BC of the point where the circumcircle of ABC is met by a line from A making an angle θ with BC. Further, O and the two corresponding points for CA and AB are the three points into which the orthocentre breaks up when θ is substituted for $\frac{\pi}{2}$, and so may be regarded as constituting a kind of generalised orthocentre.

In connection with the case just mentioned it may be observed that if O_a, O_a' be the isopoles of BC with respect to θ and $\pi - \theta$, θ being here taken as acute as may without loss of generality be done, and O_b, O_b' and O_c, O_c' be the corresponding points for CA and AB, θ being the same throughout, then these six points not only lie in pairs on the circles BHC, CHA, AHB, but all lie on the circle with centre H and radius $2R\cos\theta$.

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For the lines BO_a, BO_a' make angles $\pi - \theta$ and θ with CA. Hence angle $O_aBO_a' = \pi - 2\theta$, and is bisected by BH. Therefore,

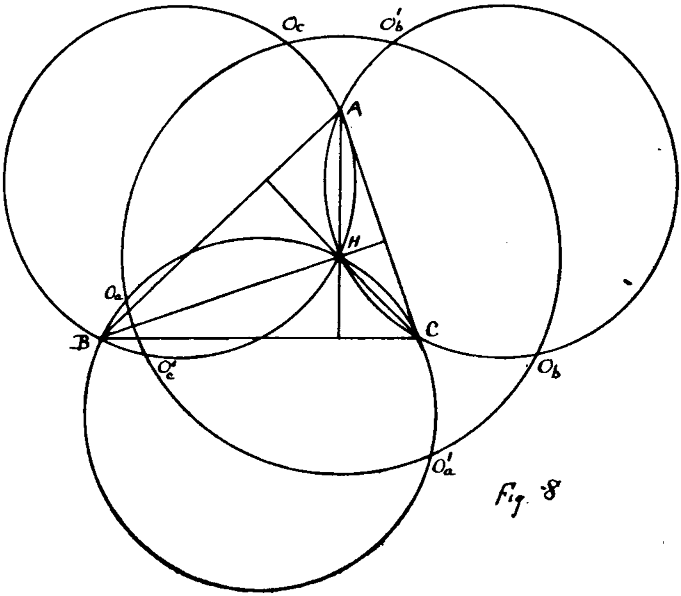

Fig. 8

since O_a, O_a' lie on the circle BHC,

$$O_aH = O_a'H = 2R\sin\left(\frac{\pi}{2} - \theta\right) = 2R\cos\theta.$$

Similarly the distance of H from any of the other O-points may be proved equal to $2R\cos\theta$.

Among the numerous other properties of this figure the following may be noticed $O_aO_a' = O_bO_b' = O_cO_c' = 2R\sin 2\theta$, 2θ being the angle which each subtends at H. The triangles $O_aO_bO_c$ and $O_a'O_b'O_c'$ are congruent, similar to ABC and circumscribed to it; the area of each $= 4\triangle ABC\cos^2\theta$. The lines O_bO_c, $O_b'O_c'$ meet the circumcircle of ABC in the images with respect to BC of O_a', O_a, with two similar statements. When $\theta = 0$, the pairs O_a, O_a'; O_b, O_b'; O_c, O_c' coincide in the points where the circle with centre H and radius 2R touches the circles BHC, CHA, AHB.

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*Generalised Brocard Points and Circle.**

(8). I now return for a little to the figure of section (2) (Figure 3), where from a point O isoclinals OL, OM, ON are drawn to the sides BC, CA, AB respectively of the triangle, making with these an angle θ. If AL, BM, CN make angles α, β, γ with BC, CA, AB respectively, then

$$\cot\theta = \cot\alpha + \cot\beta + \cot\gamma.$$

For from (1) we have

$$LB^2 - LC^2 = AB^2 - AC^2 - 4\triangle ABC\cot\alpha,$$

with two similar statements for β and γ.

Hence $\quad [LMN] = -4\triangle ABC(\cot\alpha + \cot\beta + \cot\gamma).$

But by (2), $[LMN] = -4\triangle ABC\cot\theta.$

Therefore, $\cot\theta = \cot\alpha + \cot\beta + \cot\gamma.$

In all cases where $\theta = \frac{\pi}{2}$, as for example when LMN is a true Wallace line, or AL, BM, CN are the medians, the relation becomes

$$\cot\alpha + \cot\beta + \cot\gamma = 0.$$

When L, M, N coincide with B, C, A respectively (or C, A, B) O becomes the positive (or negative) Brocard point, θ becomes the Brocard angle ω (or $\pi - \omega$), and we obtain the well known relation,

$$\cot\omega = \cot A + \cot B + \cot C.$$

Thus, being given the position of L, M, N, we can regard O as the generalised Brocard point, and θ as the generalised Brocard angle, for that position. While it is obvious from (2) that for any given position of L, M, N there is only one generalised Brocard point O, and one generalised Brocard angle θ, it is clear that for any given position of O there are an infinite number of possible values of θ, and an infinite number of different positions of L, M, N, viz. the different positions assumed by the vertices of an inscribed triangle of constant species of which O is the centre of similitude. Hence any point whatever in the plane can be regarded

* There have been various generalisations of the Brocard circle, *e.g.* that of Mr Griffiths in the *Proceedings of the London Mathematical Society*, Vol. XXV., pp. 75–85 and 376–388, and Vol. XXVI., pp. 173–183: but all those known to me differ widely from the one now given.

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as a generalised Brocard point, and with it can be associated as generalised Brocard angle any angle whatever lying between zero and π, thus determining the position of L, M, N.

The position of L, M, N being given, we can easily deduce from the relation given above connecting θ, α, β, γ, a geometrical construction determining the position of O.

When L, M, M are collinear the locus of O, as is well known, is the circumcircle of ABC, LMN being then the true or generalised (isoclinals replacing perpendiculars) Wallace line of O. When the triangle LMN is of constant shape, O, as is also well known, is fixed. It might readily be shown by using trilinear coordinates that when AL, BM, CN are concurrent, the locus of O for any given value of θ is a cubic. The coordinates of O being given, with the condition that AL, BM, CN shall be concurrent, we find a cubic equation for the determination of $\cot\theta$; hence for every position of O there are three positions of L, M, N such that AL, BM, CN are concurrent; of these positions one is always real, and for certain positions of O, but not for all, the other two also. When of two different positions of L, M, N the one set are the isotomic conjugates of the other, it is evident from (2) that the corresponding generalised Brocard angles are supplementary; this is of course the case with the true Brocard points.

The connection of all this with the well known "Point O" theorem is obvious, O being the point of intersection of the circles AMN, BNL, CLM.

It may not be amiss to give the coordinates of O for certain data. If $\text{BL} = l$, $\text{LC} = l'$, $\text{CM} = m$, $\text{MA} = m'$, $\text{AN} = n$, $\text{NB} = n'$, then the trilinear coordinates of O are

$$blm + cn'l' - all',\ cmn + al'm' - bmm',\ anl + bm'n' - cnn'.$$

If AL, BM, CN are concurrent in the point whose trilinear coordinates are λ, μ, ν, then the trilinear coordinates of O are

$$\frac{a}{b\mu + c\nu}\left(\frac{b\nu\lambda}{c\nu + a\lambda} + \frac{c\lambda\mu}{a\lambda + b\mu} - \frac{a\mu\nu}{b\mu + c\nu}\right),$$

with two similar expressions obtained by the cyclic interchange of a, b, c and λ, μ, ν.

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(9). If O, O′ be any two generalised Brocard points, θ, θ' the corresponding generalised Brocard angles, and L, M, N and L′, M′, N′ the corresponding positions of L, M, N, and if LO, L′O′ meet in A′, MO, M′O′ in B′, and NO, N′O′ in C′, then (i) A′, B′, C′, O, O′ lie on a circle, the generalised Brocard circle; (ii) perpendiculars from A′, B′, C′ to the corresponding sides of ABC cointersect in a point S on the circle, parallels through A′, B′, C′ to the corresponding sides of ABC cointersect in a point K on the circle, and generally any isoclinals from A′, B′, C′ to the corresponding sides of ABC cointersect in a point on the circle; (iii) the triangle A′B′C′ (corresponding to Brocard's first triangle) is inversely similar to ABC; (iv) perpendiculars from A, B, C to the corresponding sides of A′B′C′ cointersect in a point on the circumcircle of ABC (corresponding to Tarry's point), and generally isoclinals from A, B, C to the corresponding sides of A′B′C′ cointersect in a point on the circumcircle of ABC; (v) A′B′C′ is in perspective with the triangles LMN, L′M′N′ with O, O′ as centres of perspective; (vi) $\triangle A'BC + \triangle B'CA + \triangle C'AB = \triangle ABC$, etc., etc. In fact the correspondence between the generalised and the true Brocard circles is complete.

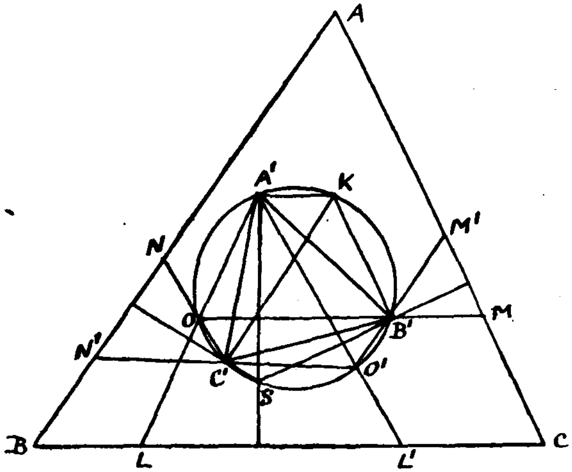

Fig.: 9

Proof. $L\widehat{A'}L' = M\widehat{B'}M' = N\widehat{C'}N' = \theta \sim \theta'$. Hence the angles OA′O′, OB′O′, OC′O′ are equal or supplementary, which establishes (i).

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Again, since AMON and A′B′OC′ are cyclic quadrilaterals, $C'\widehat{A'}B' = A$. Similarly it may be shown that $A'\widehat{B'}C' = B$ and $B'\widehat{C'}A' = C$. Thus (iii) is established.

(ii) and (iv) follow immediately from the fact referred to in (5) that if two triangles ABC, A′B′C′ be inversely similar they are reciprocally isological in an infinite number of ways, isoclinals from the vertices of either to the corresponding sides of the other co-intersecting in a point on the circumcircle of the former.

(v) is at once obvious from the figure.

(vi) is evident from the fact that the triangles A′BC, B′CA, C′AB are respectively equal to the triangles KBC, KCA, KAB.

When $\theta = \theta'$, the generalised Brocard circle obviously degenerates into the line OO′ and the line at infinity.

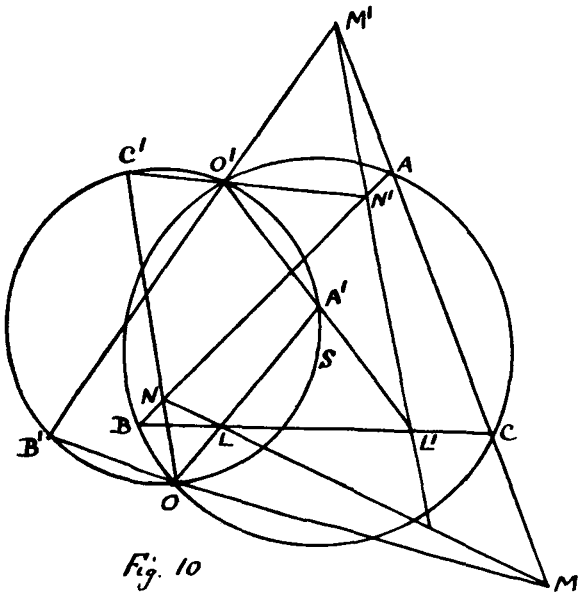

Fig. 10

When the pairs L, L′; M, M′; N, N′ are isotomic conjugates, and consequently θ and θ' are supplementary, S is obviously the circumcentre of ABC, as is the case for the true Brocard circle.

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When in addition to the isotomic relation, L, M, N are collinear, and therefore also L′, M′, N′, then not only do O, O′ lie on the circumcircle of ABC (LMN, L′M′N′ being their true or generalised Wallace lines), but OO′ subtends at the circumference of that circle an angle equal to θ, and the angle between LMN and L′M′N′ is also θ.

For in this case twice the angle subtended by OO′ at the circumference of the circumcircle $= O\widehat{S}O' = O\widehat{A'}O' = 2\theta$. Also, in the figure given, the angle between LMN and L′M′N′

$$
\begin{aligned}
&= B\widehat{N}L - B\widehat{N'}L' = \theta + O\widehat{N}L - \pi + B\widehat{N'}M' \\
&= \theta + O\widehat{B}C - \pi + \theta + O'\widehat{N'}M' \\
&= 2\theta - \pi + O\widehat{B}C + O'\widehat{A}M' \\
&= 2\theta - \pi + O\widehat{B}C + O'\widehat{B}C \\
&= 2\theta - \pi + O\widehat{B}O' = 2\theta - \pi + \pi - \theta = \theta.
\end{aligned}
$$

Any circle whatever in the plane can be taken as a generalised Brocard circle, and any two points O, O′ on it can be taken as the associated generalised Brocard points. We may fix A′ arbitrarily anywhere on this circle. Then by joining A′ to O, O′ we get L, L′ and so determine θ, θ'. The other points M, M′, N, N′, B′, C′ are then obtained by drawing isoclinals from O, O′ making angles θ, θ' respectively with CA and AB.

By treating them as generalised Brocard circles we can obtain many of the properties of the circles connected with the triangle. For example, if we regard the circumcircle as a generalised Brocard circle, we at once obtain the theorem (otherwise sufficiently obvious) that if LMN be the true or a generalised Wallace line of O, then OL, OM, ON meet the circle in the vertices of a triangle inversely congruent to ABC, and that while LMN moves parallel to itself the vertices of this triangle remain fixed.

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