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# ON THE HAUSDORFF DISTANCE BETWEEN A CONVEX SET AND AN INTERIOR RANDOM CONVEX HULL

H. BRÄKER,* *University of Bern*
T. HSING,** *Singapore National University*
N. H. BINGHAM,*** *Birkbeck College (Univ. London)*

**Abstract**

The problem of estimating an unknown compact convex set $K$ in the plane, from a sample $(X_1, \cdots, X_n)$ of points independently and uniformly distributed over $K$, is considered. Let $K_n$ be the convex hull of the sample, $\Delta$ be the Hausdorff distance, and $\Delta_n := \Delta(K, K_n)$. Under mild conditions, limit laws for $\Delta_n$ are obtained. We find sequences $(a_n)$, $(b_n)$ such that

$$(\Delta_n - b_n)/a_n \to \Lambda \qquad (n \to \infty),$$

where $\Lambda$ is the Gumbel (double-exponential) law from extreme-value theory. As expected, the directions of maximum curvature play a decisive role. Our results apply, for instance, to discs and to the interiors of ellipses, although for eccentricity $e < 1$ the first case cannot be obtained from the second by continuity. The polygonal case is also considered.

*Keywords:* Convex set; convex hull; Hausdorff metric; limit law; Gumbel distribution; extreme value theory; smooth boundary; polygon; moving boundary; home range

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## 1. Introduction

The problem of estimating the large quantiles and tail probabilities of a distribution — univariate or multivariate — has been one of the focal issues of statistical extreme value theory (cf. Dekkers and de Haan (1989), Smith (1987)). It is a natural extension of these problems to estimate a compact convex set.

Suppose that $K$ is an unknown compact convex set, in $\mathbb{R}^2$ say. We wish to estimate $K$ using a set of points $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ sampled randomly from $K$. The problem has been extensively considered in both the pure and applied literature. To illustrate the concepts, we cite here a biological motivation, radio tracking. $K$ is the territorial range of an animal. We capture the animal, tag it with a radio transmitter and release it. The $\boldsymbol{X}_i$ represent fixes on the animal's position at the times its transmitter 'bleeps' (MacDonald *et al.* 1980). Another setting would

Received 15 July 1996; revision received 2 January 1997.
* Postal address: University of Bern, 3012 Bern, Switzerland.
** Postal address: Singapore National University, 10 Kent Ridge Crescent, Singapore 119260K.
*** Postal address: Birkbeck College (Univ. London), London WC1E 7HX, UK.
E-mail address: n.bingham@statistics.bbk.ac.uk

be an estimation of the geographical distribution of a (perhaps rare) species of animal or plant, based on sightings of individuals. An alternative picture, due to D. G. Kendall, arises in estimating the 'edge of a Poisson forest'. See Ripley and Rasson (1977) for an account of this and Moore (1984) for some subsequent developments.

A natural approach is to estimate $K$ by the convex hull of the $\boldsymbol{X}_i$ (denoted by $K_n$ in this paper), which is the smallest convex set containing the sample points. A natural measure of the discrepancy between $K$ and $K_n$ is the Hausdorff distance,

$$\Delta_n = \Delta(K, K_n),$$

where $\Delta$ is the Hausdorff metric, given by

$$\Delta(A, B) := \inf\{\epsilon > 0 : A \subset B_\epsilon . B \subset A_\epsilon\}.$$

Here, $A_\epsilon$ is the '$\epsilon$-neighbourhood' of $A$, namely,

$$A_\epsilon := \{\boldsymbol{x} : |\boldsymbol{x} - \boldsymbol{y}| \leq \epsilon \text{ for some } \boldsymbol{y} \in A\}.$$

Hence,

$$\Delta_n = \inf\{\epsilon > 0 : K \subset (K_n)_\epsilon\},$$

which also happens to be the Hausdorff distance between the boundaries $\partial K$ and $\partial K_n$. The following observation is useful. For each supporting line $\ell$ of $\partial K_n$, let $\delta(\ell)$ be the vertical distance between $\ell$ and the parallel supporting line of $\partial K$ on the same side of $K_n$. Therefore, it is clear that

$$\Delta_n = \sup \delta(\ell), \tag{1}$$

where the supremum is taken over all supporting lines $\ell$ of $K_n$. The goal of this paper is to derive the asymptotic distribution of $\Delta_n$ as $n \to \infty$ for certain classes of $K$.

Problems of this sort were considered by Rényi and Sulanke (1963, 1964), who focused on the two principal cases: (a) $\partial K$ is a smooth curve with positive curvature, and (b) $K$ is a polygon, with $r$ vertices, say. They measured the discrepancy by $A - EA_n$, $L - EL_n$, where $A$, $L$ are the area and perimeter-length of $K$ and $A_n$, $L_n$ are the same for $K_n$. For subsequent work here, and extension to $\mathbb{R}^d$, see Bárány and Larman (1988), Bárány (1989), Schneider and Wieacker (1980). More recently, the asymptotic distributions of $A - A_n$ and $L - L_n$ for the polygonal case were derived by Cabo and Groeneboom (1994) and for the smooth case by Hsing (1994) and Bräker and Hsing (1995). Not only are the smooth and polygonal cases the two extreme cases of the problem, but also intermediate cases typically exhibit irregular behaviour. (For precise formulation, see Gruber (1983), Bárány and Larman (1988).) We shall focus accordingly on the smooth and polygonal cases below. Our principal technical tool is an adaptation of the method of Hsing (1994). There, the residual area $\int \delta(\ell)$ is treated by exploiting asymptotic independence in different directions and using a blocking technique. Here, we replace integration by the supremum in (1) and use extreme-value theory for weakly dependent random variables (cf. Leadbetter *et al.* (1982), O'Brien (1987), Rootzén (1988)).

This paper is organized as follows. In Section 2 we state the main result, Theorem 1, for the smooth case, give a number of examples and then state the variant, Theorem 2. Theorem 3 for the polygonal case is stated and proved in Section 3. Finally, the proof of Theorem 1 (which is lengthy) is given in Section 4; and since the proof of Theorem 2 is similar, it is merely sketched.

## 2. Asymptotic distribution of $\Delta_n$ – the smooth case

Consider a bounded convex set $K$ in $\mathbb{R}^2$. Assume that $\partial K$, the boundary of $K$, has length $L$ and parameterize it as

$$t \mapsto \boldsymbol{c}(t), \text{ where } t = \text{length from } \boldsymbol{c}(0) \text{ to } \boldsymbol{c}(t).$$

Without loss of generality, assume that the parameterization is positively oriented (i.e. counterclockwise). Assume that the curvature

$$\kappa(t) := |\ddot{\boldsymbol{c}}(t)|$$

is well-defined and bounded away from 0 and $\infty$ ( $\dot{}$ denotes $\mathrm{d}/\mathrm{d}t$), and that $\kappa$ has a bounded derivative. At $\boldsymbol{c}(t)$ define the tangent and normal vectors as follows,

$$\boldsymbol{e}_1(t) = \dot{\boldsymbol{c}}(t), \qquad \boldsymbol{e}_2(t) = (\kappa(t))^{-1}\ddot{\boldsymbol{c}}(t). \tag{2}$$

Note that for each $t$, $\boldsymbol{e}_1(t)$, $\boldsymbol{e}_2(t)$ have unit length, are orthogonal, and in fact form a positively oriented coordinate system.

Consider now points $\boldsymbol{X}_i$, $1 \le i \le n$, sampled randomly and independently from $K$. Assume that the $\boldsymbol{X}_i$ are i.i.d. with density $f(\boldsymbol{x})$, $\boldsymbol{x} \in K$, where we assume that $f(\boldsymbol{x})$ decays at a polynomial rate if $\boldsymbol{x}$ approaches $\partial K$. More precisely,

$$f(\boldsymbol{c}(t) + h\boldsymbol{e}_2(t))/h^\alpha \to g(t) \qquad (h \downarrow 0), \tag{3}$$

uniformly in $t$, for some $\alpha > -1$ and some continuous function $g$ bounded away from 0 and $\infty$. The case where the $\boldsymbol{X}_i$ are uniformly distributed is covered by $\alpha = 0$. In the following, let

$$\gamma = 1/(3 + 2\alpha).$$

Think about the case in which $K$ is the home range of an animal. $K$ tells us where the animal lives but $\alpha$ tells us something about how the animal behaves—and may be a parameter of zoological interest. Herbivores (deer, for example) will tend to stick with the safe, familiar middle ground and avoid lurking danger near the edge. On the other hand, carnivores (e.g. tigers) will guard their territory jealously, warning off other animals. They will tend to seek the edge, to claim territory (by scent-marking landmarks such as trees, by urinating or defecating). Edge-neutrality (i.e. the uniform case) will mark off edge-avoiding from edge-seeking (cf. risk-seeking, risk-averse or risk-neutral in the mathematics of finance).

Define the function

$$\lambda(t) = \frac{\sqrt{\kappa(t)}}{g(t)}, \qquad 0 \le t < L$$

and write

$$\lambda_0 := \max \lambda(t), \qquad \Lambda_0 := \operatorname{argmax} \lambda(t).$$

In order to obtain the asymptotic distribution of $\Delta_n$, we only need the following, rather weak, regularity assumption on $\lambda(t)$. For some bounded sequence of non-negative constants $\nu_n$ and positive $\mu$,

$$(\log n)^{\nu_n} \int_0^L \exp\left\{-\gamma_n \left(\frac{\lambda_0}{\lambda(t)} - 1\right) \log n\right\} g(t)\,\mathrm{d}t \;\to\; \mu \in (0, \infty) \qquad (n \to \infty) \tag{$*$}$$

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where

$$\gamma_n = \left(\gamma + (1 - \gamma - \nu_n)\frac{\log\log n}{\log n}\right).$$

This condition (for the exact form see the proof of Proposition 3) is rather general and unrestrictive, and is discussed further below. It picks out the behaviour of $\lambda(.)$ at points upon or near where it attains its maximum, the exponential having the effect of down-weighting contributions elsewhere. Write the following,

$$c_0 = \frac{\lambda_0}{2\sqrt{2}B(1/2, \alpha + 1)},$$

where, as usual, $B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$ for $a, b > 0$, and

$$c_n = c_0\gamma_n/\gamma.$$

Define

$$\tau_n(x) = n^{-2\gamma}((c_n \log n)^{2\gamma} + x(\log n)^{2\gamma-1}).$$

Our main result is stated as follows, which shows that the limit law is, to within type, the Gumbel (double exponential) law of extreme value theory.

**Theorem 1.** *Suppose* $(*)$ *holds; then*

$$\lim_{n\to\infty} P\{\Delta_n \le \tau_n(x)\} = \exp(-d_1 \mathrm{e}^{-d_2 x}), \quad x \in \mathbb{R}, \tag{4}$$

*where*

$$d_1 = \frac{c_0^{1-\gamma}\mu}{1+\alpha}, \quad d_2 = \frac{1}{2c_0^{2\gamma}}.$$

**Remark.** Condition $(*)$ subsumes two cases of particular interest. These are when $\Lambda_0$, the subset of $\partial K$ where $\lambda$ attains its maximum $\lambda_0$, is either
(a) of positive Lebesgue measure (as in Example 1), or
(b) a finite set, at whose points $\lambda$ behaves 'reasonably' (as in Example 2).

The constants $\nu_n$ reflect the rates of change of $\lambda$ at those points where $\lambda$ achieves $\lambda_0$. Consequently, the asymptotic distribution of $\Delta_n$ is completely determined by the behaviour of $\lambda(t)$, i.e. of the curvature $\kappa(t)$ and the function $g(t)$, which measures the size of the density near $\partial K$, on $\Lambda_0$. In the most important special case (the uniform law on $K$, $g \equiv 1$) the limit law of $\Delta_n$ is determined by the behaviour *in the directions of maximum curvature*.

First, we give two examples to illustrate the various elements of Theorem 1.

**Example 1.** If $\Lambda_0$ has positive Lebesgue measure then by dominated convergence one finds that $(*)$ holds with $\nu_n = 0$ and $\mu = \int_{\Lambda_0} g(t)\,\mathrm{d}t$. If, in particular, $f$ is the uniform density and $K$ is the unit disk, then $\gamma = 1/3$ and then the constants $d_1$ and $d_2$ in (4) reduce to $d_1 = (\pi/2)^{2/3}$ and $d_2 = (2/\pi)^{2/3}$.

Writing $x = (y + \log d_1)/d_2$, it is straightforward to show that Theorem 1 states that

$$\lim_{n\to\infty} P\{\Delta_n \le a_n y + b_n\} = \exp\{-\mathrm{e}^{-y}\}, \quad y \in \mathbb{R}, \tag{5}$$

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where

$$a_n = \frac{2c_0^{2\gamma}}{n^{2\gamma}(\log n)^{1-2\gamma}},$$
$$b_n = b^{(1)}\left(\frac{\log n}{n}\right)^{2\gamma} + b^{(2)}\frac{\log\log n}{n^{2\gamma}(\log n)^{1-2\gamma}} + b^{(3)}\frac{1}{n^{2\gamma}(\log n)^{1-2\gamma}},$$

with

$$b^{(1)} = c_0^{2\gamma}, \qquad b^{(2)} = 2c_0^{2\gamma}\frac{1-\gamma}{\lambda_0}, \qquad b^{(3)} = 2c_0^{2\gamma}\log\left(\frac{1}{\alpha+1}c_0^{1-\gamma}\int_{\Lambda_0} g(t)\,\mathrm{d}t\right).$$

Even the less detailed information obtained by retaining only the $b^{(1)}$ term in $b_n$ is still interesting. One obtains in the uniform case (for simplicity),

$$\Delta_n/b^{(1)}\left(\frac{\log n}{n}\right)^{2/3} \to 1 \qquad \text{in probability} \qquad (n\to\infty),$$

giving (in particular) in-probability bounds both above and below. (These are Ω-bounds rather than O-bounds, in the language of analytic number theory.) Dümbgen and Walther (1996) recently obtained an almost-sure O-bound for $\Delta_n$ (under different smoothness conditions, and in general dimension $d$). Carnal and Hüsler (1991) obtained an in-probability bound similar to the above in the disk case, and gave an application to computational complexity. Bárány (1989) showed that $E\Delta_n$ has exact order of magnitude $(\log n/n)^{2/3}$.

**Example 2.** Assume that $\Lambda_0$ is a finite set, say $\{t_j, 1\le j\le k\}$, and suppose that, for every $j$, $\lambda$ is unimodal in a neighbourhood of $t_j$ and satisfies the following Lipschitz-type condition,

$$\lim_{h\downarrow 0}\frac{\lambda_0-\lambda(t_j-h)}{h^{1/\nu_{j1}}} = \phi_1(t_j)\in(0,\infty)$$

and

$$\lim_{h\downarrow 0}\frac{\lambda_0-\lambda(t_j+h)}{h^{1/\nu_{j2}}} = \phi_2(t_j)\in(0,\infty),$$

for some positive $\nu_{j1}$ and $\nu_{j2}$. Then $(*)$ holds with the following,

$$\nu_n = \nu = \min\{\nu_{jl} : 1\le j\le k, l=1,2\}$$

and

$$\mu = \left(\frac{\lambda_0}{\gamma}\right)^{\nu}\Gamma(\nu+1)\sum_{\substack{j,l\\ \nu_{jl}=\nu}}\frac{g(t_j)}{(\phi_l(t_j))^{\nu}}.$$

If $f$ is the uniform density and $\partial K$ is an ellipse with half-axes $r<R$, then $\nu=1/2$ and the constants $d_1$ and $d_2$ in Theorem 1 become

$$d_1 = \frac{2r}{\sqrt{\pi(R^2-r^2)}}\left(\frac{\pi}{2}\right)^{2/3} \quad\text{and}\quad d_2 = \frac{1}{R}\left(\frac{2}{\pi}\right)^{2/3}.$$

A similar transformation (as in Example 1) shows that (5) holds with $b^{(2)}$ and $b^{(3)}$ replaced by

$$b^{(2)'} = 2c_0^{2\gamma}\frac{1-\gamma-\nu}{\lambda_0}$$

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and

$$b^{(3)'} = 2c_0^{2\gamma} \log\left(\left(\frac{\lambda_0}{\gamma}\right)^{\nu} \frac{\Gamma(\nu+1)}{\alpha+1} c_0^{1-\gamma} \sum_{\substack{j,l \\ \nu_{jl}=\nu}} \frac{g(t_j)}{(\phi_l(t_j))^{\nu}}\right),$$

respectively.

The sensitivity of the Theorem to the shape of $K$ is well-illustrated by the contrast between the cases where $K$ is a disk and the interior of an ellipse of eccentricity $e$ slightly less than 1. Geometrically, these two cases may be made arbitrarily close by taking $e$ close enough to 1. By contrast, $\Lambda_0$ has cardinality 2 if $e < 1$ and full measure if $e = 1$. This discontinuity in the topological nature of $\Lambda_0$ persists in the limit theorem in that the disk case cannot be obtained from the elliptical case by continuity as $e \uparrow 1$.

**Moving boundaries.** If preferred, one may avoid condition $(*)$ and the consequent discontinuity between geometrically similar situations, by working instead with 'moving boundaries'. Theorem 2 below neither implies nor is implied by Theorem 1. However, they do have very similar proofs, which are lengthy. We give the proof of Theorem 1 in full, and a sketch proof of Theorem 2, in Section 4. For every $t \in [0, L)$, let $M_n(t)$ be the vertical distance between the tangent at $\boldsymbol{c}(t)$ and the parallel supporting line of $K_n$. Also let

$$c_0(t) = \frac{\lambda(t)}{2\sqrt{2}B(1/2, 1+\alpha)} \quad \text{and} \quad c_n(t) = \frac{c_0(t)}{\gamma}\left(\gamma + (1-\gamma)\frac{\log\log n}{\log n}\right).$$

**Theorem 2.** *If $0 < \lambda(t) < \infty$, $t \in [0, L)$ and $\kappa(t)$, $g(t)$ both have a bounded derivative then,*

$$\lim_{n\to\infty} P\{M_n(t) \le a_n(t)x + b_n(t) \text{ for all } t \in [0, L)\} = \exp\{-\mathrm{e}^{-x}\},$$

*where*

$$a_n(t) = \frac{2c_0(t)^{2\gamma}}{n^{2\gamma}(\log n)^{1-2\gamma}},$$

*and*

$$b_n(t) = \left(\frac{c_n(t)\log n}{n}\right)^{2\gamma} + \frac{2c_0(t)^{2\gamma}}{n^{2\gamma}(\log n)^{1-2\gamma}} \log\left(\frac{L}{1+\alpha} c_0(t)^{1-\gamma} g(t)\right).$$

## 3. Asymptotic distribution of $\Delta_n$ – the polygonal case

Suppose now that $K$ is a polygon; we derive the asymptotic distribution of $\Delta_n$. We shall focus on the setting where the points $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ are uniformly distributed on $K$ as opposed to the more general distributional setting for the smooth case. Since the derivations here are considerably simpler than those for the smooth case, the reader will be able to adapt them to more general situations as required.

The following theorem may be summarised by saying that 'everything happens at the corners' and the relevant events corresponding to distinct corners are asymptotically independent. Note also the contrast with the $\Lambda_0$ finite case of Theorem 1. In the latter, the entire boundary contributes and the limit law is Gumbel. In the former, *only* the vertices contribute and the limit law is of product form, with one factor for each product.

**Theorem 3.** *Suppose $K$ is a convex polygon with angles $\theta_1, \dots, \theta_r$ and area $|K|$. Assume that the sample points are uniformly distributed on $K$. Then*

$$\lim_{n\to\infty} P\{\sqrt{n}\Delta_n \le x\} = \prod_{i=1}^{r}(1 - p_i(x)),$$

*where*

$$p_i(x) = \begin{cases} \displaystyle\int_0^{\theta_i} h_i(x,\theta)\,\mathrm{d}\theta + \exp\biggl\{-\frac{x^2}{2|K|}\tan\theta_i\biggr\}, & 0 < \theta_i < \tfrac{1}{2}\pi, \\ \displaystyle\int_{\theta_i-\pi/2}^{\pi/2} h_i(x,\theta)\,\mathrm{d}\theta, & \tfrac{1}{2}\pi \le \theta_i < \pi, \end{cases}$$

*with*

$$h_i(x,\theta) = \exp\biggl\{-\frac{x^2}{2|K|}\Bigl(\tan\theta + \tan(\theta_i - \theta)\Bigr)\biggr\}\frac{x^2}{2|K|}\tan^2\theta.$$

*Proof.* Let $\boldsymbol{c}_1, \dots, \boldsymbol{c}_r$ be the vertices of $K$. For $1 \le i \le r$, let $\xi_{ni}$ be the point process consisting of points $(\sqrt{n}(\boldsymbol{X}_j - \boldsymbol{c}_i), 1 \le j \le n)$, where $\xi_{ni}$ is regarded as a random element in $\mathcal{N}_i$, i.e. the space of locally finite counting measures on a cone $C_i$ with angle $\theta_i$ endowed with the vague topology and Borel $\sigma$-field.

It is then straightforward to show that

$$(\xi_{n1}, \dots, \xi_{nr}) \xrightarrow{d} (\xi_1, \dots, \xi_r),$$

in the product space, where $\xi_1, \dots, \xi_r$ are independent Poisson processes on $C_1, \dots, C_r$ respectively, and intensity measure $|K|^{-1}\times$Lebesgue measure. It is clear from (1) that

$$\sqrt{n}\Delta_n = \bigvee_{1\le i\le r} f_i(\xi_{ni}),$$

where for any measure $\eta$ in $\mathcal{N}_i$, $f_i$ maps $\eta$ to the smallest distance between the origin and the convex hull of the points of $\eta$. By the continuous mapping theorem,

$$\sqrt{n}\Delta_n \xrightarrow{d} \bigvee_{1\le i\le r} f_i(\xi_i).$$

Thus, it suffices to show

$$P\{f_i(\xi_i) > x\} = p_i(x). \tag{6}$$

Fix $i$ and assume without loss of generality that the two edges of $C_i$ have angles 0 and $\theta_i$, respectively. For $\theta \in (0, \theta_i)$, let

$$A_\theta = \{(r\cos\rho, r\sin\rho) : 0 < \rho < \theta, x < r < x/\cos(\theta-\rho)\},$$
$$B_\theta = \{(r\cos\rho, r\sin\rho) : 0 < \rho < \theta, 0 < r < x,\ \text{or}\ \theta \le \rho < \theta_i, 0 < r < x/\cos(\rho-\theta)\}.$$

Define the following,

$$T_i = \inf\{\theta > 0 : \xi_i(A_\theta) \ne 0\},$$

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and write

$$P\{f_i(\xi_i) > x\} = P\{f_i(\xi_i) > x, T_i \le \theta_i\} + P\{f_i(\xi_i) > x, T_i > \theta_i\}.$$

Observe that

$$\begin{aligned} P\{f_i(\xi_i) > x, T_i \le \theta_i\} &= \int_0^{\theta_i} P\{f_i(\xi_i) > x | T = \theta\} \,\mathrm{d}P\{T \le \theta\} \\ &= \int_0^{\theta_i} P\{\xi_i(B_\theta) = 0\} \,\mathrm{d}P\{T \le \theta\}. \end{aligned}$$

Clearly,

$$P\{\xi_i(B_\theta) = 0\} = \begin{cases} \exp\left(-\dfrac{|B_\theta|}{|K|}\right) = \exp\left\{-\dfrac{x^2}{2|K|}(\theta + \tan(\theta_i - \theta))\right\}, & \text{if } 0 < \theta_i < \tfrac{1}{2}\pi \\ & \text{or } 0 \le \theta_i - \tfrac{1}{2}\pi \\ & \quad < \theta < \tfrac{1}{2}\pi, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$P\{T_i \le \theta\} = 1 - \exp(-|A_\theta|/|K|),$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\theta} P\{T_i \le \theta\} = \frac{x^2}{2|K|} \tan^2\theta \exp\left\{-\frac{x^2}{2|K|}(\tan\theta - \theta)\right\}.$$

Thus, the following is true,

$$P\{f_i(\xi_i) > x, T_i \le \theta_i\} = \begin{cases} \displaystyle\int_0^{\theta_i} h_i(x, \theta) \,\mathrm{d}\theta, & 0 < \theta_i < \tfrac{1}{2}\pi, \\ \displaystyle\int_{\theta_i - \pi/2}^{\pi/2} h_i(x, \theta) \,\mathrm{d}\theta, & \tfrac{1}{2}\pi \le \theta_i < \pi. \end{cases}$$

Similarly,

$$P\{f_i(\xi_i) > x, T_i > \theta_i\} = \begin{cases} \exp\left\{-\dfrac{x^2}{2|K|}\tan\theta_i\right\}, & 0 < \theta_i < \tfrac{1}{2}\pi, \\ 0, & \text{otherwise.} \end{cases}$$

This proves (6), and so concludes the proof.

As may be seen from both the statement and the proof of the result above, the sides of the polygonal boundary play no role, only the vertices. By contrast, because of the positive curvature in condition $(*)$, the entire boundary contributes. One could extend the scope of Theorem 1 to cover hybrid cases (such as the boundary of a running track) where $\partial K$ contains flat stretches, but these play no role. Since no new ideas are involved, the details are omitted.

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## 4. Proof of Theorem 1

We first develop some notation. Let $(X_i(t), Y_i(t))$, $1 \le i \le n$, $0 \le t < L$, be the coordinates of the point $\boldsymbol{X}_i$ with respect to $(\boldsymbol{e}_1(t), \boldsymbol{e}_2(t))$, i.e.

$$X_i(t) = \langle \boldsymbol{X}_i - \boldsymbol{c}(t), \boldsymbol{e}_1(t)\rangle, \quad Y_i(t) = \langle \boldsymbol{X}_i - \boldsymbol{c}(t), \boldsymbol{e}_2(t)\rangle,$$

and write

$$Z_i(t) = X_i(t)\sqrt{\frac{\kappa(t)}{2Y_i(t)}}.$$

Elementary arguments from differential geometry show that

$$\sup_{1\le i\le n}\ \sup_{t\in[0,L)} |Z_i(t)| \le \sqrt{\frac{\sup\kappa(s)}{\inf\kappa(s)}} < \infty \tag{7}$$

by assumption. Denote by $Y_{(i)}(t)$ the $i$th smallest of $Y_j(t)$, $1 \le j \le n$. Ignoring ties, let $Y_{(i)}(t) = Y_j(t)$ for some $j$, and define

$$Y_{(i)}(t,\theta) = Y_j(t+\theta), \quad Z_{(i)}(t) = Z_j(t).$$

Note that we suppress the sample size $n$ in $Y_{(i)}$ and $Z_{(i)}$ to simplify notation.

We now briefly describe the steps in the proof. Let $\delta(\ell)$ be as defined in Section 1. Note that for every supporting line $\ell$ of $\partial K_n$, there exists a $t \in [0, L)$ such that

$$\delta(\ell) = Y_{(1)}(t),$$

and conversely. Hence it follows from (1) that

$$\Delta_n = \sup_{t\in[0,L)} Y_{(1)}(t). \tag{8}$$

Thus, the essential idea in the proof of Theorem 1 is the following discrete approximation,

$$P\Big\{\sup_{t\in[0,L)} Y_{(1)}(t) \le \tau_n(x)\Big\} \approx P\Big\{\bigvee_{1\le j\le N_n} Y_{(1)}(j\theta_n) \le \tau_n(x)\Big\},$$

where $N_n = [L/\theta_n]$ for some suitable $\theta_n \to 0$ (Proposition 4). We then approximate

$$P\Big\{\bigvee_{1\le j\le N_n} Y_{(1)}(j\theta_n) \le \tau_n(x)\Big\},$$

by $\exp(-d_1 \mathrm{e}^{-d_2 x})$ via quantities of the form

$$P\{Y_{(1)}(t) > \tau_n(x), \bigvee_{1\le j\le r_n} Y_{(1)}(t + j\theta_n) \le \tau_n(x)\},$$

for some $r_n = o(N_n)$, using a classical blocking argument (Propositions 2 and 3). The first order of business is to establish the asymptotic behaviour of these quantities. In the following, write

$$\eta_n(t) = (\log n)^{\nu_n} \exp\Big\{-\gamma_n\Big(\frac{\lambda_0}{\lambda(t)} - 1\Big)\log n\Big\},$$

and

$$\Lambda(x,t) = \exp\Big\{-\frac{\lambda_0 x}{2c_0^{2\gamma}\lambda(t)}\Big\}\frac{g(t)}{\alpha+1}c_0^{1-\gamma}.$$

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**Proposition 1.** *Let positive constants* $\epsilon_n$ *and positive integers* $r_n$ *be such that*

$$\epsilon_n \to 0,\ (\epsilon_n)^{1/\gamma} \log n \to \infty \ \text{and} \ r_n\epsilon_n = o(n^{\gamma 2^{-1/\gamma}} (\log n)^{1+(1-\gamma-\nu_n)2^{-1/\gamma}}) \tag{9}$$

*and define*

$$\theta_n = \frac{\epsilon_n}{n^\gamma (\log n)^{1-\gamma}}. \tag{10}$$

*Then for all* $x \in \mathbb{R}$,

$$P\left\{\bigvee_{j=1}^{r_n} Y_{(1)}(t + j\theta_n) \le \tau_n(x),\, Y_{(1)}(t) > \tau_n(x)\right\} \sim \theta_n \eta_n(t) \Lambda(x, t),$$

*uniformly in* $t$.

*Proof.* By (2),

$$\dot{\boldsymbol{e}}_1(t) = \kappa(t)\boldsymbol{e}_2(t), \quad \dot{\boldsymbol{e}}_2(t) = -\kappa(t)\boldsymbol{e}_1(t), \quad \ddot{\boldsymbol{e}}_2(t) = -\dot{\kappa}(t)\boldsymbol{e}_1(t) - \kappa^2(t)\boldsymbol{e}_2(t). \tag{11}$$

In the following, $\boldsymbol{o}(h)$ denotes a vector whose length is of the order $o(h)$. By (11) and Taylor expansion,

$$\begin{aligned}
Y_{(i)}(t, \theta) &= Y_j(t + \theta) \\
&= \langle \boldsymbol{X}_j - \boldsymbol{c}(t + \theta), \boldsymbol{e}_2(t + \theta)\rangle \\
&= \langle \boldsymbol{X}_j - \boldsymbol{c}(t) - \theta \boldsymbol{e}_1(t) - \tfrac{1}{2}\theta^2\kappa(t)\boldsymbol{e}_2(t) + \boldsymbol{o}(\theta^2), \\
&\qquad \boldsymbol{e}_2(t) - \theta\kappa(t)\boldsymbol{e}_1(t) - \tfrac{1}{2}\theta^2(\dot{\kappa}(t)\boldsymbol{e}_1(t) + \kappa^2(t)\boldsymbol{e}_2(t)) + \boldsymbol{o}(\theta^2)\rangle \\
&= Y_j(t)(1 - \kappa^2(t)\theta^2/2) - X_j(t)(\kappa(t)\theta + \dot{\kappa}(t)\theta^2/2) + \kappa(t)\theta^2/2 + o(\theta^2) \\
&= Y_{(1)}(t) - \sqrt{2\kappa(t)Y_{(1)}(t)}Z_{(1)}(t)\theta + R(t),
\end{aligned}$$

with

$$R(t) = \tfrac{1}{2}\theta^2(\kappa(t) - \kappa^2(t)Y_{(1)}(t) - \dot{\kappa}(t)\sqrt{2Y_{(1)}(t)/\kappa(t)}Z_{(1)}(t)) + o(\theta^2).$$

In particular,

$$Y_{(1)}(t, j\theta_n) = Y_{(1)}(t) - \sqrt{2\kappa(t)Y_{(1)}(t)}Z_{(1)}(t)j\theta_n + R_{nj}(t), \tag{12}$$

where

$$R_{nj}(t) = \tfrac{1}{2}(j\theta_n)^2(\kappa(t) - \kappa^2(t)Y_{(1)}(t) - \dot{\kappa}(t)\sqrt{2Y_{(1)}(t)/\kappa(t)}Z_{(1)}(t)) + o((j\theta_n)^2).$$

Observe that

$$\frac{\theta_n}{\sqrt{\tau_n(x)}} = O\left(\frac{\epsilon_n}{\log n}\right) \longrightarrow 0, \tag{13}$$

which implies

$$R_{n1}(t) = o(\tau_n(x)). \tag{14}$$

By (12) and (14), $Y_{(1)}(t,\theta_n) \le \tau_n(x)$ if and only if

$$\sqrt{Y_{(1)}(t)} \le \tfrac{1}{2}\sqrt{2\kappa(t)}Z_{(1)}(t)\theta_n + \tfrac{1}{2}\sqrt{2\kappa(t)Z^2_{(1)}(t)\theta_n^2 + 4(\tau_n(x) - R_{n1}(t))},$$

which is equivalent to,

$$\begin{aligned} Y_{(1)}(t) \le T_n(x) &:= \tau_n(x) + \kappa(t)Z^2_{(1)}(t)\theta_n^2 - R_{n1}(t) \\ &\quad + \sqrt{2\kappa(t)}Z_{(1)}(t)\theta_n\sqrt{\kappa(t)Z^2_{(1)}(t)\theta_n^2/2 + \tau_n(x) - R_{n1}(t)} \\ &= \tau_n(x) + \sqrt{2\kappa(t)}Z_{(1)}(t)\theta_n\sqrt{\tau_n(x)}(1+o(1)) \\ &= \tau_n(x) + \sqrt{2\kappa(t)}Z_{(1)}(t)\frac{\epsilon_n c_n^\gamma}{n^{2\gamma}(\log n)^{1-2\gamma}}(1+o(1)). \end{aligned} \tag{15}$$

Note that the $o(1)$ term converges to 0 uniformly over the entire probability space and hence $T_n(x) - \tau_n(x)$ has the same sign as $Z_{(1)}(t)$ for all large $n$. We will approximate

$$P\left\{\bigvee_{j=1}^{r_n} Y_{(1)}(t + j\theta_n) \le \tau_n(x),\, Y_{(1)}(t) > \tau_n(x)\right\}$$

by $P\{Z_{(1)}(t) > 0, \tau_n(x) < Y_{(1)}(t) \le T_n(x)\}$. We start with some preliminaries.

For given $t \in [0, L)$ and small $v > 0$, let $s_1(t, v)$ be the first component of the vector $(s, u)$ which solves

$$\boldsymbol{c}(t+s) = \boldsymbol{c}(t) + u\boldsymbol{e}_1(t) + v\boldsymbol{e}_2(t),\ \ s > 0,$$

and let $s_2(t, v)$ be the first component of the vector $(s, u)$ which solves

$$\boldsymbol{c}(t) = \boldsymbol{c}(t+s) + u\boldsymbol{e}_1(t) + v\boldsymbol{e}_2(t),\ \ s > 0,$$

and

$$s(t, v) = \min(s_1(t, v), s_2(t, v)).$$

Define the following,

$$t_0 = t, \quad t_1 = t + \epsilon_n s(t, \tau_n(x)), \quad t_2 = t + s(t_1, \tau_n(x + h\epsilon_n)), \tag{16}$$

where

$$h = c_0^\gamma \sqrt{2\kappa(t)(\sup \kappa(s))/(\inf \kappa(s))},$$

and $t_i$, $i \ge 3$, are constructed recursively by

$$t_3 = t_1 + s(t_1, \tau_n(x)), \qquad t_i = t_{i-1} + s(t_{i-1}, \tau_n(x)), \quad 4 \le i \le Y_{(1)}, \tag{17}$$

where

$$m_n = \vee\{i \ge 0 : t_i \le t + r_n\theta_n\}. \tag{18}$$

By Lemma 1 below, the assumption that $\kappa$ has a bounded derivative, (9) and (10), we have

$$s(t_i, \tau_n(x)) = \sqrt{\frac{2\tau_n(x)}{\kappa(t)}}(1 + o(\epsilon_n)) \quad \text{uniformly for } t \text{ in } [0, L) \text{ and } i \le m_n, \tag{19}$$

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and

$$s(t_1, \tau_n(\tau_n(x + h\epsilon_n))) = \sqrt{\frac{2\tau_n(x)}{\kappa(t)}}(1 + o(\epsilon_n)).$$

Hence, by (16) and (17), it is straightforward to verify that

$$t_3 - t_2 = \epsilon_n s(t, \tau_n(x)) + s(t_1, \tau_n(x)) - s(t_1, \tau_n(x + h\epsilon_n)) = \epsilon_n \sqrt{\frac{2\tau_n(x)}{\kappa(t)}}(1 + o(1)). \quad (20)$$

Thus, for large $n$, the $t_i$ are strictly increasing. Now fix $\delta \in (0, 1)$. We define

$$\begin{aligned}
&\Bigg| P\Big\{ Z_{(1)}(t) > \delta, \tau_n(x) < Y_{(1)}(t) \le T_n(x) \Big\} \\
&\qquad - P\Big\{ \bigvee_{j=1}^{r_n} Y_{(1)}(t + j\theta_n) \le \tau_n(x), Z_{(1)}(t) > \delta, Y_{(1)}(t) > \tau_n(x) \Big\} \Bigg| \\
&\quad \le \Bigg| P\Big\{ Z_{(1)}(t) > \delta, \tau_n(x) < Y_{(1)}(t) \le T_n(x) \Big\} \\
&\qquad - P\Big\{ \bigvee_{j=1}^{r_n} Y_{(1)}(t + j\theta_n) \le \tau_n(x), Z_{(1)}(t) > \delta, \tau_n(x) < Y_{(1)}(t) \le T_n(x) \Big\} \Bigg| \\
&\qquad + \Bigg| P\Big\{ \bigvee_{j=1}^{r_n} Y_{(1)}(t + j\theta_n) \le \tau_n(x), Z_{(1)}(t) > \delta, \tau_n(x) < Y_{(1)}(t) \le T_n(x) \Big\} \\
&\qquad - P\Big\{ \bigvee_{j=1}^{r_n} Y_{(1)}(t + j\theta_n) \le \tau_n(x), Z_{(1)}(t) > \delta, Y_{(1)}(t) > \tau_n(x) \Big\} \Bigg| \\
&\quad \le S_{n,1} + S_{n,2} + S_{n,3},
\end{aligned} \quad (21)$$

where

$$\begin{aligned}
S_{n,1} &= P\Big\{ \bigvee_{j \in (t_0/\theta_n, t_1/\theta_n]} Y_{(1)}(j\theta_n) > \tau_n(x), Z_{(1)}(t) > \delta, \tau_n(x) < Y_{(1)}(t) \le T_n(x) \Big\}, \\
S_{n,2} &= P\Big\{ \bigvee_{i=1}^{m_n} \bigvee_{j \in (t_i/\theta_n, t_{i+1}/\theta_n]} Y_{(1)}(j\theta_n) > \tau_n(x), Z_{(1)}(t) > \delta, \tau_n(x) < Y_{(1)}(t) \le T_n(x) \Big\}, \\
S_{n,3} &= P\{ Y_{(1)}(t, \theta_n) > \tau_n(x) \ge Y_{(1)}(t + \theta_n), Z_{(1)}(t) > \delta, Y_{(1)}(t) > \tau_n(x) \}.
\end{aligned}$$

The result follows from Lemmas 3, 4, 5, 6, and 7 below. Thus, the proof is complete upon showing those lemmas.

The notation of the preceding proof will be assumed in the following.

**Lemma 1.** *For a given $t \in [0, L)$ and small $v > 0$, let $(s, u)$ be a solution of*

$$\boldsymbol{c}(t + s) = \boldsymbol{c}(t) + u\boldsymbol{e}_1(t) + v\boldsymbol{e}_2(t). \quad (22)$$

*Then uniformly in $t$,*

$$s = \sqrt{\frac{2v}{\kappa(t)}} + O(v^{3/2}) \quad \text{as } v \downarrow 0.$$

*Proof.* By Taylor expansion using (11), (22) is equivalent to

$$\boldsymbol{c}(t) + s\boldsymbol{e}_1(t) + \frac{s^2}{2}\kappa(t)\boldsymbol{e}_2(t) + \frac{s^3}{3!}(\dot{\kappa}(t)\boldsymbol{e}_2(t) - \kappa^2(t)\boldsymbol{e}_1(t)) + \boldsymbol{o}(s^3),$$
$$= \boldsymbol{c}(t) + u\boldsymbol{e}_1(t) + v\boldsymbol{e}_2(t). \quad (23)$$

Therefore,

$$u \sim s \sim \sqrt{\frac{2v}{\kappa(t)}}.$$

Replacing the $\boldsymbol{O}(s^3)$ term in (23) by $\boldsymbol{O}(v^{3/2})$ gives the desired result.

Next, we show how the assumption (2) is used.

**Lemma 2.** *Let $(Y(t), Z(t))$ have the same joint distribution as $(Y_1(t), Z_1(t))$ for some $t$, then let $h_{Y(t),Z(t)}$ be the joint density of $(Y(t), Z(t))$ and $h_{Y(t)}$ the marginal density of $Y(t)$. Then, uniformly for $t \in [0, L)$ and $|z| \neq 1$,*

$$\lim_{y\to 0} y^{-(\alpha+1/2)} h_{Y(t)}(y) = B(1/2, \alpha+1)g(t)\sqrt{\frac{2}{\kappa(t)}}, \quad (24)$$

*and*

$$\lim_{y\to 0} y^{-(\alpha+1/2)} h_{Y(t),Z(t)}(y, z) = g(t)\sqrt{\frac{2}{\kappa(t)}}(1-z^2)^{\alpha} I_{[0,1)}(|z|). \quad (25)$$

*Proof.* Let $h_{X(t),Y(t)}$ be the joint density of $(X_1(t), Y_1(t))$; i.e.

$$h_{X(t),Y(t)}(x, y) = f(\boldsymbol{c}(t) + x\boldsymbol{e}_1(t) + y\boldsymbol{e}_2(t)),$$

where $f$ is the density of $\boldsymbol{X}$. By changing variables,

$$h_{Y(t),Z(t)}(y, z) = h_{X(t),Y(t)}(z\sqrt{2y/\kappa(t)}, y)\sqrt{\frac{2y}{\kappa(t)}}$$
$$= f(\boldsymbol{c}(t) + z\sqrt{2y/\kappa(t)}\boldsymbol{e}_1(t) + y\boldsymbol{e}_2(t))\sqrt{\frac{2y}{\kappa(t)}}. \quad (26)$$

First, consider the case $|z| < 1$. For small $y$ there exists (cf. Bräker and Hsing 1995) unique $u \in [0, L)$ and $v > 0$, such that

$$\boldsymbol{c}(t) + z\sqrt{2y/\kappa(t)}\boldsymbol{e}_1(t) + y\boldsymbol{e}_2(t) = \boldsymbol{c}(t+u) + v\boldsymbol{e}_2(t+u). \quad (27)$$

By arguments similar to those in Lemma 1 and the fact that $|z| < 1$,

$$v = y(1 + z^2(1 + o(1))),$$

where the $o(1)$ term tends to 0 at the rate $y^{1/2}$, uniformly for $|z| < 1$. Thus, it follows from (3), (26) and (27) that, uniformly for $|z| < 1$,

$$h_{Y(t),Z(t)}(y, z) = g(t)\sqrt{\frac{2}{\kappa(t)}} y^{\alpha+1/2}(1 - z^2(1 + o(1)))^{\alpha}.$$

Next, if $|z| > 1$, then it is clear that for small $y$ no $u \in [0, L)$, $v > 0$ will satisfy (27) and therefore the left-hand side of (27) represents a point outside of $K$ hence $h_{Y(t),Z(t)}(y, z) = 0$. This proves (25), and (24) follows from integrating out $z$.

**Lemma 3.** *As* $n \to \infty$,

$$P\{\tau_n(x) < Y_{(1)}(t) \le T_n(x), Z_{(1)}(t) > \delta\} \sim \theta_n \eta_n(t)\Lambda(x, t)(1 - \delta^2)^{\alpha+1},$$

*uniformly for* $t \in [0, L)$ *and* $\delta \in [0, 1)$.

*Proof.* For convenience, we use the notation of Lemma 2. Given

$$T_n(x, z) = \tau_n(x) + \sqrt{2\kappa(t)}z\epsilon_n c_n^{\gamma} n^{-2\gamma}(\log n)^{2\gamma-1}, \tag{28}$$

we will show that

$$P\{\tau_n(x) < Y_{(1)}(t) \le T_n(x, Z_{(1)}), Z_{(1)}(t) > \delta\} \sim \theta_n \eta_n(t)\Lambda(x, t)(1 - \delta^2)^{\alpha+1}, \tag{29}$$

from which the result follows. Since the $\boldsymbol{X}_i$ are i.i.d., each $Z_j$ is conditionally independent of $Y_i$, $i \ne j$, given $Y_j$. Therefore,

$$\begin{aligned} &P\{\tau_n(x) < Y_{(1)}(t) \le T_n(x, z), Z_{(1)}(t) > \delta\} \\ &\qquad = \int_{z>\delta} \int_{y=\tau_n(x)}^{T_n(x,z)} nP^{n-1}\{Y(t) > y\} h_{Y(t),Z(t)}(y, z)\,\mathrm{d}y\,\mathrm{d}z. \end{aligned} \tag{30}$$

By (7), for any $t$ and $y$, the point $\boldsymbol{c}(t) + z(2y/\kappa(t))^{1/2}\boldsymbol{e}_1(t) + y\boldsymbol{e}_2(t)$ is outside of $K$ if $|z| > (\sup \kappa(t)/ \inf \kappa(t))^{1/2}$, and therefore $h_{Y(t),Z(t)}(y, z) = 0$ for such $z$. On the other hand, $T_n(x, z)$ tends to 0 for bounded $z$. Hence, we can apply Lemma 2 to estimate the expression $h_{Y(t),Z(t)}(y, z)$ in the integral and conclude that, uniformly for $z \ne 1$, in the effective range of integration in (30) and $y \in (\tau_n(x), T_n(x, z))$,

$$h_{Y(t),Z(t)}(\hat{y}, z) \sim g(t)\sqrt{\frac{2}{\kappa(t)}}(\tau_n(x))^{\alpha+1/2}(1 - z^2)^{\alpha} I_{(\delta,1)}(z). \tag{31}$$

By Taylor expansion, we have (uniformly)

$$\hat{y}^{1/(2\gamma)} \sim \frac{1}{n}\left(c_n \log n + c_n^{1-2\gamma}\frac{x}{2\gamma}\right),$$

and hence it follows from Lemma 3 that

$$\begin{aligned} P^{n-1}\{Y(t) > \hat{y}\} &= (1 - \beta_2(t)y^{1/2\gamma})^{n-1} \\ &\sim \exp\left(-\beta_2(t)c_n \log n - \beta_2(t)\frac{c_n^{1-2\gamma}}{2\gamma}x\right), \end{aligned} \tag{32}$$

where

$$\beta_2(t) = \frac{\gamma\lambda_0}{c_0\lambda(t)}.$$

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Hence, by (30), (28), (31) and (32),

$$\begin{aligned}
&\int_{z>\delta}\int_{\tau_n(x)}^{T_n(x,z)} nP^{n-1}\{Y>y\}h_{Y(t),Z(t)}(y,z)\,\mathrm{d}y\,\mathrm{d}z \\
&\quad\sim \int_{z>\delta} nP^{n-1}\{Y>\tau_n(x)\}(\tau_n(x)-T_n(x,z))h_{Y(t),Z(t)}(\tau_n(x),z)\,\mathrm{d}z \\
&\quad\sim n\exp\biggl(-\beta_2(t)c_n\log n-\beta_2(t)\frac{c_n^{1-2\gamma}}{2\gamma}x\biggr)\sqrt{2\kappa(t)}c_0^{\gamma}\epsilon_n n^{-2\gamma}(\log n)^{2\gamma-1} \\
&\qquad\times g(t)\sqrt{\frac{2}{\kappa(t)}}(\tau_n(x))^{\alpha+1/2}\int_{z=\delta}^{1} z(1-z^2)^{\alpha}\,\mathrm{d}z \\
&\quad= 2c_0^{1-\gamma}g(t)\int_{\delta}^{1} z(1-z^2)^{\alpha}\,\mathrm{d}z \\
&\qquad\times\epsilon_n\exp\biggl(-\beta_2(t)c_n\log n-\beta_2(t)\frac{c_n^{1-2\gamma}}{2\gamma}x\biggr)\biggl(\frac{\log n}{n}\biggr)^{(2\alpha+1)\gamma+(2\gamma-1)} \\
&\quad= c_0^{1-\gamma}g(t)\frac{(1-\delta^2)^{\alpha+1}}{\alpha+1}\epsilon_n\exp\biggl(-\beta_2(t)c_n\log n-\beta_2(t)\frac{c_n^{1-2\gamma}}{2\gamma}x\biggr).
\end{aligned}$$

This shows (29) and concludes the proof.

**Lemma 4.** $S_{n,1}=0$ *for all large* $n$.

*Proof.* Clearly,

$$S_{n,1}\le P\biggl\{\bigvee_{j\in(0,t_1-t/\theta_n]} Y_{(1)}(t,j\theta_n)>\tau_n(x),\,Z_{(1)}(t)>\delta,\,\tau_n(x)<Y_{(1)}(t)\le T_n(x)\biggr\}. \tag{33}$$

On $\{Y_{(1)}(t)>\tau_n(x),\,0<j\theta_n\le t_1-t\}$, we have

$$\frac{R_{nj}}{\sqrt{Y_{(1)}(t)}j\theta_n}=O\biggl(\frac{j\theta_n}{\sqrt{\tau_n(x)}}\biggr)=O\biggl(\frac{\epsilon_n s_0}{\sqrt{\tau_n(x)}}\biggr)=o(1),$$

by (19). Hence, with $\tau_n(x)<Y_{(1)}(t)\le T_n(x)$, it follows from (12) that for all large enough $n$, $Y_{(1)}(t,j\theta_n)$ is decreasing in $j$ for $j\in(0,(t_1-t)/\theta_n]$. We conclude that the right-hand side of (33) is 0 for large $n$.

For $u<v$, the region between $\partial K$ and the linear segment through $\boldsymbol{c}(u)$ and $\boldsymbol{c}(v)$ is denoted by $R(u,v)$.

**Lemma 5.** $S_{n,2}=o(\theta_n\eta_n(t))$, *uniformly in* $t$.

*Proof.* First note that if some of the points $\boldsymbol{X}_i$, $1\le i\le n$, fall in $R(t_2,t_3)$, then

$$\sup_{t_1\le u\le t_3} Y_{(1)}(u)\le\tau_n(x),$$

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and similarly, if each $R(t_i, t_{i+1})$, $3 \le i \le Y_{(1)}$, contains some of the points $\boldsymbol{X}_i$, $1 \le i \le n$, then

$$\sup_{t_3 < u \le t_{m_n+1}} Y_{(1)}(u) \le \tau_n(x).$$

As a result,

$$P\left\{\bigvee_{i=1}^{m_n} \vee_{j\in(t_i/\theta_n, t_{i+1}/\theta_n]} Y_{(1)}(j\theta_n) > \tau_n(x), \tau_n(x) < Y_{(1)}(t) \le T_n(x), Z_{(1)}(t) > \delta\right\} \tag{34}$$
$$\le \sum_{i=2}^{m_n} P\left\{R(t_i, t_{i+1}) \text{ contains none of the } \boldsymbol{X}_j, \tau_n(x) < Y_{(1)}(t) \le T_n(x), Z_{(1)}(t) > \delta\right\}.$$

It can be shown that by (7) and (15), that $t_2 > s_1(t, T_n(x))$. Hence, by the arguments in Lemma 2, we have for $i \ge 2$,

$$P\{R(t_i, t_{i+1}) \text{ contains none of the } \boldsymbol{X}_j, \tau_n(x) < Y_{(1)}(t) \le T_n(x), Z_{(1)}(t) > \delta\} \tag{35}$$
$$= \int_{z>\delta} \int_{\tau_n(x)}^{T_n(x)} n P^{n-1}\{\tilde{Y}(t) > y\} h_{\tilde{Y}(t), \tilde{Z}(t)}(y, z) P\{R(t_i, t_{i+1}) \text{ contains none of the } \boldsymbol{X}_j\}\, dy dz,$$

where $h_{\tilde{Y}(t), \tilde{Z}(t)}$ and $\tilde{Y}(t)$, $\tilde{Z}(t)$ are defined the same way as $h_{Y(t), Z(t)}$ and $Y(t)$, $Z(t)$ in Lemma 2, with the conditional distribution of $\boldsymbol{X}$, given $\boldsymbol{X} \notin R(t_i, t_{i+1})$, replacing that of $\boldsymbol{X}$. Arguments similar to those in Lemma 2 show that, uniformly in $t$,

$$P\{\boldsymbol{X}_1 \in R(u, v)\} \sim \beta_1(t)(v-u)^{1/\gamma}, \qquad \text{as } u, v \to t, \tag{36}$$

where

$$\beta_1(t) = 2^{-1/\gamma}\left(\frac{\kappa(t)}{2}\right)^{1/(2\gamma)} \frac{\gamma\lambda_0}{c_0\lambda(t)}.$$

Then, arguing in the same way as Lemmas 2 and 3, the right-hand side of (34) is asymptotically equivalent to

$$\rho_n\theta_n\eta_n(t)\Lambda(x, t)(1-\delta^2)^{\alpha+1},$$

where

$$\rho_n = \exp\left\{-\beta_1(t)\left(\frac{2}{\kappa(t)}\right)^{1/(2\gamma)} \epsilon_n^{1/\gamma} c_n \log n\right\} + (m_n - 2)\exp\left\{-\beta_1(t)\left(\frac{2}{\kappa(t)}\right)^{1/(2\gamma)} c_n \log n\right\}.$$

The first term in $\rho_n$ tends to 0 by (9), whereas the second term in $\rho_n$ tends to 0, since by (18), (19), and (9),

$$\begin{aligned} m_n \exp\left\{-\beta_1(t) c_n \left(\frac{2}{\kappa(t)}\right)^{1/(2\gamma)} \log n\right\} &\sim \frac{r_n\epsilon_n\sqrt{\kappa(t)/2}}{c_n^{\gamma}\log n} \exp\left\{-\gamma_n 2^{-1/\gamma}\frac{\lambda_0}{\lambda(t)}\log n\right\} \\ &= \frac{r_n\epsilon_n\sqrt{\kappa(t)/2}}{c_n^{\gamma}\log n}(n^{\gamma}(\log n)^{1-\gamma-\nu_n})^{-2^{-1/\gamma}\lambda_0/\lambda(t)} \\ &\le r_n\epsilon_n \frac{\sqrt{\kappa(t)/2}}{c_n^{\gamma}} n^{-\gamma 2^{-1/\gamma}}(\log n)^{-(1+(1-\gamma-\nu_n)2^{-1/\gamma})} \\ &= o(1). \end{aligned}$$

This concludes the proof.

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**Lemma 6.** *$S_{n,3} = o(\theta_n \eta_n(t))$, uniformly in $t$.*

*Proof.* Firstly,

$$S_{n,3} = P\{Z_{(1)}(t) > \delta, Y_{(1)}(t + \theta_n) \le \tau_n(x), Y_{(1)}(t) > T_n(x)\}.$$

From the derivations leading up to (15) in the proof of Theorem 1, there exist finite positive constants $c_1 < c_2$ such that,

$$S_{n,3} \le P\left\{\tau_n(x) + \frac{c_1\epsilon_n}{n^{2\gamma}(\log n)^{1-2\gamma}} < Y_{(1)}(t) < Y_{(2)}(t) < \tau_n(x) + \frac{c_2\epsilon_n}{n^{2\gamma}(\log n)^{1-2\gamma}}\right\}.$$

Thus, computations similar to those in the proof of Lemma 3 give the desired result.

**Lemma 7.**

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \frac{1}{\theta_n\eta_n(t)} P\left\{\bigvee_{j=1}^{r_n} Y_{(1)}(t + j\theta_n) \le \tau_n(x), Z_{(1)}(t) \le \delta, Y_{(1)}(t) > \tau_n(x)\right\} = 0.$$

*Proof.* First,

$$\begin{aligned} P&\left\{\bigvee_{j=1}^{r_n} Y_{(1)}(t + j\theta_n) \le \tau_n(x), Z_{(1)}(t) \le \delta, Y_{(1)}(t) > \tau_n(x)\right\} \\ &\le P\left\{Y_{(1)}(t + \theta_n) \le \tau_n(x), Z_{(1)}(t) \le \delta, Y_{(1)}(t) > \tau_n(x)\right\} \\ &\le P\left\{Y_{(1)}(t, \theta_n) \le \tau_n(x), 0 < Z_{(1)}(t) \le \delta, Y_{(1)}(t) > \tau_n(x)\right\} \\ &\quad + P\left\{\bigwedge_{j=2}^{n} Y_{(j)}(t, \theta_n) \le \tau_n(x), Y_{(1)}(t) > \tau_n(x)\right\}. \end{aligned} \tag{37}$$

In analysing the first term on the right, note that it is equal to

$$P\{0 < Z_{(1)}(t) \le \delta, \tau_n(x) < Y_{(1)}(t) \le T_n(x)\},$$

and so by Lemma 3 the following is obtained,

$$\begin{aligned} &\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \frac{1}{\theta_n\eta_n(t)} P\{Y_{(1)}(t, \theta_n) \le \tau_n(x), 0 < Z_{(1)}(t) \le \delta, Y_{(1)}(t) > \tau_n(x)\} \\ &= \lim_{\delta\downarrow 0} \Lambda(x, t)(1 - (1 - \delta^2)^{\alpha+1}) \\ &= 0. \end{aligned} \tag{38}$$

On the other hand, using arguments similar to those in the proof of Lemma 6, we now obtain

$$\limsup_{n\to\infty} \frac{1}{\theta_n\eta_n(t)} P\left\{\bigwedge_{j=2}^{n} Y_{(j)}(t, \theta_n) \le \tau_n(x), Y_{(1)}(t) > \tau_n(x)\right\} = 0. \tag{39}$$

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The result follows from (37), (38), and (39).

Define $N_n = [L\theta_n^{-1}]$. We now show that the random variables $Y_{(1)}(j\theta_n)$, $1 \le j \le N_n$ are weakly dependent in some sense. We use a version of a standard blocking argument tailored for this problem.

Given positive integers $r_n$ and $\ell_n$ where $\ell_n = o(r_n)$, define

$$k_n = \left[\frac{L}{r_n\theta_n}\right].$$

Let $w_0 = 0$, and

$$w_j = jr_n\theta_n,\ w'_j = (jr_n - \ell_n)\theta_n, \quad 1 \le j \le k_n.$$

Define 'big' and 'small' intervals $I_j$ and $I'_j$ by

$$\begin{aligned} I_j &= (w_{j-1}\theta_n^{-1}, w'_j\theta_n^{-1}] = ((j-1)r_n, jr_n - \ell_n], \quad 1 \le j \le k_n, \\ I'_j &= (w'_j\theta_n^{-1}, w_j\theta_n^{-1}] = (jr_n - \ell_n, jr_n], \quad 1 \le j \le k_n - 1, \\ I'_{k_n} &= (w'_{k_n}\theta_n^{-1}, N_n] = (k_n r_n - \ell_n, N_n]. \end{aligned}$$

Each $I_j$ contains $r_n - \ell_n$ integers and all $I'_j$ except $I'_{k_n}$ contain $\ell_n$ integers, while $I'_{k_n}$ contains $(N_n - k_n r_n) + \ell_n$ integers.

**Proposition 2.** *There exist positive numbers $\epsilon_n \to 0$ and positive integers $r_n \to \infty$ such that (9) holds and*

$$P\left\{\bigvee_{i=1}^{N_n} Y_{(1)}(i\theta_n) \le \tau_n(x)\right\} - \prod_{j=1}^{k_n} P\left\{\bigvee_{i=(j-1)r_n+1}^{jr_n} Y_{(1)}(i\theta_n) \le \tau_n(x)\right\} \to 0.$$

*Proof.* Let $r_n \sim n^p$ and $\ell_n \sim n^q$ where $0 < q < p < \gamma 2^{-\gamma}$. Clearly (9) is satisfied for this $r_n$ and any $\epsilon_n$ satisfying $\epsilon_n \to 0$ and $\epsilon_n^{1/\gamma}\log n \to \infty$. Let $N_{n,j}$ be the number of $\boldsymbol{X}_i$, $1 \le i \le n$, in

$$R\left(\frac{w'_{j-1} + w_{j-1}}{2}, \frac{w'_j + w_j}{2}\right).$$

Also let

$$Q_{nj} = \{i : |i/(np_{nj}) - 1| \le 1/2\},$$

where, by (36),

$$p_{nj} = P\left\{\boldsymbol{X}_1 \in R\left(\frac{w'_{j-1} + w_{j-1}}{2}, \frac{w'_j + w_j}{2}\right)\right\} \sim \beta_1(w_j)(r_n\theta_n)^{1/\gamma}. \tag{40}$$

Similarly,

$$\begin{aligned} q_{nj} &= P\left\{\boldsymbol{X}_1 \in R\left(\frac{w'_{j-1} + w_{j-1}}{2}, w_{j-1}\right)\right\} \sim P\left\{\boldsymbol{X}_1 \in R\left(w'_j, \frac{w'_j + w_j}{2}\right)\right\} \\ &\sim \beta_1(w_j)2^{-1/\gamma}(\ell_n\theta_n)^{1/\gamma}. \end{aligned} \tag{41}$$

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Let us define the following events,

$$A_{n,j} = \left( \{\boldsymbol{X}_i\}_{i=1}^n \cap R\left( \frac{w'_{j-1} + w_{j-1}}{2}, w_{j-1} \right) \neq \phi \right) \cap \left( \{\boldsymbol{X}_i\}_{i=1}^n \cap R\left( w'_j, \frac{w'_j + w_j}{2} \right) \neq \phi \right),$$

$$B_{n,j} = (N_{n,j} \in Q_{nj}),$$

$$A_n = \bigcap_{j=1}^{k_n} A_{n,j},$$

$$B_n = \bigcap_{j=1}^{k_n} B_{n,j}.$$

Using (40) and (41) it can be shown, along similar lines as Hsing (1994; Lemma 3.2 and Lemma 3.3, p. 486), that there exists finite constants $C_1$ and $C_2$ such that

$$\bigvee_{i=1}^{k_n} k_n [P(A_{n,j}^c) + P(B_{n,j}^c)] \leq C_1 k_n \quad \text{and} \quad \bigvee_{i=1}^{k_n} (\mathrm{e}^{-nq_{nj}} + \mathrm{e}^{-C_2 n p_{nj}}) \to 0, \tag{42}$$

by the choice of various constants and since $\beta_1(t)$ is bounded away from 0. Now write

$$P\left\{ \bigvee_{i=1}^{N_n} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right\} - \prod_{j=1}^{k_n} P\left\{ \bigvee_{i=(j-1)r_n+1}^{jr_n} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right\} = \sum_{i=1}^{6} S_{n,i},$$

where

$$S_{n,1} = P\left\{ \bigvee_{i=1}^{N_n} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right\} - P\left\{ \bigvee_{j=1}^{k_n} \bigvee_{i \in I_j} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right\},$$

$$S_{n,2} = P\left\{ \bigvee_{j=1}^{k_n} \bigvee_{i \in I_j} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right\} - P\left\{ \left( \bigvee_{j=1}^{k_n} \bigvee_{i \in I_j} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right) \cap A_n \cap B_n \right\},$$

$$S_{n,3} = P\left\{ \left( \bigvee_{j=1}^{k_n} \bigvee_{i \in I_j} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right) \cap A_n \cap B_n \right\} - \prod_{j=1}^{k_n} P\left\{ \left( \bigvee_{i \in I_j} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right) \cap A_{n,j} \cap B_{n,j} \right\},$$

$$S_{n,4} = \prod_{j=1}^{k_n} P\left\{ \left( \bigvee_{i \in I_j} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right) \cap A_{n,j} \cap B_{n,j} \right\} - \prod_{j=1}^{k_n} P\left\{ \bigvee_{i \in I_j} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right\},$$

$$S_{n,5} = \prod_{j=1}^{k_n} P\left\{ \bigvee_{i \in I_j} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right\} - \prod_{j=1}^{k_n} P\left\{ \bigvee_{i \in I_j \cup I'_j} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right\},$$

$$S_{n,6} = \prod_{j=1}^{k_n} P\left\{ \bigvee_{i \in I_j \cup I'_j} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right\} - \prod_{j=1}^{k_n} P\left\{ \bigvee_{i=(j-1)r_n+1}^{jr_n} Y_{(1)}(i\theta_n) \leq \tau_n(x) \right\}.$$

It can be shown that $S_{n,1}$, $S_{n,5}$, $S_{n,6}$ tend to 0, since by Lemma 3

$$k_n \ell_n P\{Y_{(1)}(t) > \tau_n(x)\} \sim O(n^{q-p}(\log n)^{\nu_n} \epsilon_n^{-1}) \to 0,$$

uniformly in $t$, whereas $S_{n,2}$ and $S_{n,4}$ tend to 0 by (42). It therefore remains to show $S_{n,3} \to 0$. Observe that if $A_n$ holds then for each $j$, $Y_{(1)}(i\theta_n)$, $i \in I_j$ are determined by the $\boldsymbol{X}_i$ in

$$R\left(\frac{w'_{j-1} + w_{j-1}}{2}, \frac{w'_j + w_j}{2}\right)$$

only. It follows from arguments similar to those in Hsing (1994; Theorem 4.1, pp. 491–492) that for some constant $C$,

$$\begin{aligned} |S_{n,3}| &\le Cn \sum_{1 \le i < j \le k_n} p_{ni} p_{nj} \\ &= O(nk_n^2 (r_n\theta_n)^{2/\gamma}) \qquad \text{by (40)} \\ &= O(n(r_n\theta_n)^{2/\gamma - 2}) \to 0, \end{aligned}$$

by the choice of constants. This concludes the proof.

**Proposition 3.** *Suppose* $(*)$ *holds. Then*

$$\lim_{n\to\infty} P\left\{\bigvee_{i=1}^{N_n} Y_{(1)}(i\theta_n) \le \tau_n(x)\right\} = \exp(-d_1 \mathrm{e}^{-d_2 x}), \quad x \in \mathbb{R},$$

*where* $d_1$, $d_2$ *are as given in Theorem 1.*

*Proof.* We apply the essence of the proof of O'Brien (1987; Theorem 2.1) (cf. Rootzén 1988). Let $r_n$, $k_n$ be given by Proposition 2. The proof of O'Brien's result can be modified to suit the present situation if we can produce another sequence $\tilde{r}_n$ such that $r_n = o(\tilde{r}_n)$ and such that

$$P\left\{\bigvee_{i=1}^{N_n} Y_{(1)}(i\theta_n) \le \tau_n(x)\right\} - \prod_{j=1}^{\tilde{k}_n} P\left\{\bigvee_{i=(j-1)\tilde{r}_n+1}^{j\tilde{r}_n} Y_{(1)}(i\theta_n) \le \tau_n(x)\right\} \to 0,$$

where $\tilde{k}_n = [N_n/\tilde{r}_n]$. In view of how the various constants were chosen in the proof of Proposition 2, this $\tilde{r}_n$ sequence clearly exists. Thus by O'Brien's result, by Proposition 1 and continuity,

$$\begin{aligned} &\lim_{n\to\infty} P\left\{\bigvee_{i=1}^{N_n} Y_{(1)}(i\theta_n) \le \tau_n(x)\right\} \\ &\quad = \lim_{n\to\infty} \exp\left\{-\sum_{s=1}^{N_n} P\{Y_{(1)}(s\theta_n) > \tau_n(x), \bigvee_{i=s+1}^{s+r_n} Y_{(1)}(i\theta_n) \le \tau_n(x)\}\right\} \\ &\quad = \lim_{n\to\infty} \exp\left\{-(\log n)^{\nu_n} \int_0^L \exp\left\{-\gamma_n\left(\frac{\lambda_0}{\lambda(t)} - 1\right)\log n\right\}\Lambda(x,t)\,\mathrm{d}t\right\}. \end{aligned}$$

For any $\epsilon > 0$,

$$(\log n)^{\nu_n} \int_0^L I\left(\frac{\lambda_0}{\lambda(t)} - 1 > \epsilon\right) \exp\left\{-\gamma_n\left(\frac{\lambda_0}{\lambda(t)} - 1\right) \log n\right\} \Lambda(x, t)\,\mathrm{d}t \longrightarrow_{n\to\infty} 0.$$

Hence, by the continuity of $\lambda$ and $(*)$, it is clear that

$$\begin{aligned}
&\lim_{n\to\infty} (\log n)^{\nu_n} \int_0^L \exp\left\{-\gamma_n\left(\frac{\lambda_0}{\lambda(t)} - 1\right) \log n\right\} \Lambda(x, t)\,\mathrm{d}t \\
&\quad = \exp\left\{-\frac{x}{2c_0^{2\gamma}}\right\} \frac{c_0^{1-\gamma}}{\alpha+1} \lim_{n\to\infty} (\log n)^{\nu_n} \int_0^L \exp\left\{-\gamma_n\left(\frac{\lambda_0}{\lambda(t)} - 1\right) \log n\right\} g(t)\,\mathrm{d}t \\
&\quad = \exp\left\{-\frac{x}{2c_0^{2\gamma}}\right\} \frac{c_0^{1-\gamma}\mu}{\alpha+1}.
\end{aligned}$$

This concludes the proof.

**Proposition 4.** *For each $x \in \mathbb{R}$,*

$$P\left\{\sup_{0\le i\le N_n} Y_{(1)}(i\theta_n) \le \tau_n(x)\right\} - P\left\{\sup_{0\le t<L} Y_{(1)}(t) \le \tau_n(x)\right\} \to 0. \tag{43}$$

*Proof.* The absolute difference in (43) is bounded by

$$\sum_{s=1}^{N_n} P\left\{Y_{(1)}((s-1)\theta_n) \vee Y_{(1)}(s\theta_n) \le \tau_n(x), \sup_{(s-1)\theta_n<t<s\theta_n} Y_{(1)}(t) > \tau_n(x)\right\}.$$

Observe that the event

$$(Y_{(1)}((s-1)\theta_n) \vee Y_{(1)}(s\theta_n) \le \tau_n(x)) \cap \left(\sup_{(s-1)\theta_n<t<s\theta_n} Y_{(1)}(t) > \tau_n(x)\right)$$

occurs only if the directional minima $Y_{(1)}((s-1)\theta_n)$ and $Y_{(1)}(s\theta_n)$ correspond to different sample points, say, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, respectively. By (15) we then must have

$$|Y_i(s\theta_n) - \tau_n(x)| = O\left(\frac{\epsilon_n}{n^{2\gamma}(\log n)^{1-2\gamma}}\right), \; i = 1, 2.$$

The conclusion now follows in the same way as in the proof of Lemma 6.

The proof of Theorem 1 is now complete.

*Proof of Theorem 2.* Let $\tau_n(x)$ be replaced by

$$\tau_n(x, t) = n^{-2\gamma}((c_n'(t)\log n)^{2\gamma} + x(\log n)^{2\gamma-1}),$$

in the above proof, and let $\epsilon_n, r_n$ and $\theta_n$ be as above, except let $\nu_n = 0$ in (9). The proof of Proposition 4 can be modified to obtain

$$P\left\{Y_{(1)}(t) > \tau_n(x, t), \bigvee_{j=1}^{r_n} Y_{(1)}(t + j\theta_n) \le \tau_n(x, t)\right\} \sim \theta_n d_1(t)\mathrm{e}^{-d_2(t)x}, \tag{44}$$

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where

$$d_1(t) = \frac{g(t)}{1+\alpha} c_0'(t)^{1-\gamma}, \qquad d_2(t) = \tfrac{1}{2} c_0'(t)^{-2\gamma}.$$

Replacing $x$ in (44) by $(x + \log[Ld_1(t)])/d_2(t)$ yields

$$P\left\{ Y_{(1)}(t) > a_n(t)x + b_n(t), \bigvee_{j=1}^{r_n} Y_{(1)}(t + j\theta_n) \le a_n(t)x + b_n(t) \right\} \sim L^{-1}\theta_n \mathrm{e}^{-x}.$$

The rest of the proof parallels the proofs of Proposition 2, 3 and 4.

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