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A NOTE ON DIVISIBILITY IN $H^\infty(X)$

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1. Let X be a Riemann surface, and $H^\infty(X)$ the ring of bounded holomorphic functions in X. We offer here a question on divisibility in $H^\infty(X)$, and then give in Section 2 a condition in which the answer is yes (Corollary 2 to Lemma 1). In Section 3 we use part 2 to prove a theorem on the separation of points by $H^\infty(X)$. In Section 4 we study $X/H^\infty(X)$.

If f is meromorphic in X and $z \in X$, then by $o(f, z)$ we mean the order of f at z. (We agree that $o(f, z) = \infty$ if $f \equiv 0$.) Let h be memomorphic in X; then h might be said to be of bounded type if $h = f/g$ where $f, g \in H^\infty(X)$, $g \neq 0$.

The question on divisibility is this. Let $x \in X$ and let h be of bounded type. If $o(h, x)$, $\geqq 0$, do we then have $h = f/g$ where $f, g \in H^\infty(X)$ with $g(x) \neq 0$?

There is a simple necessary condition. Put

$$O_x = \{o(f, x):f \in H^\infty(X), f \neq 0\};$$

then

$$(1.1) \quad O_x = \{js:j = 0, 1, 2, \dots\}$$

if the answer to the question on divisibility is yes.

To see this let z of bounded type be of least positive order at x (without loss of generality $H^\infty(X) \neq \mathbf{C}$). Put $s = o(z, x)$. Let h be of bounded type, $h \neq 0$; then by the division algorithm the order of h at x is a multiple of s, i.e.,

$$o(h, x) = js, \quad j \in \mathbf{Z}.$$

Suppose now the answer to the question on divisibility is yes. Then we may take z in $H^\infty(X)$; this gives (1.1).

2. A lemma on divisibility in $H^\infty(X)$. Let θ be meromorphic in X and proper at 0. By proper at 0 we mean $0 \in \theta(X)$, $\theta \not\equiv 0$, and $\{ |\theta| < 1\}$ is bounded in X. (A set is bounded if it is contained in a compact set.) Let $x \in X$, and let $\theta(x) = 0$. We now come to our lemma on divisibility.

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LEMMA 1. *Let* $f \in H^\infty(X)$, $f \not\equiv 0$. *Let s and t be any integers such that*

(2.1) $so(\theta, x) = to(f, x)$.

Then there are functions g and h in $H^\infty(X)$ *such that*

(2.2) $g(x)h(x) \neq 0$ *and* $f^t g = \theta^s h$.

(We might point out that the condition (2.1) on s and t is necessary; i.e., if (2.2) holds, then (2.1) holds.)

Proof. We have

$$\theta^{-1}(0) = \{z_1, \dots, z_p, z_{p+1}, \dots, z_q\}$$

where

(a) if $g \in H^\infty(X)$, then $g(z_j) = g(x)$, $1 \leqq j \leqq p$;

(b) if $p + 1 \leqq j \leqq q$, then there is a g_j in $H^\infty(X)$ with $g_j(z_j) \neq g_j(x)$.

To see this we need only write

$$\theta^{-1}(0) = \Delta \cup (\theta^{-1}(0)\backslash\Delta)$$

where

$$\Delta = \{z \in \theta^{-1}(0): g(z) = g(x) \text{ if } g \in H^\infty(X)\},$$

and then notice that $\theta^{-1}(0)$ is bounded in X, hence finite (since $\theta \not\equiv 0$).

Put $t_j = o(\theta, z_j)$. Then $t_j \geqq 1$, $1 \leqq j \leqq q$. Put

$$s_j = o(f, z_j), \quad 1 \leqq j \leqq p.$$

We will prove that

(2.3) $s_1/t_1 = s_2/t_2 = \dots = s_p/t_p$.

Without loss of generality let

$$s_1/t_1 \leqq s_2/t_2, \dots, s_p/t_p.$$

By (b) there is a g in $H^\infty(X)$ with $g(x) \neq 0$ and

$$g(z_{p+1}) = g(z_{p+2}) = \dots = g(z_q) = 0.$$

Choose integers $\alpha > 0$, $\beta \geqq 0$ such that $\alpha s_1 = \beta t_1$, and let

(2.4) $h = f^\alpha g^\gamma / \theta^\beta$

where $\gamma \geqq \beta t_{p+1}, \beta t_{p+2}, \dots, \beta t_q$. The order of h at z_j is equal to $\alpha s_j - \beta t_j$ if $1 \leqq j \leqq p$; if $p + 1 \leqq j \leqq q$, its order at z_j is $\geqq 0$. Let $1 \leqq j \leqq p$; then

$$\alpha s_j - \beta t_j \geqq 0.$$

(Otherwise, $s_i/t_i < \beta/\alpha = s_1/t_1$.) Thus h is holomorphic in X. Continuing, let

$$D = \{z \in X:|\theta(z)| < 1\}.$$

Then D is bounded in X; hence in D, $|h| \leqq A$. In $X \backslash D$,

$$|h| \leqq |f|^{\alpha}|g|^{\gamma}.$$

Thus $h \in H^{\infty}(X)$. We have $h(z_1) \neq 0$; hence by (a),

$$h(z_j) \neq 0 \quad \text{if } 1 \leqq j \leqq p.$$

This proves that $\alpha s_j - \beta t_j = 0$, i.e.,

$$s_j/t_j = \beta/\alpha = s_1/t_1;$$

this is (2.3).

By (2.3), in (2.4) we may take $\alpha = |t|$, $\beta = |s|$. This gives (2.2).

COROLLARY 1. *If $H^{\infty}(X) \neq \mathbf{C}$, then θ is of bounded type. In fact, $\theta = f/g$ where $f, g \in H^{\infty}(X)$ with $g(x) \neq 0$.*

Proof. Let f in $H^{\infty}(X)$ be such that $1 \leqq o(f, x) < \infty$ (here we use $H^{\infty}(X) \neq \mathbf{C}$), and put $s = o(f, x)$, $t = o(\theta, x)$. By the lemma there are functions g and h in $H^{\infty}(X)$ with $h(x) \neq 0$ and $f^t g = \theta^s h$. Then $(\theta h)^s \in H^{\infty}(X)$, hence $\theta h \in H^{\infty}(X)$.

COROLLARY 2. *Let h be of bounded type. If $o(h, x) \geqq 0$, then $h = f/g$ where $f, g \in H^{\infty}(X)$ with $g(x) \neq 0$.*

Proof. Without loss of generality $h \neq 0$. Put $t = o(\theta, x)$ and apply the lemma separately to the numerator and denominator of h. This gives functions f_1 and f_2 in $H^{\infty}(X)$ with $f_1(x) \neq 0$ and $h^t f_1 = \theta^s f_2$ where $s = o(h, x)$. Without loss of generality $H^{\infty}(X) \neq \mathbf{C}$. Then by the first corollary, $\theta^s = f_3/f_4$ where $f_3, f_4 \in H^{\infty}(X)$ with $f_4(x) \neq 0$. Then $(hf_1 f_4)^t \in H^{\infty}(X)$, hence $hf_1 f_4 \in H^{\infty}(X)$.

Corollary 2 states that the answer to the question on divisibility is yes (if there is a θ that is meromorphic in X, proper at 0, and vanishes at x). The next corollary completes the lemma.

COROLLARY 3. *Let f be of bounded type, $f \neq 0$. Let s and t be any integers such that*

$$so(\theta, x) = to(f, x).$$

Then there are functions g and h in $H^{\infty}(X)$ such that

$$g(x)h(x) \neq 0 \quad \textit{and} \quad f^t g = \theta^s h.$$

Proof. Without loss of generality $o(f, x) \geqq 0$. Then by Corollary 2, $f = f_1/f_2$ where $f_1, f_2 \in H^{\infty}(X)$ with $f_2(x) \neq 0$. We have $o(f_1, x) = o(f, x)$, hence (by the lemma) there are functions g and h in $H^{\infty}(X)$ such that

$$g(x)h(x) \neq 0 \quad \text{and} \quad f_1^t g = \theta^s h.$$

Then $f^t f_2^t g = \theta^s h$.

COROLLARY 4. *Let y be any point in X with $f(y) = f(x)$ whenever $f \in H^\infty(X)$. That is to say, any point that $H^\infty(X)$ identifies with x.*

(i) *If $f \in H^\infty(X)$, $f \not\equiv 0$, then*

$$o(f, x)o(\theta, y) = o(f, y)o(\theta, x).$$

(ii) *If $H^\infty(X) \neq \mathbf{C}$, then $\theta(y) = 0$.*

Proof. Put $s = o(f, x)$, $t = o(\theta, x)$. Then (2.2) gives (i); if $s > 0$, it gives (ii).

To prove the next corollary we need a (known) lemma on valuation rings. It is this.

LEMMA 2. *Let F be a field and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be valuation rings in F with $\mathcal{O}_1 \subset \mathcal{O}_2$. If $\mathcal{O}_1$ is discrete, then either $\mathcal{O}_2 = \mathcal{O}_1$ or $\mathcal{O}_2 = F$.*

Proof. Put $\mathcal{P}_k$ = the ideal of nonunits of $\mathcal{O}_k$. If $f \in \mathcal{P}_2$, then $1/f \notin \mathcal{O}_1$, hence $f \in \mathcal{P}_1$. This proves that

$$\mathcal{P}_2 \subset \mathcal{P}_1 \subset \mathcal{O}_1 \subset \mathcal{O}_2,$$

hence $\mathcal{P}_2$ is a prime ideal in $\mathcal{O}_1$, $\neq \mathcal{O}_1$. Since $\mathcal{O}_1$ is discrete, its prime ideals are 0, $\mathcal{P}_1$, and itself. If $\mathcal{P}_2 = 0$, then $\mathcal{O}_2 = F$. If $\mathcal{P}_2 = \mathcal{P}_1$, then $\mathcal{O}_2 = \mathcal{O}_1$.

Put

$$M_x = \{f \in H^\infty(X): f(x) = 0\};$$

then M_x is a maximal ideal in the ring $H^\infty(X)$.

COROLLARY 5. *The prime ideal M_x is minimal, i.e., if P is a prime ideal in $H^\infty(X)$ with $P \subset M_x$, then either $P = 0$ or $P = M_x$.*

Proof. Without loss of generality $H^\infty(X) \neq \mathbf{C}$. Let

$$H_x^\infty = \{f/g: f, g \in H^\infty(X), g(x) \neq 0\},$$

$$H_P^\infty = \{f/g: f, g \in H^\infty(X), g \notin P\},$$

and

$$F = \{f/g: f, g \in H^\infty(X), g \neq 0\};$$

then $H_x^\infty \subset H_P^\infty \subset F$. By Corollary 2,

$$H_x^\infty = \{h \in F: o(h, x) \geqq 0\},$$

hence H_x^∞ is a discrete valuation ring in the field F (here we use $H^\infty(X) \neq \mathbf{C}$). Then by the lemma, the local ring H_P^∞ is either equal to H_x^∞ or to F. If the first alternative holds, $P = M_x$, while if the second holds, $P = 0$.

2.1. Let X be any Riemann surface in which $H^{\infty}(X) \neq \mathbf{C}$ and in which $H^{\infty}(X)$ does not separate points. Let x and y be any pair of distinct points in X that $H^{\infty}(X)$ identifies. Let θ be meromorphic in X with $\theta(x) = 0$. If θ is proper at 0, then by part (ii) of the fourth corollary, $\theta(y) = 0$. Thus we see there does not exist a θ that is meromorphic in X, proper at 0, and vanishes only at x. (This may be proved directly.) On the other hand there is, for example, a g in $\mathcal{O}(X)$ that vanishes only at x; if we wish, it vanishes there to order 1. If there are infinitely many distinct points $y_1, y_2, \dots$ that $H^{\infty}(X)$ identifies with x, then we see there does not exist a θ that is meromorphic in X, proper at 0, and vanishes at x. Let $Y = X \setminus y$. Then by Corollary 1 there does not exist a θ that is meromorphic in Y, proper at 0, and vanishes at x.

2.2. Let W be a compact Riemann surface, X a region in W, and $x \in X$. Then (e.g. by [**1**, p. 106]) there is an f meromorphic in W with a pole at x and no other poles. Let $t > 0$; then

$$\{ |t/f| \leqq 1\} \subset X$$

if t is sufficiently large. Thus we see:

(2.5) there is a θ that is meromorphic in X, proper at 0, and vanishes only at x.

We will use (2.5) in part 3.

Regions in compact surfaces are rich in bounded functions (provided $H^{\infty} \neq \mathbf{C}$). The precise statement is this. (We include it for comparison with the theorems in part 3.)

THEOREM 1. *Let X be a region in a compact Riemann surface W.*

(i) *Let $x \in X$ and let h be of bounded type in X. If $o(h, x) \geqq 0$, then $h = f/g$ where $f, g \in H^{\infty}(X)$ with $g(x) \neq 0$. This is to say, in X the answer to the question on divisibility is yes (for each x in X).*

(ii) *If $H^{\infty}(X) \neq \mathbf{C}$, then $H^{\infty}(X)$ separates the points of X.*

(iii) *Let h be meromorphic in W. If $H^{\infty}(X) \neq \mathbf{C}$, then h is of bounded type in X.*

(iv) *Let $x \in X$. If $H^{\infty}(X) \neq \mathbf{C}$, then there is an f in $H^{\infty}(X)$ with $o(f, x) = 1$.*

Proof. (2.5) gives (i) and (ii).

To prove (iii), let $x \in X$. We have $h = f/g$ where f and g are meromorphic in W with poles at x and no other poles. Then (by the first corollary) f and g are of bounded type in X. This gives (iii).

To prove (iv), there is (e.g. by [**1**, pp. 135-136]) an h meromorphic in W with $o(h, x) = 1$. By (iii) and (i), $h = f/g$ where $f, g \in H^{\infty}(X)$ with $g(x) \neq 0$. Then $o(f, x) = o(h, x)$.

We might paraphrase (ii) in this way. Let X be a Riemann surface in which $H^\infty(X)$ does not separate points. If $H^\infty(X) \neq \mathbf{C}$, then X cannot be imbedded in a compact Riemann surface. That is to say there is no univalent holomorphic map of X into a compact Riemann surface. (On the other hand there may be a proper holomorphic map of X into a compact surface.)

3. A theorem on the separation of points by bounded holomorphic functions. Let X be a Riemann surface; then one may ask if $H^\infty(X)$ separates points in X. We will prove the following.

THEOREM 2. *Let X be any Riemann surface in which there is a proper holomorphic map φ onto a region G in a compact Riemann surface. If $H^\infty(X)$ separates the points of at least one fiber of φ in which there are no branch points and if $H^\infty(X) \neq \mathbf{C}$, then* (i) *$H^\infty(X)$ separates the points of X, and* (ii) *to each x in X there is an f in $H^\infty(X)$ with $o(f, x) = 1$.*

We might recall that a mapping $\varphi:X \to G$ of topological spaces X and G is said to be proper if inverse images of bounded sets are bounded, i.e., if $\varphi^{-1}(E)$ is bounded in X whenever E is bounded in G.

Our proof is elementary. Its main ingredient is one or the other of the corollaries to the lemma on divisibility (Section 2).

3.1. We now come to the proof of Theorem 2. Accordingly, X is a Riemann surface, G a region in a compact Riemann surface, and φ a proper holomorphic map of X onto G.

Let F be the field of fractions of the ring $H^\infty(X)$, i.e.,

$$F = \{f/g:f, g \in H^\infty(X), g \neq 0\}.$$

In the terminology of Section 1, F is the field of functions in X of bounded type. Let $x \in X$, choose z in $F \setminus 0$ of least positive order at x (here we use $H^\infty(X) \neq \mathbf{C}$), and put

$$R_x = \{f \in F:o(f, x) \geqq 0\}.$$

Let $f \in R_x$. Put $\alpha_0 = f(x), f_1 = (f - \alpha_0)/z$; then $f_1 \in R_x$. Likewise put $\alpha_1 = f_1(x), f_2 = (f_1 - \alpha_1)/z$; then $f_2 \in R_x$. Continuing in this way we define sequences $\{\alpha_j\}$ in $\mathbf{C}$ and $\{f_j\}$ in R_x by

$$(3.1) \quad \alpha_j = f_j(x), f_{j+1} = (f_j - \alpha_j)/z.$$

The second of these gives

$$(3.2) \qquad f = \sum_{j=0}^{k} \alpha_j z^j + z^{k+1} f_{k+1}.$$

Let $s =$ the order of z at x. Then we may choose a local coordinate ξ with

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(3.3) $z(\xi) = \xi^s, |\xi| < \epsilon.$

(Here it is understood that x is the center of the disc $\{ |\xi| < \epsilon\}$.) If f in R_x is holomorphic in the disc $\{ |\xi| < \epsilon\}$, then it is equal there to a convergent power series, say

$$\sum_{j=0}^{\infty} c_j \xi^j.$$

We have (by (3.2) and (3.3))

$$f(\xi) = \sum_{j=0}^{k} \alpha_j \xi^{js} + \xi^{(k+1)s} f_{k+1}(\xi)$$

in the disc $\{ |\xi| < \epsilon\}$, hence

$$\sum_{j=0}^{k} \alpha_j \xi^{js} = \sum_{j=0}^{ks} c_j \xi^j$$

(otherwise the difference is a polynomial of degree $\leqq ks$ with a zero of order $\geqq ks + 1$). Thus the formal series

$$\sum_{j=0}^{\infty} \alpha_j z^j$$

converges to f in the disc $\{ |\xi| < \epsilon\}$. (What we have done here is standard pràctice; see e.g. [**2**] or [**3**].)

Let $y \in X$, $y \neq x$, and suppose $H^\infty(X)$ does not separate x and y. By (2.5) there is a θ that is meromorphic in G, proper at 0, and vanishes at $\varphi(x)$; then $\theta(\varphi)$ is meromophic in X, proper at 0, and vanishes at x. Then by either the second or fourth corollary to the lemma on divisibility, $R_x = R_y$; hence

(3.4) $P_x = P_y$

where

$$P_x = \{g \in F{:}o(g, x) > 0\}.$$

Thus if $g \in R_x$, then $g - g(x) \in P_y$, hence

(3.5) $g(y) = g(x).$

By (3.4), z being of least positive order at x is of least positive order at y. Let $t =$ the order of z at y, and choose a local coordinate ζ with

$$z(\zeta) = \zeta^t, |\zeta| < \delta.$$

(Here it is understood that y is the center of the disc $\{ |\zeta| < \delta\}$.) Without

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loss of generality we may assume that the discs $\{ |\xi| < \epsilon\}$ and $\{ |\zeta| < \delta\}$ are disjoint (in X) and that $\epsilon^s = \delta^t$. Let $|\xi| < \epsilon$; then there is a ζ, $|\zeta| < \delta$, with $z(\xi) = z(\zeta)$. Thus if $f \in H^{\infty}(X)$, then by (3.1) and (3.5)

$$f(\xi) = \sum_{j=0}^{\infty} \alpha_j z(\xi)^j = \sum_{j=0}^{\infty} \alpha_j z(\zeta)^j = f(\zeta).$$

The mapping φ, being holomorphic and proper, is of constant valence ($<\infty$), say n. Thus if $w \in G$, then

(3.6) $\varphi^{-1}(w) = \{x_1, \dots, x_n\}$

counting multiplicities. Let $f \in \mathcal{O}(X)$, $w \in G$, and put

$$D(w) = \prod_{j \neq k} (f(x_j) - f(x_k))$$

where $\{x_k\}$ = the right side of (3.6). (Let $D \equiv 1$ if $n = 1$.) Then $D \in \mathcal{O}(G)$. We will call D the discriminant of f.

We have proved that $H^{\infty}(X)$ identifies each point in $\{ |\xi| < \epsilon\}$ with a point in $\{ |\zeta| < \delta\}$ (and vice versa), hence by (2.5) and one or the other of the corollaries to Lemma 1, $H^{\infty}(X) \cup \{\varphi\}$ identifies each point in the first disc with a point in the second. This in turn proves that if $f \in H^{\infty}(X)$, then the discriminant of f vanishes in $\{\varphi(\xi):|\xi| < \epsilon\}$, hence it vanishes identically. But then $H^{\infty}(X)$ cannot separate points in any fiber of φ in which there are no branch points. This proves (i).

By (i), the order of z at x is equal to 1, while by the second corollary to the lemma on divisibility, $z = f/g$ where $f, g \in H^{\infty}(X)$ with $g(x) \neq 0$. Then $o(f, x) = 1$. This is (ii).

3.2. We do not know if Theorem 2 holds if we replace the compactly imbedded surface G by a Riemann surface Y in which $H^{\infty}(Y)$ separates points. But we do have this.

THEOREM 3. *Let X and Y be Riemann surfaces in which there is a proper holomorphic map φ of X onto Y. Suppose*: (i) *$H^{\infty}(Y)$ separates the points of Y*; (ii) *if $\xi \in Y$ and h is of bounded type in Y with $o(h, \xi) \geqq 0$, then $h = f/g$ where $f, g \in H^{\infty}(Y)$ with $g(\xi) \neq 0$ (i.e., for each ξ in Y, the answer to the question on divisibility in $H^{\infty}(Y)$ is yes). Then $H^{\infty}(X)$ separates the points of X, and to each x in X there is an f in $H^{\infty}(X)$ with $o(f, x) = 1$, if $H^{\infty}(X)$ separates the points of at least one fiber of φ in which there are no branch points.*

We will give the proof in brief. Three lemmas are needed; we omit the proof of the first.

LEMMA 3. *Let* κ *be a field, F a vector space over* κ, $\mathcal{O}_1, \dots, \mathcal{O}_n$ *subspaces of F, and* $f_1, \dots, f_n$ *vectors in F. Suppose* $f_j \notin \mathcal{O}_j$, $1 \leqq j \leqq n$. *If* $n < \#(\kappa)$ (*in particular if* κ *is infinite*), *then we may choose scalars* $t_1, \dots, t_n$ *such that*

$$\sum t_k f_k \notin \mathcal{O}_j, \quad 1 \leqq j \leqq n.$$

LEMMA 4. *Let X be a Riemann surface, and* $x_1, \dots, x_n$ *points in X. Put*

$$R_j = \{f : f \text{ is of bounded type in } X \text{ with } o(f, x_j) \geqq 0\}$$

and

$$S = \bigcap_{j=1}^{n} R_j.$$

Then $R_1 = S_1$ *where*

$$S_1 = \{f/g : f, g \in S, g(x_1) \neq 0\}.$$

Proof. We have $S_1 \subset R_1$; it is to be proved that $R_1 \subset S_1$. Without loss of generality $R_1 \not\subset R_j$ if $j \neq 1$. Then by Lemma 3 there is a g in R_1 with $g \notin R_j$, $2 \leqq j \leqq n$. If $g(x_1) = 0$, replace g by $1 + g$; then $g(x_1) \neq 0$. Put $\varphi = 1/g$; then $\varphi \in S$ with $\varphi(x_1) \neq 0$ and

$$\varphi(x_2) = \dots = \varphi(x_n) = 0.$$

Now let $f \in R_1$; then $\varphi^k f \in S$ if

$$k o(\varphi, x_j) + o(f, x_j) \geqq 0, \quad 2 \leqq j \leqq n.$$

This proves that $f \in S_1$.

LEMMA 5. *Let X and Y be Riemann surfaces in which there is a proper holomorphic map* φ *of X onto Y. Let* $x \in X$, *and put* $\xi = \varphi(x)$. *If* (i) *the answer to the question on divisibility in* $H^\infty(Y)$ *for the point* ξ *is yes, then* (ii) *it is yes in* $H^\infty(X)$ *for x.*

Proof. (We might point out there may not be a θ that is meromorphic in Y, proper at 0, and vanishes at ξ; then we cannot compose with φ and use Corollary 2.) Let $F(Y)$ = the field of functions in Y of bounded type, and identify $F(Y)$ with $\{f(\varphi) : f \in F(Y)\}$. Put

$$R_\xi = \{f \in F(Y) : o(f, \xi) \geqq 0\},$$

and let S = the integral closure of R_ξ in $F(X)$. Then since φ is proper,

$$S = \bigcap_{\varphi(y)=\xi} R_y$$

where

$$R_y = \{f \in F(X) : o(f, y) \geqq 0\};$$

hence by Lemma 4,

(3.7) $R_x = S_x$

where

$$S_x = \{f/g: f, g \in S, o(g, x) = 0\}.$$

Let $f \in S$; then f is integral over R_ξ, hence by (i) there is a γ in $H^\infty(Y)$ with $\gamma(\xi) \neq 0$ and γf integral over $H^\infty(Y)$. Then $\gamma f \in H^\infty(X)$. This proves that

$$S = \{g/\gamma : g \in H^\infty(X), \gamma \in H^\infty(Y), \gamma(\xi) \neq 0\},$$

hence

(3.8) $S_x = \{f/g: f, g \in H^\infty(X), g(x) \neq 0\}.$

By (3.7) and (3.8), (ii) holds.

We now come to the proof of Theorem 3. Let $x, y \in X$, $x \neq y$, and suppose $H^\infty(X)$ does not separate x and y. Then by Lemma 5, $R_x = R_y$. We may now repeat (without change) the proof of Theorem 2.

The question then is may we prove Theorem 3 without (ii)? Or better yet, may we prove (ii)?

Other proofs of Theorem 3, with special Y, are in [**4**], [**5**], [**6**] (in [**5**] and [**6**], $Y = \mathbf{D}$). These proofs, which are not easy but give more, will not work here.

There is a corollary to Lemma 5. Let X and Y be Riemann surfaces in which there is a proper holomorphic map φ of X onto Y. Let $x \in X$, and put $\xi = \varphi(x)$. Suppose the answer to the question on divisibility in $H^\infty(X)$ for the point x is yes. Then it is yes in $H^\infty(Y)$ for ξ (we omit the proof), hence (this is the corollary) it is yes in $H^\infty(X)$ for y in $\varphi^{-1}(\xi)$.

4. Is $X/H^\infty(X)$ a Riemann surface? If we look over the proof of Theorem 2, we find the following.

THEOREM 4. *Let X be a Riemann surface in which to each x in X there is a θ that is meromorphic in X, proper at* 0, *and vanishes at x. Then $X/H^\infty(X)$ is a Riemann surface (provided $H^\infty(X) \neq$* **C**). *By this we mean there is a Riemann surface Y and a holomorphic map σ of X onto Y such that σ and $H^\infty(X)$ identify the same pairs of points in X.*

Proof (in brief; see also [**2**], [**3**]). Put $Y = X/H^\infty(X)$, and

$$\sigma(x) = \{y \in X: f(y) = f(x) \quad \text{if } f \in H^\infty(X)\}.$$

We give Y the largest topology in which σ is continuous, i.e., $E \subset Y$ is open if $\sigma^{-1}(E)$ is open. Then by the proof of Theorem 2, the map σ is open (hence Y is Hausdorff).

Let $f \in H^{\infty}(X)$; then $f = \hat{f}(\sigma)$ where $\hat{f} \in C(Y)$. Let $x \in X$. Then, using the proof of Theorem 2,

$$f = \sum_{j=0}^{\infty} \alpha_j z^j$$

in the disc $\{ |\xi| < \epsilon \}$. Without loss of generality (by Corollary 2) $z \in H^{\infty}(X)$. Put

$$D = \sigma(\{ |\xi| < \epsilon \}).$$

Then D is open in Y, and in D,

$$\hat{f} = \sum_{j=0}^{\infty} \alpha_j \hat{z}^j;$$

hence $\hat{z}$ is a homeomorphism of D and the disc $\{ |\xi| < \epsilon^s \}$.

Now let $x_j \in X, j = 1, 2$. Put $D_j = \sigma(\{ |\xi| < \epsilon_j \})$. Suppose $D_1 \cap D_2 \neq \phi$, and define

$$\varphi : \hat{z}_1(D_1 \cap D_2) \rightarrow \hat{z}_2(D_1 \cap D_2)$$

by $\varphi(\hat{z}_1) = \hat{z}_2$. Then in $\sigma^{-1}(D_1 \cap D_2)$, $\varphi(z_1) = z_2$. This proves that φ is holomorphic. Thus we see that Y is a Riemann surface and (since $z = \hat{z}(\sigma)$) that σ is holomorphic.

Let $\xi \in Y$; then $\xi = \sigma(x)$, $x \in X$. By the hypothesis, there is a θ that is meromorphic in X, proper at 0, and vanishes at x. By Corollary 1, θ is of bounded type in X; hence $\theta = \hat{\theta}(\sigma)$ where $\hat{\theta}$ is of bounded type in Y. Then $\hat{\theta}$ is proper at 0 and vanishes at ξ. This is to say, the hypothesis holds in Y (where H^{∞} separates points). Put

$$G_{\xi} = \{ |\hat{\theta}| < 1 \};$$

then $\{G_{\xi}:\xi \in Y\}$ is an open cover of Y with $\sigma^{-1}(G_{\xi})$ bounded in X. This proves that σ is proper. Hence σ (being holomorphic) is of constant valence $<\infty$; in other words, there is a positive integer n such that for each x in X the fiber

$$\{ y \in X : f(y) = f(x) \text{ if } f \in H^{\infty}(X) \}$$

consists of precisely n points counting multiplicities.

Finally, to each ξ in Y there is an f in $H^{\infty}(Y)$ with $o(f, \xi) = 1$.

COROLLARY (to Theorem 4). *Let X be a Riemann surface in which there is a proper holomorphic map φ onto a region G in a compact Riemann surface. Then $X/H^{\infty}(X)$ is a Riemann surface* (*provided* $H^{\infty}(X) \neq \mathbf{C}$).

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Proof. (2.5) + φ.

The corollary with $G = \mathbf{D}$ is in [**6**]; the proof there will not work here.

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