CONVOLUTION OF FUNCTIONALS OF DISCRETE-TIME NORMAL MARTINGALES

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Abstract

Let $M = (M_n)_{n \in \mathbb{N}}$ be a discrete-time normal martingale satisfying some mild requirements. In this paper we show that through the full Wiener integral introduced by Wang et al. (‘An alternative approach to Privault’s discrete-time chaotic calculus’, J. Math. Anal. Appl. 373 (2011), 643–654), one can define a multiplication-type operation on square integrable functionals of $M$, which we call the convolution. We examine algebraic and analytical properties of the convolution and, in particular, we prove that the convolution can be used to represent a certain family of conditional expectation operators associated with $M$. We also present an example of a discrete-time normal martingale to show that the corresponding convolution has an integral representation.

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1. Introduction


Let $M = (M_n)_{n \in \mathbb{N}}$ be a discrete-time normal martingale satisfying some mild requirements. In [7] the authors introduced a Wiener-type integral with respect to the noise associated with $M$, which is called the full Wiener integral. In this paper...
we show that through the full Wiener integral, one can define a multiplication-type operation on square integrable functionals of $M$, which we call the convolution. We examine algebraic and analytical properties of the convolution and, in particular, we prove that the convolution can be used to represent a certain family of conditional expectation operators associated with $M$. We also present an example of discrete-time normal martingale to show that the corresponding convolution has an integral representation.

The paper is organised as follows. Section 2 recalls some basic notions and facts such as discrete-time normal martingales, the full Wiener integral, and the chaotic representation property. Sections 3 and 4 state our main results. We first define the convolution on square integrable functionals of a discrete-time normal martingale $M$. Then we examine its algebraic and analytical properties and show its interesting link with a certain family of conditional expectation operators associated with $M$. Finally, we present an example of a discrete-time normal martingale to show that the corresponding convolution has an integral representation.

**Notation and conventions.** Let $\mathbb{N}$ be the set of all nonnegative integers. For a subset $S \subset \mathbb{N}$, we define $\Gamma(S)$ as the finite power set of $S$, namely

$$\Gamma(S) = \{\sigma \mid \sigma \subset S \text{ and } \#\sigma < \infty\},$$

where $\#\sigma$ means the cardinality of $\sigma$ as a set. If $S = \{0, 1, \ldots, k\}$, then we simply write $\Gamma_k = \Gamma(S)$. We set $\Gamma_{-1} = \Gamma(\emptyset)$, where $\emptyset$ denotes the empty set.

We write $\Gamma = \Gamma(\mathbb{N})$ for brevity. (Clearly, $\Gamma$ is countable.) As usual, $l^2(\Gamma)$ denotes the space of square summable real-valued functions on $\Gamma$.

**2. Normal martingale**

Let $(\Omega, \mathcal{F}, P)$ be a probability space with $\mathbb{E}$ denoting the expectation with respect to $P$. We use $L^2(\Omega)$ to mean $L^2(\Omega, \mathcal{F}, P)$ if there is no risk of confusion.

**Definition 2.1.** An $L^2$-stochastic process $M = (M_n)_{n \in \mathbb{N}}$ on $(\Omega, \mathcal{F}, P)$ is called a discrete-time normal martingale if it satisfies:

(i) $\mathbb{E}[M_0|\mathcal{F}_{-1}] = 0$ and $\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}$ for $n \geq 1$;

(ii) $\mathbb{E}[M_n^2|\mathcal{F}_{n-1}] = 1$ and $\mathbb{E}[M_n^2|\mathcal{F}_{n-1}] = M_{n-1}^2 + 1$ for $n \geq 1$,

where $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(M_k; 0 \leq k \leq n)$ for $n \in \mathbb{N}$.

Let $M = (M_n)_{n \in \mathbb{N}}$ be a discrete-time normal martingale. Then, from $M$, we can construct another stochastic process $Z = (Z_n)_{n \in \mathbb{N}}$ as follows:

$$Z_0 = M_0, \quad Z_n = M_n - M_{n-1}, \quad n \geq 1. \tag{2.1}$$

We may view $Z$ as a noise in discrete time, which we call the noise associated with $M$. It can be verified that, as a process on $(\Omega, \mathcal{F}, P)$, $Z$ admits the following two properties:

(i) for each $n \in \mathbb{N}$, $Z_n$ is conditionally centred, that is,

$$\mathbb{E}[Z_n|\mathcal{F}_{n-1}] = 0; \tag{2.2}$$
(ii) for each $n \in \mathbb{N}$, $Z_n$ has a standard conditional quadratic variation, that is, 

$$\mathbb{E}[Z_n^2|\mathcal{F}_{n-1}] = 1.$$ 

Here $\mathcal{F}_n$ is the same as in Definition 2.1.

Recall that $\Gamma$ is the finite power set of $\mathbb{N}$. The next lemma shows that, from the noise $Z$, one can construct an orthonormal system for $L^2(\Omega)$, which is indexed by $\sigma \in \Gamma$.

**Lemma 2.2.** Let $Z_\emptyset = 1$, where $\emptyset$ denotes the empty set and 

$$Z_\sigma = \prod_{i \in \sigma} Z_i, \quad \sigma \in \Gamma, \, \sigma \neq \emptyset.$$ (2.3)

Then the set $\{Z_\sigma \mid \sigma \in \Gamma\}$ forms a countable orthonormal system of $L^2(\Omega)$.

For a proof of this lemma, we refer to [2, 5] or [7]. Using this lemma and related general results in functional analysis [1], we come to the next lemma.

**Lemma 2.3.** There exists a unique isometry $\mathcal{J} : L^2(\Gamma) \to L^2(\Omega)$ such that 

$$\mathcal{J}(f) = \sum_{\sigma \in \Gamma} f(\sigma)Z_\sigma, \quad f \in L^2(\Gamma),$$ (2.4)

where the series is convergent in the norm of $L^2(\Omega)$.

The isometry $\mathcal{J}$ mentioned in Lemma 2.3 is referred to as the full Wiener integral operator [7] and $\mathcal{J}(f)$ the full Wiener integral of $f$.

**Definition 2.4.** The noise $Z$ is said to have the chaotic representation property if the set $\{Z_\sigma \mid \sigma \in \Gamma\}$ is total in $L^2(\Omega)$.

So if the noise $Z$ has a chaotic representation property, the set $\{Z_\sigma \mid \sigma \in \Gamma\}$ actually forms an orthonormal basis of $L^2(\Omega)$. In that case, the full Wiener integral operator $\mathcal{J} : L^2(\Gamma) \to L^2(\Omega)$ becomes an isometric isomorphism.

**Lemma 2.5** [7]. Let the noise $Z$ have the chaotic representation property. Then for each $k \in \mathbb{N}$, there exists a bounded operator $\partial_k$ on $L^2(\Omega)$ such that 

$$\partial_k Z_\sigma = 1_{\sigma\setminus k}(k)Z_{\sigma\setminus k}, \quad \sigma \in \Gamma,$$

where $\sigma \setminus k$ stands for $\sigma \setminus \{k\}$.

The operator $\partial_k$ is called the annihilation operator at $k$ and its dual $\partial_k^*$ the creation operator. As its name suggests, the creation operator has the following property:

$$\partial_k^* Z_\sigma = (1 - 1_{\sigma}(k))Z_{\sigma \cup k}, \quad \sigma \in \Gamma,$$

where $\sigma \cup k$ means $\sigma \cup \{k\}$. See [7] for details about annihilation and creation operators.

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3. Convolution

In this section, we always assume that $M = (M_n)_{n \in \mathbb{N}}$ is a given discrete-time normal martingale on the probability space $(\Omega, \mathcal{F}, P)$. We also assume that the noise $Z = (Z_n)_{n \in \mathbb{N}}$ associated with $M$ has a chaotic representation property (see (2.1) for the meaning of $Z_n$).

So the set $\{Z_\sigma \mid \sigma \in \Gamma\}$ forms an orthonormal basis of $L^2(\Omega)$, where $Z_\sigma$ is defined by (2.3). This means that $\mathcal{F}$ is generated by the noise $Z$ (equivalently, by the normal martingale $M$). Thus we may call random variables on $(\Omega, \mathcal{F}, P)$ functionals of the normal martingale $M$ or functionals of the noise $Z$.

Note that the full Wiener integral operator $J : l^2(\Gamma) \mapsto L^2(\Omega)$ is an isometric isomorphism (see (2.4) for its definition) and $l^2(\Gamma)$ forms an algebra with the usual product given by

$$(fg)(\sigma) = f(\sigma)g(\sigma), \quad \sigma \in \Gamma,$$

where $f, g \in l^2(\Gamma)$. In view of these two facts, we come to the next definition.

**Definition 3.1.** Let $\xi, \eta \in L^2(\Omega)$. Then the convolution $\xi * \eta$ of $\xi$ and $\eta$ is defined as

$$\xi * \eta = J(fg),$$

where $f = J^{-1}(\xi)$ and $g = J^{-1}(\eta)$.

Thus we have an operation $*$ on $L^2(\Omega)$, which we call the convolution. The next two propositions show that, with the convolution as multiplication, $L^2(\Omega)$ becomes a commutative Banach algebra.

**Proposition 3.2.** Let $\xi, \eta, \zeta \in L^2(\Omega)$ and $s, t \in \mathbb{R}$ (the real numbers). Then:

(i) $\xi * \eta = \eta * \xi$;
(ii) $\xi * (\eta * \zeta) = (\xi * \eta) * \zeta$;
(iii) $\xi * (s\eta + t\zeta) = s(\xi * \eta) + t(\xi * \zeta)$.

**Proof.** The proof is straightforward. \(\square\)

**Proposition 3.3.** The convolution is continuous with respect to the norm of $L^2(\Omega)$; more precisely,

$$\|\xi * \eta\| \leq \|\xi\| \|\eta\|, \quad \xi, \eta \in L^2(\Omega),$$

where $\|\cdot\|$ denotes the $L^2(\Omega)$-norm.

**Proof.** Take $f, g \in l^2(\Gamma)$ such that $\xi = J(f)$ and $\eta = J(g)$. Then, by the isometric property of $J$,

$$\|\xi * \eta\| = \|J(fg)\| = \|fg\|_{l^2(\Gamma)} \leq \|f\|_{l^2(\Gamma)}\|g\|_{l^2(\Gamma)} = \|\xi\| \|\eta\|.$$

This completes the proof. \(\square\)
We now use this property to verify (3.3). Let
\[ \xi \in L^2(\Omega). \]
Then for each \( \sigma \in \Gamma \),
\[ \xi \ast Z_{\sigma} = \langle \xi, Z_{\sigma} \rangle Z_{\sigma}, \tag{3.1} \]
where \( \langle \xi, Z_{\sigma} \rangle = \mathbb{E}[\xi Z_{\sigma}] \).

**Proof.** Take \( f \in L^2(\Gamma) \) such that \( \xi = \mathbb{J}(f) \). Then, noticing that \( Z_{\sigma} = \mathbb{J}(1_{\{\sigma\}}) \),
\[ \xi \ast Z_{\sigma} = \mathbb{J}(f 1_{\{\sigma\}}) = \sum_{\tau \in \Gamma} f(\tau) 1_{\{\sigma\}}(\tau) Z_{\tau} = f(\sigma) Z_{\sigma}, \]
which together with \( \langle \xi, Z_{\sigma} \rangle = f(\sigma) \) gives (3.1). \( \square \)

In the following we write \( \mathcal{F}_{-1} = \{\emptyset, \Omega\} \) and \( \mathcal{F}_n = \sigma(M_k; 0 \leq k \leq n) \) for \( n \in \mathbb{N} \). In this way \( (\mathcal{F}_n)_{n \geq -1} \) forms a filtration on \( (\Omega, \mathcal{F}, \mathbb{P}) \). We note that \( \mathcal{F}_n \) can also be expressed in terms of \( Z \), namely \( \mathcal{F}_n = \sigma(Z_k; 0 \leq k \leq n) \).

For \( k \in \mathbb{N} \), we define a functional \( \psi_k \) as
\[ \psi_k = \sum_{\sigma \in \Gamma_k} Z_{\sigma}, \tag{3.2} \]
where \( \Gamma_k = \{\sigma \mid \sigma \subset \{0, 1, \ldots, k\}\} \). Clearly \( \psi_k \in L^2(\Omega) \) for each \( k \in \mathbb{N} \). The next proposition is one of our main results, showing that the conditional expectation operator \( \mathbb{E}[\cdot \mid \mathcal{F}_k] \) can be represented by \( \psi_k \) through the convolution.

**Proposition 3.5.** Let \( \xi \in L^2(\Omega) \) and \( k \in \mathbb{N} \). Then
\[ \mathbb{E}[\xi | \mathcal{F}_k] = \xi \ast \psi_k. \tag{3.3} \]

**Proof.** We first show that \( \mathbb{E}[Z_{\sigma} \mid \mathcal{F}_k] = 0 \) if \( \sigma \in \Gamma \setminus \Gamma_k \). In fact, if \( \sigma \in \Gamma \setminus \Gamma_k \), then \( \sigma \neq \emptyset \) and \( n = \max \{\sigma > k\} \); hence, by the conditionally centred property of \( Z_n \) (see (2.2)),
\[ \mathbb{E}[Z_{\sigma} \mid \mathcal{F}_k] = \mathbb{E}[Z_{\sigma} \mid \mathcal{F}_{n-1}] = 0. \]

We now use this property to verify (3.3). Let \( \xi = \mathbb{J}(f) \) with \( f \in L^2(\Gamma) \). Then
\[ \mathbb{E}[\xi \mid \mathcal{F}_k] = \sum_{\sigma \in \Gamma_k} f(\sigma) \mathbb{E}[Z_{\sigma} \mid \mathcal{F}_k] + \sum_{\sigma \in \Gamma \setminus \Gamma_k} f(\sigma) \mathbb{E}[Z_{\sigma} \mid \mathcal{F}_k] = \sum_{\sigma \in \Gamma_k} f(\sigma) Z_{\sigma}. \]

On the other hand, by Proposition 3.4, we find that
\[ \xi \ast \psi_k = \sum_{\sigma \in \Gamma_k} \xi \ast Z_{\sigma} = \sum_{\sigma \in \Gamma_k} \langle \xi, Z_{\sigma} \rangle Z_{\sigma} = \sum_{\sigma \in \Gamma_k} f(\sigma) Z_{\sigma}. \]
Thus (3.3) holds. \( \square \)

The next proposition suggests that the sequence \( \psi_k, k \in \mathbb{N} \), can be viewed as an approximate identity of the Banach algebra \( (L^2(\Omega), \ast) \).
Proposition 3.6. Let $\xi \in L^2(\Omega)$. Then
\[
\lim_{k \to \infty} \|\xi * \psi_k - \xi\| = 0. \tag{3.4}
\]

Proof. Set $\xi_k = \xi * \psi_k$, $k \in \mathbb{N}$. Then it follows from Proposition 3.5 that
\[
\xi_k = E[\xi|F_k], \quad k \in \mathbb{N}.
\]
It is easy to see that $F = \sigma(\bigcup_{k \in \mathbb{N}} F_k)$. Thus by the well-known martingale convergence theorem (see, for example, [3]) we come to (3.4). \hfill \Box

As an immediate consequence of Proposition 3.5, we have the following version of the Clark formula in discrete time (see, for example, [7]).

Corollary 3.7. For each $\xi \in L^2(\Omega)$,
\[
\xi = E\xi + \sum_{k \in \mathbb{N}} Z_k \partial_k (\xi * \psi_k) = E\xi + \sum_{k \in \mathbb{N}} Z_k [(\partial_k \xi) * \psi_{k-1}],
\]
where $\psi_k$ is defined by (3.2).

4. Integral representation

In this section, we present an example of a discrete-time normal martingale to show that the corresponding convolution has an integral representation.

Let $\Omega = \{-1, 1\}^\mathbb{N}$, the set of all mappings $\omega : \mathbb{N} \to \{-1, 1\}$. Then $\Omega$ is a commutative group with the natural product given by
\[
(\omega_1 \omega_2)(n) = \omega_1(n) \omega_2(n), \quad n \in \mathbb{N},
\]
where $\omega_1, \omega_2 \in \Omega$. Note that this group has 1 as its identity and, moreover, each $\omega \in \Omega$ has itself as its inverse, namely $\omega^{-1} = \omega$.

Let $(Z_n)_{n \in \mathbb{N}}$ be the sequence of canonical projections on $\Omega$ given by
\[
Z_n(\omega) = \omega(n), \quad \omega \in \Omega.
\]
Denote by $\mathcal{F}$ the $\sigma$-field generated by the sequence $(Z_n)_{n \in \mathbb{N}}$. Then (see [5]) there exists a unique probability measure $P$ on $\mathcal{F}$ such that
\[
P \circ (Z_{n_1}, Z_{n_2}, \ldots, Z_{n_k})^{-1}[\{e_1, e_2, \ldots, e_k\}] = \frac{1}{2^k}
\]
for $n_j \in \mathbb{N}$, $e_j \in \{-1, 1\}$ ($1 \leq j \leq k$) with $n_i \neq n_j$ when $i \neq j$ and $k \in \mathbb{N}$ with $k \geq 1$. Note that $P$ is also the only invariant probability measure on the group $\Omega$.

So we come to a probability measure space $(\Omega, \mathcal{F}, P)$ and a sequence $Z = (Z_n)_{n \in \mathbb{N}}$ of independent random variables on it. Define
\[
M_n = \sum_{k=0}^{n} Z_k, \quad n \in \mathbb{N}.
\]
Then \( M = (M_n)_{n \in \mathbb{N}} \) is a discrete-time normal martingale on \((\Omega, \mathcal{F}, P)\). Thus \( Z \) is the noise associated with \( M \), which we call the Bernoulli noise. It can be shown [5] that the Bernoulli noise \( Z \) has the chaotic representation property.

**Proposition 4.1.** Let \( \xi, \eta \in \mathcal{L}^2(\Omega) \). Then the convolution \( \xi \ast \eta \) defined as in Definition 3.1 has the following integral representation:

\[
\xi \ast \eta(\omega_1) = \int_{\Omega} \xi(\omega)\eta(\omega_1 \omega) \, dP(\omega), \quad \text{for } P\text{-almost all } \omega_1 \in \Omega.
\]

**Proof.** Define an operation \( \ast \) on \( \mathcal{L}^2(\Omega) \) as follows:

\[
\xi \ast \eta(\omega_1) = \int_{\Omega} \xi(\omega)\eta(\omega_1 \omega) \, dP(\omega), \quad \omega_1 \in \Omega,
\]

where \( \xi, \eta \in \mathcal{L}^2(\Omega) \). It can be shown that \( \xi \ast \eta \in \mathcal{L}^2(\Omega) \) whenever \( \xi, \eta \in \mathcal{L}^2(\Omega) \) and, moreover, \( \mathcal{L}^2(\Omega) \) becomes a commutative Banach algebra with the operation \( \ast \) as multiplication.

Now let \( \xi, \eta \in \mathcal{L}^2(\Omega) \). To complete the proof, we need only verify that \( \xi \ast \eta = \xi \ast \eta \).

Take \( g \in l^2(\Gamma) \) such that

\[
\eta = \sum_{\sigma \in \Gamma} g(\sigma)Z_\sigma.
\]

For each \( \sigma \in \Gamma \), noticing that \( Z_\sigma(\omega_1 \omega_1) = Z_\sigma(\omega)Z_\sigma(\omega_1) \), \( \omega, \omega_1 \in \Omega \), we have

\[
\xi \ast Z_\sigma(\omega_1) = \int_{\Omega} \xi(\omega)Z_\sigma(\omega_1 \omega) \, dP(\omega) = Z_\sigma(\omega_1) \int_{\Omega} \xi(\omega)Z_\sigma(\omega) \, dP(\omega), \quad \omega_1 \in \Omega,
\]

which together with Proposition 3.4 gives

\[
\xi \ast Z_\sigma = \xi \ast Z_\sigma.
\]

Thus by the continuity of both \( \ast \) and \( \circ \) we get

\[
\xi \ast \eta = \sum_{\sigma \in \Gamma} g(\sigma)\xi \ast Z_\sigma = \sum_{\sigma \in \Gamma} g(\sigma)\xi \ast Z_\sigma = \xi \ast \eta.
\]

This completes the proof. \( \square \)

**References**


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