LINE FORMATION IN MULTI-DIMENSIONAL MEDIA

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ABSTRACT

The flux divergence technique of Athay and Skumanich (1967) is generalized for application to media whose properties vary in more than one spatial dimension. In this method, the flux divergence is viewed as an integro-differential functional of the source function. The source function is then expanded in terms of basis functions along characteristic paths, and, with the help of various interpolations, the flux divergence is converted to an approximate linear algebraic operator on a discrete spatial grid. A large but finite set of linear, inhomogeneous, simultaneous algebraic equations with known matrix coefficients is thus generated and is solved by direct matrix inversion for the source function at each point of the spatial grid.

Some aspects of the accuracy, stability, and computational convenience of the technique are discussed. Sample solutions for depth dependent, axially symmetric variations of temperature are shown.

Key words: radiative transfer in inhomogeneous media, line formation in multi-dimensional media, numerical methods in transfer, matrix methods for integro-differential operators.

I. INTRODUCTION

A casual inspection of a spectroheliogram at almost any wavelength reveals that the radiation field of the solar atmosphere has considerable horizontal structure. The ordinary plane-parallel

idealization is thus at best an average description of the medium, and a more realistic treatment can be provided by model atmospheres whose properties are allowed to vary in two or three dimensions.

The formation of spectrum lines in such media has, until recently, received little attention. Rybicki (1965) has developed Fourier transform techniques suitable for media in which the absorption coefficient is constant with position. Wilson (1968) has published results obtained with a trial and error scheme making use of a three-dimensional equivalent of the Eddington approximation. Avery and House (1968) have adapted the Monte Carlo technique for use in line formation problems and have paid particular attention to spicular geometries.

In the present paper, we will present a new numerical technique for dealing with line formation in multi-dimensional media. The method is a generalization of a one-dimensional technique developed independently by Kuhn (1966) and by Athay and Skumanich (1967). In contrast with the Monte Carlo scheme, it is "deterministic" or non-statistical. The method may be used with variable absorption coefficient and is, in principle, non-iterative for linear problems.

II. BASIC EQUATIONS

We write the time independent transfer equation as

$$\frac{1}{K_\nu(\vec{x})}\,\hat{n}\cdot\vec{\nabla}\,I_\nu(\vec{x},\hat{n}) = S_\nu(\vec{x}) - I_\nu(\vec{x},\hat{n}) \quad . \tag{1}$$

We have used the following notation: $\vec{x}$ is the position vector of a point in the region of interest; $\hat{n}$ is a unit direction vector; K_ν is the absorption coefficient at frequency ν; S_ν is the source function; I_ν is the specific intensity.

In the one-dimensional case it is customary to use a geometric position variable which increases outward (i.e., away from the region of interest) and an optical depth scale increasing in the opposite sense; this results in the multiplication of the right hand side of (1) by -1. We have not adopted

https://doi.org/10.1017/S0252921100151231 Published online by Cambridge University Press

this convention here. As a result certain sign differences will appear when the following analysis is compared with the equations of Athay and Skumanich (1967).

We define two angular moments of the specific intensity by

$$J_\nu(\vec{x}) = \frac{1}{4\pi} \int I_\nu(\vec{x},\hat{n})\, d\Omega \qquad (2)$$

and

$$\vec{H}_\nu(\vec{x}) = \frac{1}{4\pi} \int \hat{n}\, I_\nu(\vec{x},\hat{n})\, d\Omega \qquad (3)$$

where the integrations are over the full domain of the solid angle Ω. If one integrates Eq. (1) over solid angle, one obtains

$$\frac{1}{K_\nu} \vec{\nabla}\cdot\vec{H}\,(\vec{x}) = S_\nu(\vec{x}) - J_\nu(\vec{x}) \quad . \qquad (4)$$

We wish to consider the equation of transfer at frequencies centered about a spectrum line with maximum absorption coefficient at $\nu = \nu_o$. We split the absorption coefficient into the sum of the part arising from bound-bound processes in the line ($K_L(\nu)$) and the part arising from all other processes occurring at the same frequency ($K_c(\nu)$). Thus, at any point $\vec{x}$, we write

$$K_\nu = K_L(\nu) + K_c(\nu) \quad . \qquad (5)$$

To a good approximation $K_c(\nu)$ is independent of frequency at frequencies where $K_L(\nu)$ is appreciable. Defining $K_c = K_c(\nu_o)$ and $r_\nu = K_c/K_L(\nu)$ we have

$$K_\nu = K_L(\nu)(1 + r_\nu) \quad . \qquad (6)$$

Letting $r_o = r_{\nu_o}$, $K_o = K_L(\nu_o)$, and $\Phi_\nu = \dfrac{K_L(\nu)}{K_o}$ we have

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$$K_\nu(\vec{x}) = K_o(\vec{x})(\Phi_\nu(\vec{x}) + r_o(\vec{x})). \qquad (7)$$

Separating the emission coefficient in similar fashion and noting that the source function is the ratio of emission to absorption one obtains

$$S_\nu(\vec{x}) = \frac{\phi_\nu(\vec{x})S_{L,\nu}(\vec{x}) + r_o S_{c,\nu}(\vec{x})}{\Phi_\nu(\vec{x}) + r_o(\vec{x})} \qquad (8)$$

where $S_{L\nu}$ and $S_{c\nu}$ are the line and continuum source functions. For this paper, we will take $S_{c,\nu}(\vec{x}) = B_{\nu_o}(T_e(\vec{x})) = B(\vec{x})$; i.e., for frequencies around the line, the continuum source function is the Planck function at frequency ν_o and local electron temperature $T_e(\vec{x})$. If one assumes "complete redistribution" in frequency of scattered radiation in the rest frame of the atom, the line source function is frequency independent. For a two-level atom the statistical equilibrium equation may be written in the form

$$S_L = \frac{\bar{J} + \varepsilon B}{1 + \varepsilon} \qquad (9)$$

where

$$\bar{J} = \int_o^\infty \Phi_\nu J_\nu \, d\nu \qquad (10)$$

and

$$\varepsilon = \frac{C_{UL}}{A_{UL}} \left(1 - \exp\left\{ - h\nu_o/kT_e \right\}\right) . \qquad (11)$$

Here, Φ_ν is the profile coefficient normalized to unity $\left(\Phi_\nu = \phi_\nu / \int_o^\infty \phi_\nu d\nu\right)$, C_{UL} is the collisional rate per atom from the upper to lower levels in the atom, and A_{UL} is the Einstein A coefficient.

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To obtain a single equation for the source function, one can use the transfer equation to eliminate $\bar{J}$ from Eq. (9). Operating on (1) with

$$\int_0^\infty d\nu \Phi_\nu ,$$

we have that

$$\int \frac{\Phi_\nu}{K_\nu} \vec{\nabla}\cdot\vec{H}_\nu d\nu = \int \Phi_\nu S_\nu d\nu - \bar{J}. \tag{12}$$

Substituting (8) and (9) into (12) one has that

$$\int \frac{\Phi_\nu}{K_\nu} \vec{\nabla}\cdot\vec{H}_\nu d\nu = S_L \int \frac{\Phi_\nu \phi_\nu d\nu}{\phi_\nu + r_o} - (1+\varepsilon) S_L + \varepsilon B + B \int \frac{r_o \Phi_\nu d\nu}{\phi_\nu + r_o} \quad . \tag{13}$$

We define

$$\delta = \int \frac{r_o \Phi_\nu d\nu}{\phi_\nu + r_o}$$

and note that since

$$\frac{\phi_\nu}{\phi_\nu + r_o} = 1 - \frac{r_o}{\phi_\nu + r_o} ,$$

$$\int \frac{\phi_\nu \Phi_\nu d\nu}{\phi_\nu + r_o} = 1 - \delta \quad . \tag{14}$$

Putting (14) into (13) and combining terms, one finally has that

$$S_L + \frac{1}{\varepsilon+\delta} \int_0^\infty \frac{\Phi_\nu}{K_\nu} \vec{\nabla}\cdot\vec{H}_\nu d\nu = B \quad . \tag{15}$$

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We will use (15) as the basic equation in developing our multi-dimensional technique. It is incomplete, however, until we relate $\vec{\nabla}\cdot\vec{H}_\nu$ to S_L and B i.e., until we express $\vec{H}_\nu$ as a functional of S_L and B. To find the needed relation we must turn to the formal solution of the equation of transfer.

At any point $\vec{x}$ we form characteristics or paths, $p(\vec{x},\hat{n})$, which are straight lines passing through $\vec{x}$ in direction $\hat{n}$. If s represents a geometric length along the path, measured from some arbitrary origin and increasing in direction $\hat{n}$, the vector $\vec{x}^P(s)$ of any point on the path is given by

$$\vec{x}^P(s) = \vec{x} + \hat{n}\ (s-s_o) \qquad (16)$$

where s_o is the distance from $\vec{x}$ to the point at which s = o. Figure 1 illustrates the path geometry.

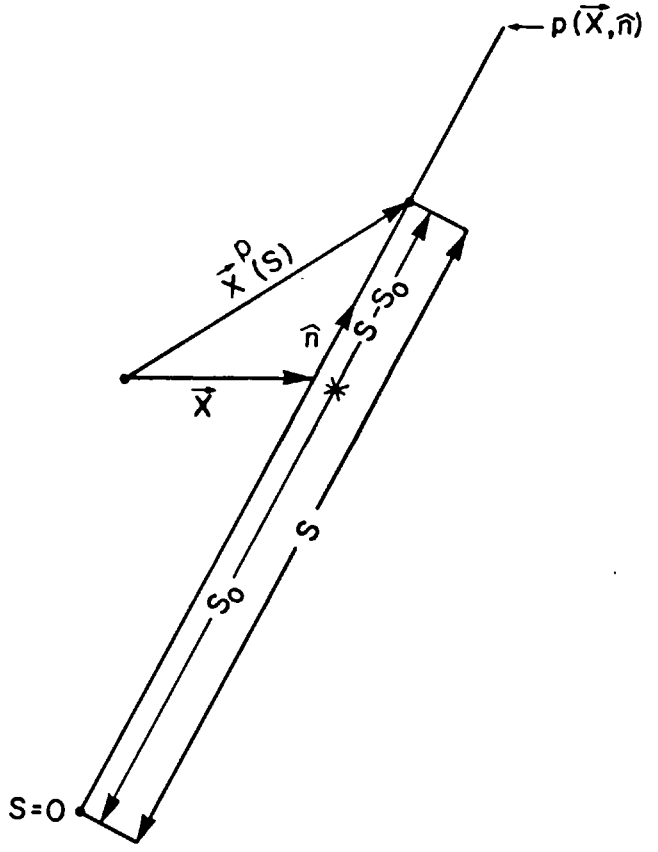

Figure 1. Geometry of a path $p(\vec{x},\hat{n})$ passing through $\vec{x}$ in direction $\hat{n}$. Geometric length s is measured from an arbitrary origin and increases in the same sense as $\hat{n}$. s_o is the length from the origin to $\vec{x}$, and $\vec{x}^P(s)$ is the position vector of an arbitrary point on the path.

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We define an optical path length, t_ν, by

$$t_\nu(s) = \int_0^s K_\nu(\vec{x}^p(s'))\, ds' \tag{17}$$

and adopt the following conventions: a quantity evaluated along the path will be denoted by a script letter. If it is regarded as a function of geometric path length, a superscript s will be included, while if no superscript appears, the quantity is understood to be a function of optical path length at frequency ν.
Thus

$$S_\nu(\vec{x}^p(s)) = \mathcal{S}_\nu^s(s) = \mathcal{S}_\nu(t_\nu(s)), \tag{18}$$

$$I_\nu(\vec{x}^p(s)) = \mathcal{I}_\nu^s(s) = \mathcal{I}_\nu(t_\nu(s)), \tag{19}$$

$$K_\nu(\vec{x}^p(s)) = k_\nu^s(s) = k_\nu(t_\nu(s)). \tag{20}$$

It must be remembered that our path notation implies that a script quantity is an implicit function in $\vec{x}$ and $\hat{n}$ and explicit in path length.

With these definitions and conventions, the equation of transfer (1) may be rewritten as

$$\frac{1}{k_\nu(s)} \frac{d\mathcal{I}_\nu^s}{ds} = \frac{d\mathcal{I}_\nu}{dt_\nu} = \mathcal{S}_\nu - \mathcal{I}_\nu . \tag{21}$$

Equation (21) has the well-known solution for $\mathcal{I}_\nu$ in terms of $\mathcal{S}_\nu$

$$\mathcal{I}_\nu(t_\nu) = e^{-t_\nu}\mathcal{I}_\nu(o) + e^{-t_\nu}\int_0^{t_\nu} \mathcal{S}_\nu(t'_\nu) e^{t'_\nu}\, dt'_\nu . \tag{22}$$

We will assume that $\vec{x}^p(s=0)$ is at a boundary and either $\mathcal{I}_\nu(0) = 0$ or is specified by appropriate boundary conditions. Since we shall be considering half-spaces we shall assume that $\mathcal{I}_\nu(0)$ is suffi-

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ciently bounded so that $e^{-t_\nu}\mathcal{J}_\nu(0)\to 0$ for $t_\nu\to\infty$.

In Eq. (15) we are interested in the quantity

$$\int_{-\infty}^{\infty} \frac{\Phi_\nu}{k_\nu} \vec{\nabla}\cdot\vec{H}_\nu d\nu \ .$$

We note that

$$\frac{1}{K_\nu} \vec{\nabla}\cdot\vec{H}_\nu = \frac{1}{4\pi} \int \frac{d\mathcal{J}_\nu}{dt_\nu} d\Omega \ . \tag{23}$$

Note that $d\mathcal{J}_\nu/dt_\nu$ depends implicitly on direction and location. Our final functional relation is that

$$\int^{\infty} \frac{\Phi_\nu}{K_\nu} \vec{\nabla}\cdot\vec{H}_\nu d\nu =$$

$$= \frac{1}{4\pi} \int_0^{\infty} \int_\Omega \frac{\Phi_\nu}{K_\nu} \frac{d}{dt_\nu} \left(e^{-t_\nu} \int_0^{t_\nu} \mathcal{J}_\nu(t'_\nu) e^{t'_\nu} \, dt'_\nu \right) d\Omega d\nu . \tag{24}$$

An alternate expression can be obtained by transforming the integration in (24) to a volume integration. For example, one can show (cf. among others, Rybicki 1965) that

$$\vec{H}_\nu(\vec{x}) = \frac{1}{4\pi} \int_V K_\nu(\vec{x}') S_\nu(\vec{x}') e^{-\tau_\nu(\vec{x},\vec{x}')} \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} d^3\vec{x}' \tag{25}$$

where V is the volume of interest. Here, $\tau_\nu(\vec{x},\vec{x}')$ is the optical path length along the characteristic between the "field point" $\vec{x}$ and the "source point" $\vec{x}'$. In this case, Eq. (24) becomes

$$\int_{-\infty}^{\infty} \frac{\Phi_\nu}{K_\nu} \vec{\nabla}\cdot\vec{H}_\nu d\nu =$$

$$= \frac{1}{4\pi} \int_0^{\infty} \frac{\Phi_\nu}{K_\nu} \vec{\nabla}\cdot \int_V S_\nu(\vec{x}') K_\nu(\vec{x}') e^{-\tau_\nu(\vec{x},\vec{x}')} \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} d^3\vec{x}' .$$

(26)

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In either case, we see that the weighted frequency integral of the flux divergence is an integro-differential operator on the total monochromatic source function. We call this operator α_V if (26) is to be used and α_p if (24) is used; more precisely, we write

$$\frac{1}{\epsilon+\delta}\int_{-\infty}^{\infty}\frac{\Phi_\nu}{K_\nu}\nabla\cdot\vec{H}_\nu d\nu = \alpha_V(S_\nu) = \alpha_p(\mathscr{A}_\nu).$$

Note particularly that α is a linear operator, i.e.,

$$\alpha(a\,S_\nu^1 + b\,S_\nu^2) = a\alpha(S_\nu^1) + b\alpha(S_\nu^2) \qquad (27)$$

where a and b are arbitrary constants, α is α_V or α_p and S_ν^1 and S_ν^2 are arbitrary functions of appropriate arguments. The linearity of α holds only so long as k_ν is independent of S_ν, a condition that is approximately satisfied in some, but by no means all, situations of astrophysical interest. If the operator is non-linear, then iterative techniques must be employed.

In general form, then (15) may be written

$$S_L + \alpha_p\mathscr{A}_\nu = S_L + \alpha_V S_\nu = B. \qquad (28)$$

Using (8) and (27) we have that*

$$S_L + \alpha_V\left(\frac{\phi_\nu S_L}{\phi_\nu + r_o}\right) = B - \alpha_V\left(\frac{r_o B}{\phi_\nu + r_o}\right) = B'. \qquad (29)$$

Letting $\alpha_V^\circ(S_L) = \alpha_V\left(\frac{\phi_\nu S_L}{\phi_\nu + r_o}\right)$ and $\alpha_V'(B) = \alpha_V\left(\frac{r_o B}{\phi_\nu + r_o}\right)$,

we have

$$(\underline{1} + \alpha_V^\circ)\,S_L = (\underline{1} - \alpha_V')B = B' \qquad (30)$$

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* Note that (29) holds even if α is non-linear since it remains linear in the factor S_ν.

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where $\underline{1}$ is the unit operator. We regard, ε, δ, and B' as known functions and wish to solve for S_L. Formally,

$$S_L(\vec{x}) = (\underline{1} + \alpha_\nu^\circ)^{-1} B'(\vec{x}) \quad . \tag{31}$$

We now seek an algebraic representation of α. The basic step is to assume that we may represent S_ν in terms of its values on a finite grid of space points $\{\vec{x}_n;\ n = 1,2, \ldots N\}$, i.e., we assume

$$S_\nu(\vec{x}) \simeq \sum_{n'=1}^{N} P_{\nu,n'}(\vec{x}) S_\nu(\vec{x}_{n'}) \tag{32}$$

where the functions $\{P_{\nu,n}(\vec{x});\ n = 1,2, \ldots, N\}$ are presumed known** but are left unspecified for the moment. We let

$$\alpha^\circ_n(\vec{x}) = \alpha_V\left(\frac{\phi_\nu}{\phi_\nu + r_o} P_{\nu,n}\right) = \alpha_p\left(\frac{\phi_\nu}{\phi_\nu + r_o} P_{\nu,n}(\vec{x}^p(t_\nu))\right)$$

and

$$\alpha'_n(\vec{x}) = \alpha_V\left(\frac{r_o}{\phi_\nu + r_o} P_{\nu,n}\right) = \alpha_p\left(\frac{r_o}{\phi_\nu + r_o} P_{\nu,n}(\vec{x}^p(t_\nu))\right) \quad .$$

Then (29) becomes

$$S_L(\vec{x}) + \sum_{n'=1}^{N} \alpha^\circ_{n'}(\vec{x}) S_L(\vec{x}_{n'}) = B'(\vec{x}) \tag{33}$$

with

$$B'(\vec{x}) = B(\vec{x}) - \sum_{n'=1}^{N} \alpha'_{n'}(\vec{x}) B(\vec{x}_{n'}) \quad . \tag{34}$$

** If the $P_{\nu,n}(\vec{x})$ are known in terms of the spectral functions of the operator α, then the truncation error in (32) is, in principle, known.

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Setting $B'(x_n) = B'_n$, $S_L(x_n) = S_n$, and evaluating (33) at each of the grid points one obtains the N-fold set of equations

$$S_n + \sum_{n'=1}^{N} \alpha^{\circ}_{n'}(\vec{x}_n) S_{n'} = B'_n \quad , \quad n = 1, \ldots, N. \tag{35}$$

Letting $\alpha^{\circ}_{nn'} = \alpha^{\circ}_{n'}(\vec{x}_n)$, we have

$$\sum_{n'} (\delta_{nn'} + \alpha^{\circ}_{nn'}) S_{n'} = B'_n \tag{36}$$

where $\delta_{nn'}$ is the Kronecker delta. Clearly, (36) is the algebraic analog of (30). Defining $Q_{nn'} = \delta_{nn'} + \alpha^{\circ}_{nn'}$, one finally has

$$\sum_{n'} Q_{nn'} S_{n'} = B'_n \tag{37}$$

We have thus expressed αS at any point of our grid as a linear combination of S's at every other point on the grid; i.e., we have formed a set of N inhomogeneous, simultaneous, linear, algebraic equations with known coefficients (provided we can evaluate $\alpha^{\circ}_{n'}(\vec{x}_n)$) which may be cast in matrix-vector form and solved by some suitable standard matrix inversion technique. The method is non-iterative if a direct inversion scheme such as Gauss elimination is used.

The method is limited in practice by the size of the matrices that can be calculated, stored, and inverted. Even a modest grid may produce quite a large matrix; for example, for a 30 × 30 two-dimensional grid, the matrix Q has order of approximately 1,000. If the matrix is full, roughly 10^6 numbers must be calculated and stored and a 1,000 × 1,000 matrix must be inverted. A high speed, large-scale computer is thus a necessity.

In view of the scale of the computations, a desirable characteristic of the method is that it be spatially stable, that is, the solutions should be relatively insensitive to the "fineness" of the discrete grid. One reason for choosing (15) as the basic operator equation is that previous experience in one-dimension (cf. Athay and Skumanich 1967) indicates just this kind of stability.

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III. EVALUATION OF THE FLUX DIVERGENCE OPERATOR

The method outlined in the preceding section illustrates the form of the procedure we have used but masks considerable detail. In particular, we have not said how to evaluate $\alpha(P_{\nu,n}(\vec{x}))$; nor have we specified what functions $P_{\nu,n}(\vec{x})$ are suitable for our purposes. We proceed to examine both of these points in more detail.

To evaluate $\alpha(P_{\nu,n})$ for any known function $P_{\nu,n}(\vec{x})$ we may use either (24) or (26). Equation (26) has the more concise analytic form and may appear to be more suitable at first sight. However, useful closed-form expressions for $\alpha_n(\vec{x})$ are difficult to obtain even for simple functions $P_{\nu,n}$ and standard two-dimensional geometries. They are nearly impossible to obtain if one considers a wide class of problems with variable absorption coefficient (possibly given in tabular form, for example). We have tried approximating volume integrals of the form in (25) by certain types of quadrature formulae and have met with some success. However, we have not yet been able to satisfactorily carry out the full set of operations in (26). Further, the methods so far devised have proven extremely inefficient with regard to computing time. Some of the difficulty may be seen in the behavior of the kernel

$$e^{-\tau_\nu(\vec{x},\vec{x}')} \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

which has a singularity (although integrable) at $\vec{x} = \vec{x}'$ and whose variation differs considerably in different parts of the range of $\vec{x} - \vec{x}'$.

We have found equation (24) to be more suitable for numerical purposes. The operations involved here are fairly simple and are in a form where variable absorption coefficient can be handled more conveniently.

The disadvantage in the use of (24) is that the dependence of the operators on the basic spatial coordinates $\vec{x}$ is now implicit. One must thus be careful to preserve the order of the operations and must be prepared to relate functions evaluated along a path to those evaluated in the basic coordinate system.

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Though much of what follows is applicable to general geometries, we now confine our attention to a two-dimensional, axisymmetric medium. The atmosphere is bounded by a plane at $z = 0$ and is semi-infinite in the z-coordinate. We take z to increase into the atmosphere. The axis of symmetry is perpendicular to the boundary, distance from the axis is denoted by r, and $0 \leq r \leq \infty$. Directions $\hat{n}$ at points (r,z) in the medium are specified by two angles, θ and ϕ, such that $\hat{n}\cdot\hat{r} = \sin\theta\cos\phi$ and $\hat{n}\cdot\hat{z} = \cos\theta$, where r and z are local unit vectors in the direction of increasing coordinate.

We assume, as we have indicated before, that we can write along a path

$$\mathcal{J}_\nu(t_\nu) \simeq \sum_{\ell=1}^{N} c_\ell f_\ell(t_\nu) \tag{38}$$

where the f_ℓ are the quadratic basic functions of Avrett and Loeser (1963),

$$\begin{aligned} f_1(t) &= 1 \\ f_\ell(t) &\begin{cases} = (1 - \frac{t}{t_\ell})^2 & o \leq t \leq t_\ell \\ = 0 & t > t_\ell \end{cases} \quad \ell = 2,3,\ldots,N-1 \\ f_N(t) &= t. \end{aligned} \tag{39}$$

We choose the constants c_ℓ such that

$$\mathcal{J}_\nu(t_m) = \sum_{\ell=1}^{N} c_\ell\, f_\ell(t_m) \equiv \sum_{\ell=1}^{N} F_{m\ell} c_\ell \quad . \tag{40}$$

Thus

$$c_\ell = \sum_{m=1}^{N} F_{\ell m} \mathcal{J}_\nu(t_m) \tag{41}$$

or

$$\mathcal{J}_\nu(t_\nu) \simeq \sum_{m=1} \left\{ \sum_{\ell=1} f_\ell(t_\nu) F_{\ell m}^{-1} \right\} \mathcal{J}_\nu(t_m). \tag{42}$$

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For a given set of t_m, the $F_{\ell m}^{-1}$ are easy to compute (cf. Athay and Skumanich 1967). We shall use the same set of t_m for each frequency. The geometric point along the path corresponding to a given t_m is therefore a function of frequency and we relate $\mathcal{J}_\nu(t_m)$ to $\mathcal{J}^S_\nu(s_k)$ for a fixed set of s_k by an interpolatory transformation due to Kalkofen (1967), i.e.,

$$\mathcal{J}_\nu(t_m) = \sum_k V_{mk}(\nu,\vec{x},\hat{n})\mathcal{J}^S_\nu(s_k) \tag{43}$$

where the transformation depends on the frequency ν and the path $p(\vec{x},\hat{n})$. We note here that any spatial variations in k_ν are introduced via V_{mk}.

Similarly, we transform $\mathcal{J}^S_\nu(s_k)$ to $S_\nu(\vec{x}_{n'})$ by an interpolatory routine such that

$$\mathcal{J}^S_\nu(s_k) = \sum_{n'} T_{kn'}(\vec{x},\hat{n})S_\nu(\vec{x}_{n'}). \tag{44}$$

Thus we have that

$$\mathcal{J}_\nu(t_\nu) \simeq \sum_{\ell,m,k,n'} f_\ell(t_\nu)F_{\ell m}^{-1}V_{mk}T_{kn'}S_\nu(\vec{x}_{n'})$$

$$= \sum_{n'} P_{\nu,n'}(t_\nu)S_\nu(\vec{x}_{n'}) = \sum_{n'} P_{\nu,n'}(\vec{x}^P(t_\nu))S(\vec{x}_{n'}). \tag{45}$$

The elements of V and T will be discussed in a later section. We now proceed to evaluate $\alpha_p(P_{\nu,n'}(t_\nu))$.

According to (24) the first step is to find

$$\mathcal{J}_\nu(t_\nu) = e^{-t_\nu}\int_0^{t_\nu}\mathcal{J}_\nu(t'_\nu)e^{t_\nu'}dt'_\nu \quad . \tag{46}$$

To accomodate the semi-infinite geometry more conveniently, we will now restrict our paths to have directions $\hat{n}$ with $0 \leq \theta \leq \pi/2$ and will evaluate $\mathcal{J}_\nu(t_\nu)$ in both the $+\hat{n}$ and $-\hat{n}$ directions. We therefore define

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$$\left.\begin{aligned} \mathcal{I}_\nu^+(t_\nu^+) &= \mathcal{I}_\nu(t_\nu)_{(\vec{x},\theta,\phi)} \\ \mathcal{I}_\nu^-(t_\nu^+) &= \mathcal{I}_\nu(t_\nu)_{(\vec{x},\pi-\theta,\phi+\pi)} \end{aligned}\right\} 0 \leq \theta \leq \frac{\pi}{2} \qquad (47)$$

where the subscripts give the arguments defining the path. The origin of geometric and optical path length is taken to be at the intersection of the path and the boundary plane. The notation t_ν^+ implies optical path length measured in the above way and increasing into the medium, while t_ν on the right hand side of (47) implies optical path length increasing in the same sense as $\hat{n}$ implied in the subscripts. Thus

$$\frac{d\mathcal{I}_\nu(t_\nu)}{dt_\nu}\Bigg)_{(x,\pi-\theta,\phi+\pi)} = -\frac{d\mathcal{I}_\nu^-(t_\nu^+)}{dt_\nu^+}\Bigg)_{(\vec{x},\theta,\phi)} .$$

We will henceforth drop the + from t_ν, always understand that our paths are now restricted, and remember that we must use

$$-\frac{d\mathcal{I}_\nu^-}{dt_\nu}$$

in evaluating our operator.

Now Eq. (46) yields

$$\mathcal{I}_\nu^+ = e^{-t_\nu}\int_0^{t_\nu} \mathcal{J}_\nu(t'_\nu)e^{t'_\nu}dt'_\nu \equiv \mathcal{L}_\nu^+(\mathcal{J}_\nu) \qquad (48)$$

and

$$\mathcal{I}_\nu^{-,} = e^{t_\nu}\int_{t_\nu}^{\infty} \mathcal{J}_\nu(t'_\nu)e^{-t_\nu}\,dt'_\nu \equiv \mathcal{L}_\nu^-(\mathcal{J}_\nu) \qquad (49)$$

Letting $G_\ell^\pm(t_\nu) = \mathcal{L}_\nu^\pm(f_\ell)$, we have that

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$$\mathcal{D}_{\nu}^{\pm}(t_{\nu}) = \sum_{\ell,m,k,n'} \mathcal{G}_{\ell}^{\pm}(t_{\nu}) F_{\ell m}^{-1} V_{mk} T_{kn'} S_{\nu}(\vec{x}_{n'}) \qquad (50)$$

where

$$\mathcal{G}_{1}^{+}(t_{\nu}) = 1 - e^{-t_{\nu}}$$

$$\mathcal{G}_{\ell}^{+}(t_{\nu}) =$$

$$= \begin{cases} \left(1-\frac{t_{\nu}}{t_{\ell}}\right)^2 + \frac{2}{t_{\ell}}\left(1-\frac{t_{\nu}}{t_{\ell}}\right) + \frac{2}{t_{\ell}^2} - e^{-t_{\nu}}\left(1+\frac{2}{t_{\ell}} + \frac{2}{t_{\ell}^2}\right); 0 \leq t_{\nu} \leq t_{\ell} \\ \frac{2e^{-(t_{\nu}-t_{\ell})}}{t_{\ell}^2} - e^{-t_{\nu}}\left(1+\frac{2}{t_{\ell}} + \frac{2}{t_{\ell}^2}\right); \; t_{\ell} < t_{\nu} < \infty \end{cases}$$

$$\ell = 2,3,\ldots N-1 \qquad (51)$$

$$\mathcal{G}_{N}^{+}(t_{\nu}) = e^{-t_{\nu}} + t_{\nu} - 1$$

and

$$\mathcal{G}_{1}^{-}(t_{\nu}) = 1$$

$$\mathcal{G}_{\ell}^{-}(t_{\nu}) =$$

$$= \begin{cases} \frac{-2e^{-(t_{\ell}-t_{\nu})}}{t_{\ell}^2} + \left[\left(1-\frac{t_{\nu}}{t_{\ell}}\right)^2 - \frac{2}{t_{\ell}}\left(1-\frac{t_{\nu}}{t_{\ell}}\right) + \frac{2}{t_{\ell}^2}\right]; 0 \leq t_{\nu} \leq t_{\ell} \\ 0; \; t_{\ell} < t_{\nu} < \infty \end{cases}$$

$$\ell = 2,3,\ldots N-1 \qquad (52)$$

$$\mathcal{G}_{N}^{-}(t_{\nu}) = 1 + t_{\nu} \quad .$$

We note that $G^{\pm}$ is the "path" analog of the G-function introduced by Athay and Skumanich (1967).

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We can now take the derivative of (50) by differentiating (51) and (52). Note that the f_ℓ in (38) are well suited for this procedure since both the f_ℓ and their first derivatives are continuous over the entire range of the argument. Thus, we have that, letting

$$d\mathcal{G}_\ell^{\pm}(t_\nu)/dt_\nu = \dot{\mathcal{G}}_\ell^{\pm} \quad ,$$

$$\frac{d\mathcal{J}_\nu^{\pm}(t_\nu)}{dt_\nu} = \sum_{\ell,m,k,n'} \dot{\mathcal{G}}_\ell^{\pm}(t_\nu) F_{\ell m}^{-1} V_{mk} T_{kn'} S_\nu(\vec{x}_{n'}) \qquad (53)$$

$$\dot{\mathcal{G}}_1^{+}(t_\nu) = e^{-\tau_\nu} \qquad (54)$$

$$\mathcal{G}_\ell^{+}(t_\nu) = \begin{cases} e^{-t_\nu}\left(1+\frac{2}{t_\ell}+\frac{2}{t_\ell^2}\right)-\frac{2}{t_\ell}\left(1-\frac{t_\nu}{t_\ell}+\frac{1}{t_\ell}\right); 0\leq t_\nu \leq t_\ell \\ e^{-t_\nu}\left(1+\frac{2}{t_\ell}+\frac{2}{t_\ell^2}\right)-\frac{2e^{-(t_\nu-t_\ell)}}{t_\ell^2}; \; t_\ell < t_\nu < \infty \end{cases}$$

$$\ell = 2,3,\ldots N-1$$

$$\dot{\mathcal{G}}_N^{+}(t_\nu) = 1 - e^{-t_\nu}$$

and that

$$\dot{\mathcal{G}}_1^{-}(t_\nu) = 0$$

$$\dot{\mathcal{G}}_\ell^{-}(t_\nu) =$$

$$\begin{cases} -2\left[\frac{e^{-(t_\ell-t_\nu)}}{t_\ell}-\frac{1}{t_\ell^2}+\frac{1}{t_\ell}\left(1-\frac{t_\nu}{t_\ell}\right)\right]; \; 0 \leq t_\nu \leq t_\ell \\ 0; \; t_\ell < t_\nu < \infty \end{cases}$$

$$\ell = 2,3,\ldots,N-1 \qquad (55)$$

$$\dot{\mathcal{G}}_N^{-}(t_\nu) = 1 \quad .$$

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As indicated previously (cf. pg. 151) we have transformed $\mathcal{I}$ defined on an "internal" space of path grid points $t_\nu = \{t_\ell, \ell = 1,\ldots,N\}$ to an "external" space $s = \{s_\ell, \ell = 1,\ldots N\}$ via $\mathcal{I} = V\mathcal{I}^s$. If we were to proceed in strict analogy to the one-dimensional case, we would evaluate $\mathcal{G}^{\pm}_\ell(t_\nu)$ at each of the internal points $\{t_n, n = 1,\ldots,N\}$. Then, letting $(\mathcal{G}^{\pm}F^{-1})_{n,m} = \sum_\ell \mathcal{G}^{\pm}_\ell(t_n)F^{-1}_{\ell m}$, we would transform the vector $(\mathcal{G}^{\pm}F^{-1})V\mathcal{I}^s$, which is defined on the internal grid to a vector on the external grid by some other (inverse) interpolatory scheme represented by a matrix W. In other words,

$$(\dot{\mathcal{G}}^{\pm}F^{-1})_{ext} = W(\dot{\mathcal{G}}^{\pm}F^{-1})_{int}V \quad ,$$

(cf. Kalkofen 1968). Because of the simple nature of the dependence of $\mathcal{G}^{\pm}_\ell(t_\nu)$ on t_ν, we have not found this necessary. Instead, we calculate $t_\nu(\vec{x},\hat{n})$ for every ν, $\vec{x}$, and $\hat{n}$ and proceed analytically. This is done conveniently by precalculating the following quantities:

$$h_m = \sum_{\ell=2}^{N-1} F^{-1}_{\ell m}\left(1 + \frac{2}{t_\ell} + \frac{2}{t_\ell^2}\right)$$

$$U^+_{1m} = 0$$

$$U^+_{nm} = 2\sum_{\ell=2}^{n} \frac{F^{-1}_{\ell m}\, e^{-(t_\ell - t_{n+1})}}{t_\ell^2}$$

$$U^-_{nm} = 2\sum_{\ell=n+1}^{N-1} \frac{F^{-1}_{\ell m}\, e^{-(t_\ell - t_{n+1})}}{t_\ell^2}$$

$$V^+_{nm} = 2\sum_{\ell=n+1}^{N-1} \frac{F^{-1}_{\ell m}}{t_\ell^2}$$

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$$W^{+}_{nm} = 2 \sum_{\ell=n+1}^{N-1} \frac{F^{-1}_{\ell m}}{t_\ell} \left(1 + \frac{1}{t_\ell}\right)$$

$$W^{-}_{nm} = W^{+}_{nm} - 2V^{+}_{nm}$$

$$U^{-}_{N-1,m} = U^{-}_{N,m} = V^{+}_{N-1,m} = V^{+}_{N,m} = W^{\pm}_{N-1,m} = W^{\pm}_{N,m} = 0 \quad .$$

Using the above quantities, one can show that

$$(\dot{\mathcal{G}}^{+}F^{-1})_{t_\nu,m} = \sum_{\ell=1}^{N} \dot{\mathcal{G}}^{+}_{\ell}(t_\nu) F^{-}_{\ell m} = e^{-t_\nu}\left(h_m + F^{-1}_{1m} - F^{-1}_{Nm}\right)$$
$$- e^{-(t_\nu - t_n)} U^{+}_{nm} + t_\nu V^{+}_{nm}$$
$$- W^{+}_{nm} + F^{-1}_{Nm}$$

and

$$(\dot{\mathcal{G}}^{-}F^{-1})_{t_\nu,m} = \sum_{\ell=1}^{N} \dot{\mathcal{G}}^{-}_{\ell}(t_\nu) F^{-1}_{\ell m} = -e^{-(t_{n+1} - t_\nu)} U^{-}_{nm} + t_\nu V^{+}_{n,m}$$
$$- W^{-}_{n,m} + F^{-1}_{N,m}$$

where $t_n < t_\nu \leq t_{n+1}$. Thus $(\dot{\mathcal{G}}^{\pm}F^{-1})_{t_\nu,m}$ can be evaluated for any t_ν by calculating three exponentials, finding n (i.e., the interval in which t_ν lies), and performing a small number of multiplications and additions.

The remaining operations are integrations over solid angle and frequency. We first consider

$$\frac{1}{K_\nu} \vec{\nabla} \cdot \vec{H}_\nu = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \left(\frac{d\mathcal{J}_\nu}{dt_\nu}\right)_{(\vec{x},\theta,\phi)} \quad . \qquad (56)$$

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Noting that a characteristic passes through the same r and z points for both $+\phi$ and $-\phi$, we have

$$\frac{1}{K_\nu}\nabla\cdot\vec{H}_\nu = \frac{1}{2\pi}\int_0^\pi d\phi \int_0^\pi \sin\theta\, d\theta \left(\frac{d\ell_\nu}{dt_\nu}\right)_{(\vec{x},\theta,\phi)} . \quad (57)$$

We approximate both integrals by quadrature formulae. Although many schemes are possible, we have used Gauss-Legendre quadrature for the θ - integration and Gauss-Chebyshev quadrature for the ϕ -integration (cf. *Handbook of Mathematical Functions*, AMS 55, 1964). Thus

$$\frac{1}{K_\nu}\nabla\cdot\vec{H} = \frac{1}{2\pi}\sum_{\mu=1}^{M}\sum_{\lambda=1}^{L} W_\mu W_\lambda \left[\left(\frac{d\ell_\nu}{dt_\nu}\right)_{(\vec{x},\cos^{-1}(\gamma_\mu),\phi_\lambda)} + \left(\frac{d\ell_\nu}{dt_\nu}\right)_{(\vec{x},\cos^{-1}(-\gamma_\nu),\phi_\lambda)}\right] . \quad (58)$$

In (58) 2M is the order of the Legendre quadrature, L is the order of the Chebyshev quadrature, γ_μ are the appropriate roots of Legendre polynomials, and $\cos\phi_\lambda$ are roots of Chebyshev polynomials. Noting that $\cos^{-1}(-\gamma_\mu) = \pi - \cos^{-1}(\gamma_\mu)$, we have

$$\frac{1}{K_\nu}\vec{\nabla}\cdot\vec{H} = \frac{1}{2\pi}\sum_{\mu=1}^{M}\sum_{\lambda=1}^{L} W_\mu W_\lambda \left[\left(\frac{d\ell_\nu^+}{dt_\nu}\right)_{(\vec{x},\cos^{-1}(\gamma_\mu),\phi_\lambda)} - \left(\frac{d\ell_\nu^-}{dt_\nu}\right)_{(\vec{x},\cos^{-1}(\gamma_\mu),\pi-\phi_\lambda)}\right] . \quad (59)$$

Recalling that $\frac{d\ell_\nu^\pm}{dt_\nu}$ is even in ϕ and noting that $\phi_\lambda = \pi\frac{(2\lambda-1)}{2L}$ we see that $\pi-\phi_\lambda = +\phi_{L+1-\lambda}$. The W_λ are

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equal ($W_\lambda = \frac{\pi}{L}$) and since we sum over all λ, (57) becomes

$$\frac{1}{K}\vec{\nabla}\cdot\vec{H} = \frac{1}{2\pi}\sum_{\mu,\lambda} W_\mu W_\lambda \left[\frac{d\mathfrak{J}_\nu^+}{dt_\nu} - \frac{d\mathfrak{J}_\nu^-}{dt_\nu}\right]_{\mu,\lambda}$$

where

$$\left(\frac{d\mathfrak{J}_\nu^\pm}{dt_\nu}\right)_{\mu,\lambda} =$$

$$= \sum_{\ell,m,k,n'} \dot{\mathfrak{G}}_\ell^\pm(t_\nu) F_{\ell m}^{-1} V_{mk}(\nu,\vec{x},\hat{n}_{\mu\lambda}) T_{kn'}(\vec{x}\,\hat{n}_{\mu\lambda}) S_\nu(\vec{x}_{n'}). \tag{60}$$

The frequency integral is reduced to the following quadrature formula

$$\int \phi_y f(y)dy = \sum_{\zeta=1}^{x} c_\zeta f(y_\zeta) + c_{x+1} f_\infty \tag{61}$$

where f_∞ is the asymptotic value of $f(y)$ as $y \to \infty$. The weights c_ζ are computed following a quadrature procedure developed by one of us (Skumanich 1966) in which $f(y)$ is approximated by piecewise linear segments with the subsequent integrals performed analytically. We have simply chosen y_{x+1} to be sufficiently large so that

$$\int \phi_y f(y)dy = \sum_{\zeta=1}^{x+1} c_\zeta f_\zeta \quad . \tag{62}$$

Here

$$y = (\nu-\nu_o)/\Delta\nu_D \quad . \tag{63}$$

Thus

$$\Phi_y = \frac{\phi_y}{\int_\infty^\infty \phi_\nu d\nu} = \frac{\Delta\nu_D \phi_y}{n} \tag{64}$$

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where

$$n = \int_{-\infty}^{\infty} \phi_y dy \quad . \tag{65}$$

Then

$$\int_0^{\infty} d\nu \frac{\Phi_\nu}{K_\nu} \vec{\nabla}\cdot\vec{H}_\nu = \frac{2}{n} \int_{-\infty}^{\infty} dy \frac{\phi_y}{K_y} \vec{\nabla}\cdot\vec{H}_y \simeq \frac{2}{n} \sum_{\zeta=1}^{x+1} \frac{c_\zeta}{K_\zeta} \vec{\nabla}\cdot\vec{H}_{y_\zeta} . \tag{66}$$

Recalling the definitions of $\alpha_n^{0'}(\vec{x})$ and $\alpha_n^{1'}(\vec{x})$ we have

$$\alpha_n^{0'}(\vec{x}) = \frac{1}{\pi n(\varepsilon(\vec{x}) + \delta(\vec{x}))} \sum_{\zeta,\mu,\lambda} \sum_{\ell,m,k} c_\zeta(\vec{x}) W_\lambda W_\mu$$

$$\times \left[\dot{G}_\ell^+(t_\nu(\vec{x},\hat{n}_{\mu\lambda})) - G_\ell^-(t_\nu(\vec{x},\hat{n}_{\mu\lambda}))\right] \tag{67}$$

$$\times \; F_{\ell m}^{-1} V_{mk}(\nu,\vec{x},\hat{n}_{\mu\lambda}) T_{kn'}(\vec{x},\hat{n}_{\mu\lambda}) \frac{\phi_\zeta(x_{n'})}{\phi_\zeta(\vec{x}_{n'}) + r_o(\vec{x}_{n'})} \quad .$$

Note that $\alpha_n^{1'}(\vec{x})$ is given by the same formula with

$$\frac{\phi_\zeta(\vec{x}_{n'})}{\phi_\zeta(\vec{x}_{n'}) + r_o(\vec{x}_{n'})}$$

replaced by $r_o(\vec{x}_{n'})/(\phi_\zeta(\vec{x}_{n'}) + r_o(\vec{x}_{n'}))$. We then evaluate $\alpha_n^{0'}$ and $\alpha_n^{1'}$ at all points $(\vec{x}_n)$ and proceed to form and solve the linear algebraic equations as outlined in §II.

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IV. INTERPOLATORY TRANSFORMATIONS

For the transformation of functions of a single variable we have used a subroutine, MAPPAR, written by Kalkofen (1967). The basic step is to suppose that a function in a given domain is well represented by an interpolation function. A "backward" parabola is fitted such that for a function f(x') on the interval $x_j \leq x' < x_{j+1}$

$$f(x') \simeq \frac{(x'-x_j)(x'-x_{j+1})}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})} f_{j-1} + \frac{(x'-x_{j-1})(x'-x_{j+1})}{(x_j-x_{j-1})(x_j-x_{j+1})} f_j$$

$$+ \frac{(x'-x_{j-1})(x'-x_j)}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)} f_{j+1} \quad . \tag{68}$$

For x' in the same interval a "forward" parabola is given by

$$f(x') \simeq \frac{(x'-x_{j+1})(x'-x_{j+2})}{(x_j-x_{j+1})(x_j-x_{j+2})} f_j + \frac{(x'-x_j)(x'-x_{j+2})}{(x_{j+1}-x_j)(x_{j+1}-x_{j+2})} f_{j+1}$$

$$+ \frac{(x'-x_j)(x'-x_{j+1})}{(x_{j+2}-x_j)(x_{j+2}-x_{j+1})} f_{j+2} \quad . \tag{69}$$

In MAPPAR, except for the end intervals, a weighted sum of the two is taken; the backward weight is P, the forward Q = 1-P. Then, for $x_j \leq x'_i \leq x_{j+1}$

$$f(x'_i) \simeq \sum_{j'=j-1}^{j+2} V_{ij'} f_{j'} \tag{70}$$

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where

$$V_{ij-1} = P \frac{(x_i'-x_j)(x_i'-x_{j+1})}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})} \quad , \tag{71}$$

$$V_{ij} = P\frac{(x_i'-x_{j-1})(x_i'-x_{j+1})}{(x_j-x_{j-1})(x_j-x_{j+1})} + Q\frac{(x_i'-x_{j+1})(x_i-x_{j+2})}{(x_j-x_{j+1})(x_j-x_{j+2})} \quad , \tag{72}$$

$$V_{ij+1} = \tag{73}$$

$$= P \frac{(x_i'-x_{j-1})(x_i'-x_j)}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)} + Q\frac{(x_i'-x_j)(x_i'-x_{j+2})}{(x_{j+1}-x_j)(x_{j+1}-x_{j+2})} \quad ,$$

and

$$V_{ij+2} = Q \frac{(x_i'-x_j)(x_i'-x_{j+1})}{(x_{j+2}-x_j)(x_{j+2}-x_{j+1})} \quad . \tag{74}$$

Note particularly that the V's can be expressed in matrix form and depend only on the relation of x_i' to x_j and not on the f's. Also, the x_i' and x_j must refer to the same variable so that in our problem, where we wish to find $\mathscr{L}^s(s)$, we must first find either $s(t_\nu)$ or $t_\nu(s)$, where

$$s(t_\nu) = \int_0^{t_\nu} \frac{dt_\nu'}{K_\nu(\vec{x}^p(t'_\nu))} \quad . \tag{75}$$

$$t_\nu(s) = \int_0^s K_\nu(\vec{x}^p(s'))ds' \quad . \tag{76}$$

A similar but somewhat simpler formula is used for finding T, the transformation between $\mathscr{L}^s(s)$ and S(r,z). We assume that it is approximately valid to write

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$$\mathcal{S}^s(s) = S(r^p(s),z^p(s)) = \frac{(r^p(s)-r_{i+1})(z^p(s)-z_{j+1})}{(r_i-r_{i+1})(z_j-z_{j+1})} S_{i,j}$$

$$+ \frac{(r^p(s)-r_{i+1})(z^p(s)-z_j)}{(r_i-r_{i+1})(z_{j+1}-z_j)} S_{i,j+1}$$

$$+ \frac{(r^p(s)-r_i)(z^p(s)-z_{j+1})}{(r_{i+1}-r_i)(z_j-z_{j+1})} S_{i+1,j}$$

$$+ \frac{(r^p(s)-r_i)(z^p(s)-z_j)}{(r_{i+1}-r_i)(z_{j+1}-z_j)} S_{i+1,j+1} \quad (77)$$

whenever $r_i \leq r^p(s) < r_{i+1} \leq r_I$ and $z_j \leq z^p(s) \leq z_{j+1} \leq z_J$. Letting $n = i + I(j-1)$ we have

$$S(\vec{x}^p(s_k)) = \sum_n T_{kn} S_n \quad (78)$$

where the non-zero elements of T are given from (77).

In the cylindrical geometry which we have chosen

$$r^p(s) =$$

$$[r^2+(z \tan\theta - s\sin\theta)^2 - 2r(z\tan\theta - s\sin\theta)\cos\phi]^{1/2} \quad (79)$$

and

$$z^p(s) = s\cos\theta \quad (80)$$

where r and z are coordinates through which the path $p(r,z,\theta,\phi)$ passes. To change geometries, one must carefully choose the s = 0 point with respect to the boundary configuration and appropriately alter the $\vec{x}^p(s)$ functions.

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Boundary conditions on the present geometry are that $S_L(r,z) \to B_\infty(z)$ as $z \to \infty$, where $B_\infty(z) = \lim_{z\to\infty} B(r,z)$ and is independent of r. Also $S_L(r,z) \to S_L(\infty,z)$ as $r \to \infty$, i.e., for large r the source function reverts to the one-dimensional case. These conditions are used in the mappings by assuming that S_L is linear (as is B) in τ_c for $z \geq z_J$ and $S_L(r_I,z)$ for $r \geq r_I$. Thus we must be sure to make r_I and z_J sufficiently large.

V. RESULTS AND CONCLUSIONS

Some sample solutions obtained with the above procedure are shown in Figures 2-7. In each case, all parameters except temperature are constant with

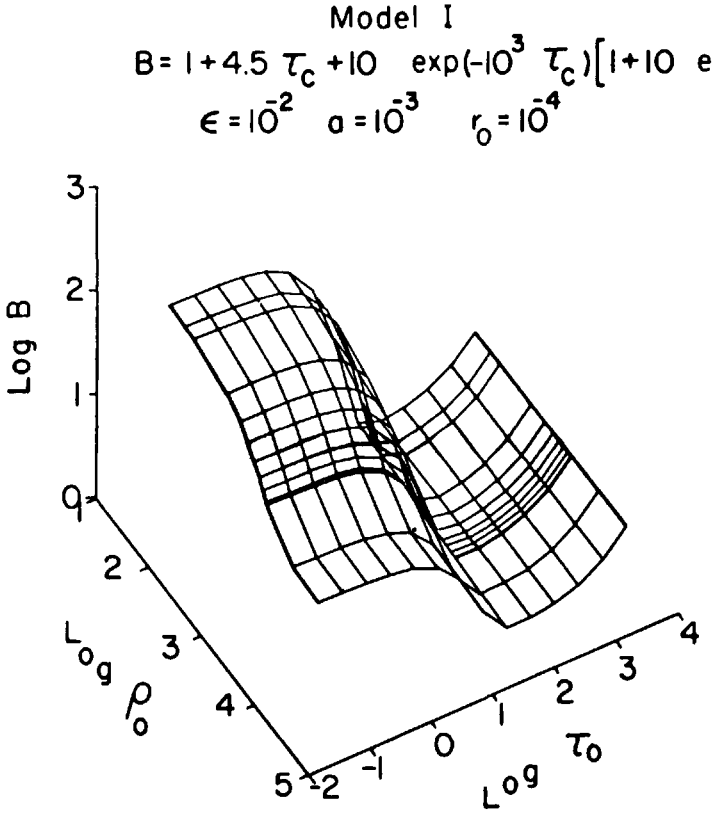

Figure 2. Planck function surface for Model I. Log $B(\rho_o,\tau_o)$ is plotted vs. log ρ_o and log τ_o. τ_o is the total optical depth at line center and ρ_o is the total line center radial optical distance from the axis.

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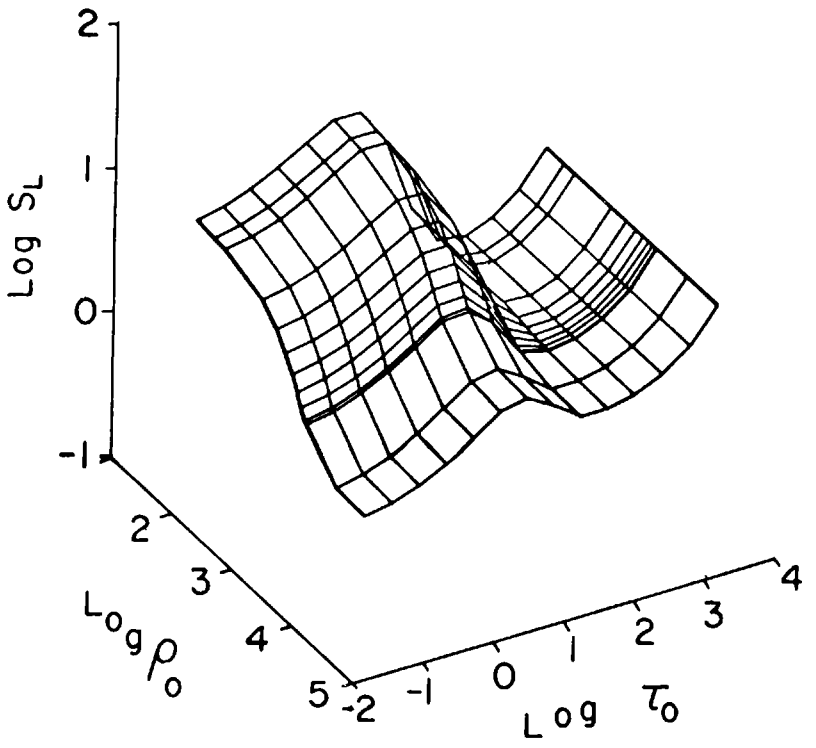

Figure 3. Source function surface, $S_L(\rho_o,\tau_o)$ for Model I.

position. We have arbitrarily chosen $\varepsilon = 10^{-2}$, $a = 10^{-3}$, and $r_o = 10^{-4}$ while the Planck functions has form

$$B(\rho_c,\tau_c) = 1 + Ae^{-c\tau_c}\left(1 + Be^{-(d\rho_c)^m}\right) + \beta\tau_c$$

where τ_c is the continuous vertical optical depth and ρ_c is the continuous radial optical depth measured from the axis. The absorption coefficient at line center is assumed to be 1.

We have in mind a temperature structure designed to mimic a hot "chromospheric" column ($\rho_c<1/d$) imbedded in a cooler ambient "chromosphere" ($\rho_c>1/d$) and overlying a homogeneous, plane-parallel "photosphere" $\tau_c \gg 1/c$ with linear vertical gradient in B. The model is merely for numerical testing, however, and we will not attempt to draw from our results any direct conclusions about lines actually observed in the solar atmosphere. The models are nevertheless not unlike possible choices one might make in attempting to treat certain solar lines.

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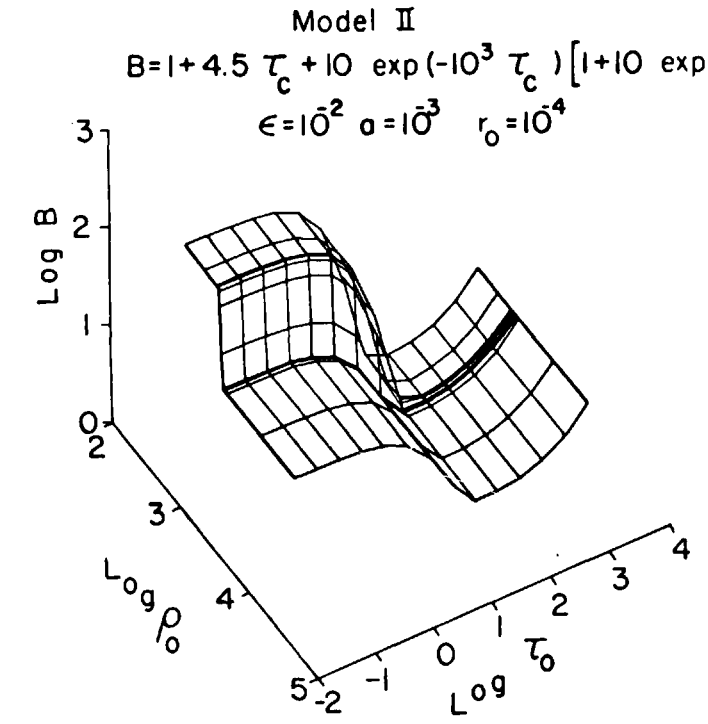

Figure 4. Planck function for Model II.

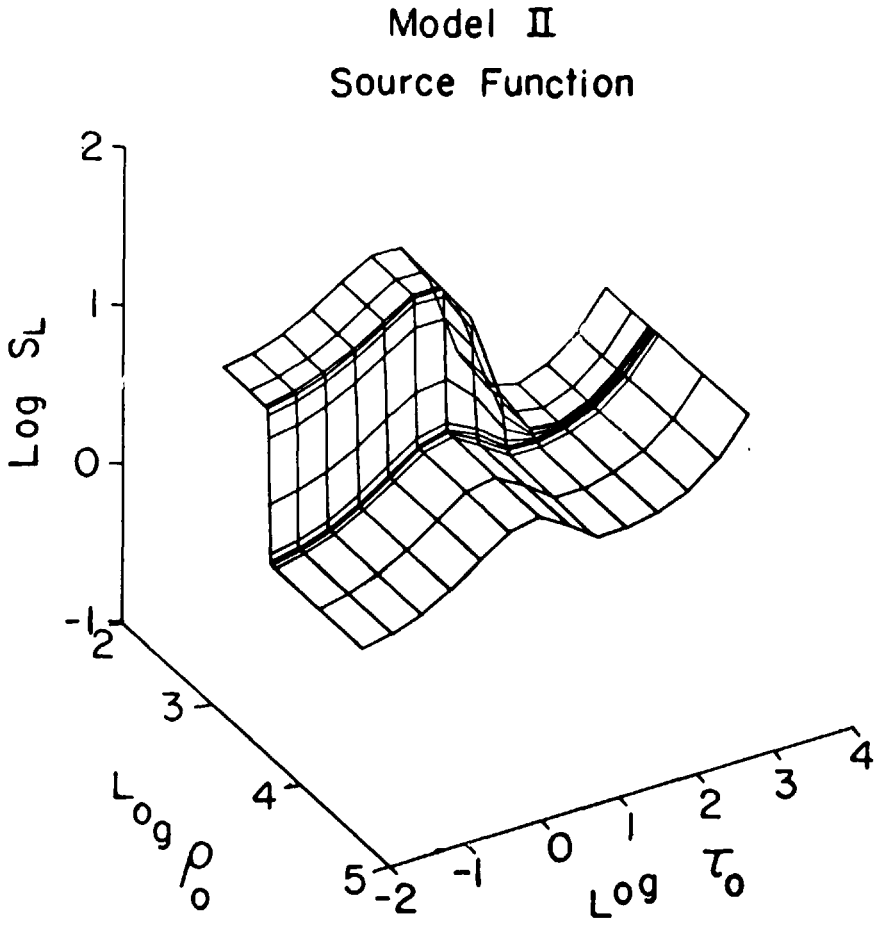

Figure 5. Source function for Model II.

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The figures show B and S_L surfaces with the vertical axis being log B (or log S_L) and the horizontal axis being the logarithm of the respective total optical coordinate at line center. In all cases A = B = 10, and c = 10^3 (corresponding to a "chromospheric" optical thickness of about 10 at line center). The parameter d controls the radial thickness of the hot inhomogeneity while m and d control the "edge scale of the radial variation of B.

In the model I solution, the radial thickness is 10^3 at line center and m = 1 so that the edge scale is also of order 10^3. All horizontal scales are thus larger than a thermalization length ($\sim 10^2$). It is not surprising that the two-dimensional solution for any value of ρ is very nearly that which would be obtained from a strictly plane-parallel atmosphere with the vertical behavior of the Planck function the same as at the given radius in the two-dimensional case.

Model II differs from Model I in that m = 50, or the edge scale is ~ 20. Detailed examination of the solution shows that the source function reaches its asymptotic radial behavior at a distance of roughly a thermalization length on either side of the region of rapid radial variation of the Planck function. (This is not evident in Figure 5 due to the large scale of the ρ-axis compared to a thermalization length). Even though the edge scale for B is less than a thermalization length, the radial dependence of S_L is suprisingly similar to that of B. It is possible that the solution is incorrect. However, both the radial thickness and the edge scale are larger than the vertical thickness of the "chromosphere." Since both scales are coupled in the term $\vec{\nabla} \cdot \vec{H}_\nu$, a possible explanation of the result is that the vertical scale still dominates. In terms of photon diffusion, a photon originating at a given radius can escape from the surface of the atmosphere or diffuse to the homogeneous photosphere more readily than it can diffuse to a different radius with noticeably different temperature.

In Model III, m = 1 and the radial thickness is 1. The radial scale should dominate, then, at least at intermediate depths. The solution, however, is unstable in the sense that oscillations appear which have no apparent physical cause. One would also expect the source function near the axis to be less for a radially thin region than for a radially thick one. However, the sharp peak in S_L in Model III at $\tau_o \sim 10$ near the axis is about twice the value of S_L for Model I in the corresponding region.

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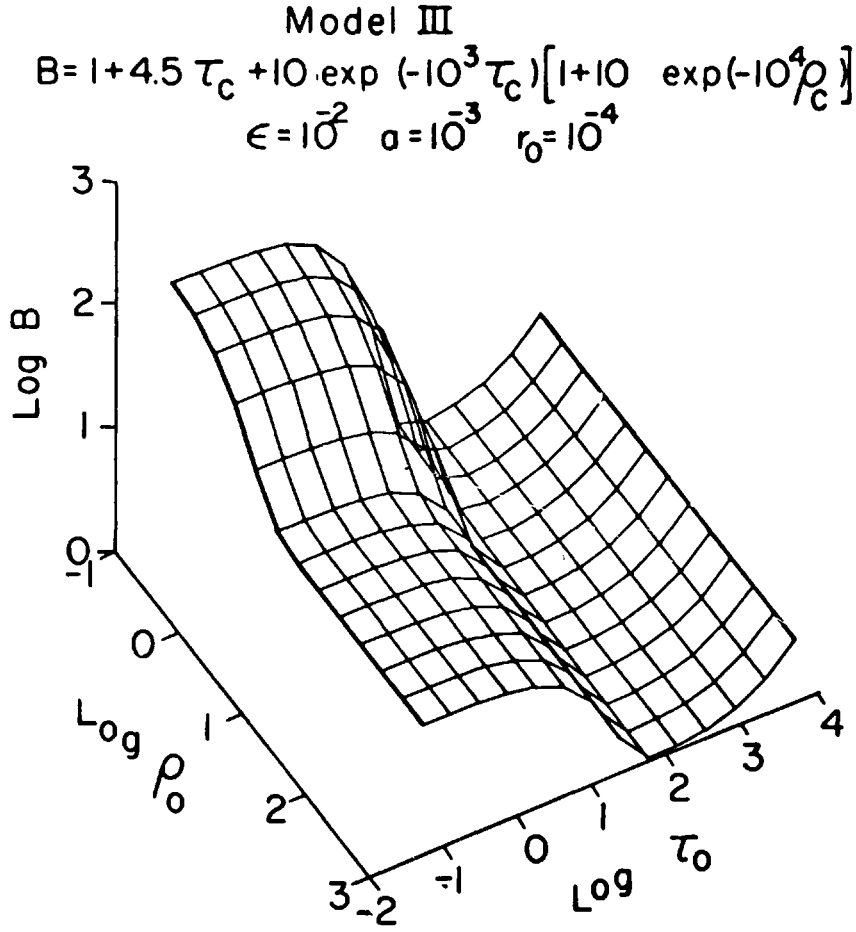

Figure 6. Planck function for Model III.

The source function behavior can be made smoother with some adjustment of grids, but the peak remains. Apparently, then, no physically realistic solution has been obtained for the third model.

Some simple tests of the method have been made. First, we must obtain the correct solution in the plane-parallel case. We have compared our technique with independent plane-parallel calculations for $B = 1 + Ae^{-c\tau_c} + \beta\tau_c$ with $c = 10^3$, $\beta = 4.5$, the same ϵ, a, and r_o as before, and A = 10, 100, and 1000. The two-dimensional code with a 10 × 10 $\rho \times \tau$ grid agrees to better than 10 percent in all cases with a typical error of less than 4 percent.

For a thick inhomogeneity, the solutions near the axis and at very large radii must also reduce to the corresponding plane-parallel limit. Again, our solutions satisfy this requirement to about the accuracy indicated above.

With the exception that points must be nested

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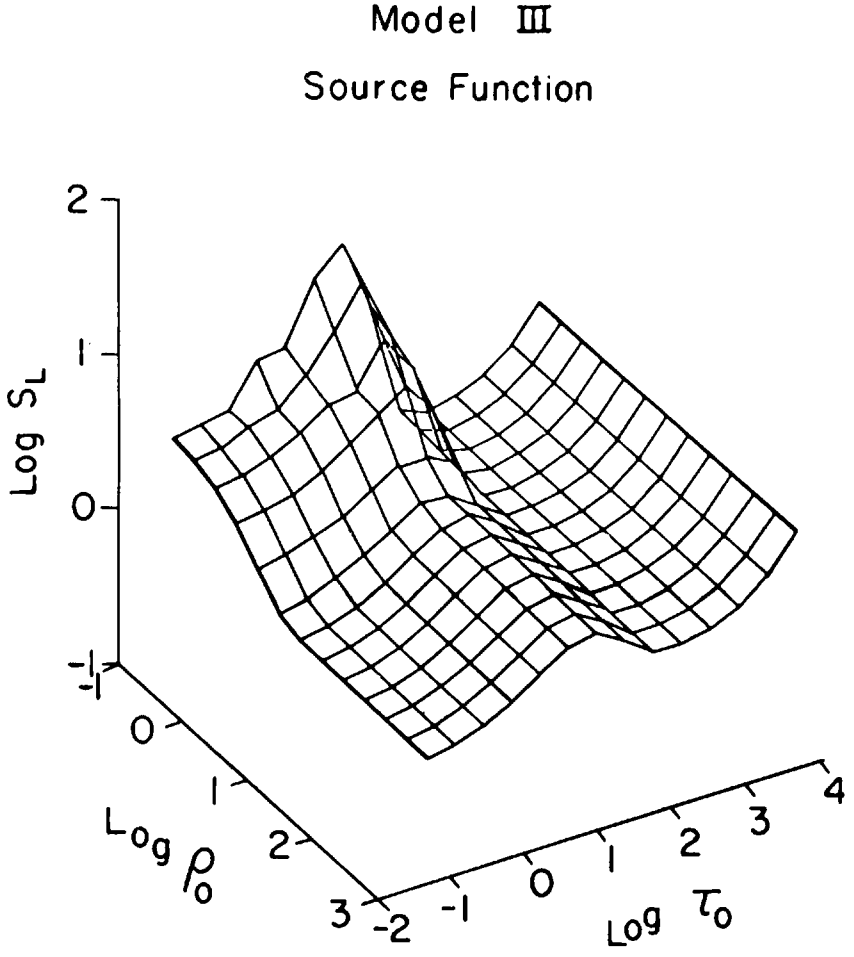

Figure 7. Source function for Model III.

so that a reasonable number lie in regions of Planck function variation, the nesting of the spatial grid has not proven critical. Different types and orders of angular quadratures have been tried and again, do not have much effect on the solution to within the sort of accuracy indicated above.

The selections of the t_ℓ (the path grid) does seem to be important in determining the solution. In every case that has been so far attempted, oscillatory solutions appear at intermediate depths ($\tau_o \gtrsim 2$) whenever the radial behavior of the Planck function is sharp enough to dominate; these are just the solutions which are sensitive to choice of path grid. Numerical experiments to determine more exactly how different choices of path origin and path grid affect the numerical stability of the technique are in progress.

The technique is quite convenient for media with an absorption coefficient which is constant or varies

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only with depth. Central processing time for such a problem with a 10 × 10 grid is about 45 seconds on the CDC 6600 and increases linearly with the total number of space points (i.e., to about 90 seconds for a 14 × 14 grid). Fully variable opacity will of course cause an increase in computing time.

In spite of the stability problems encountered to date, we feel the method holds great promise. In principle, it is able to deal with quite arbitrary spatial behavior and boundary configurations and is particularly well-suited for media with very large optical extent.

The authors wish to thank Professor R. D. Richtmyer for valuable discussion in the early stages of this work.

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DISCUSSION

Underhill: Did you use depth-dependent functions in your calculations?

Skumanich: We considered only horizontal

changes in the examples shown. But we have also calculated other models, as indicated in the text.

Underhill: Can you include the granulation?

Skumanich: We have thought it best to calculate simple nonstatistical structures initially. The granulation problem, if modeled by periodic structures, can and will be included ultimately.

https://doi.org/10.1017/S0252921100151231 Published online by Cambridge University Press