Normal and invertible composition operators

R.K. Singh and D.K. Gupta

Let \( N \) denote the set of natural numbers and let \( \phi \) be a mapping from \( N \) into itself. Then the composition transformation \( C_\phi \) on the weighted \( l^2 \) space with weights \( a^{2n} \), where \( n \in N \) and \( 0 < a < 1 \) is defined by \( C_\phi f = f \circ \phi \). If \( C_\phi \) is a bounded operator, then it is called a composition operator. The adjoint of the composition operator \( C_\phi \) is computed, and it is used to characterise normal, unitary, isometric, and co-isometric composition operators. Not every invertible \( \phi \) induces an invertible composition operator, as is shown by examples. At the end of this note all invertible composition operators are characterised.

1. Preliminaries

Let \( N \) denote the set of non-zero positive integers and let \( \lambda \) be the measure on \( N \) defined by \( \lambda(\{n\}) = \lambda_n = a^{2n} \) for every \( n \in N \), where \( 0 < a < 1 \). Let \( l^2_a \) denote the space of all complex sequences such that

\[
l^2_a = \{ g : N \rightarrow \mathbb{C} \ | \sum_{n=1}^{\infty} \lambda_n |g(n)|^2 < \infty \}.
\]

Then \( l^2_a \) is a Hilbert space under pointwise addition and scalar multiplication with the inner product defined by

\[
\langle f, g \rangle = \sum_{n=1}^{\infty} \lambda_n f(n) \overline{g(n)}.
\]

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If \( \phi \) is a mapping from \( N \) into itself, we define a composition transformation \( C_\phi \) on \( l^2_a \) into the space of all complex valued functions on \( N \) by

\[
C_\phi f = f \circ \phi \quad \text{for all } f \in l^2_a.
\]

If the range of \( C_\phi \) is in \( l^2_a \) and \( C_\phi \) is bounded, then we call \( C_\phi \) a composition operator induced by \( \phi \). By \( B\{l^2_a\} \) we mean the Banach algebra of bounded linear operators on \( l^2_a \).

In Section 2 of this paper we compute the adjoint of \( C_\phi \) and, using this, we characterise normal, unitary, and isometric composition operators. In Section 3 of this paper invertible composition operators are characterised.

If \( \phi \) is a mapping on \( N \) into itself such that \( C_\phi \in B\{l^2_a\} \), then the measure \( \lambda\phi^{-1} \) is absolutely continuous with respect to \( \lambda \). We denote the Radon-Nikodym derivative of \( \lambda\phi^{-1} \) with respect to \( \lambda \) by \( f_0 \). In the case of \( l^2 \) there is a \( \phi \) such that \( \phi \) is not the identity map, but \( f_0 = 1 \) (for example any bijection other than the identity). In the case of \( l^2_a \), it is not so, as is proved in the following lemma.

**Lemma 1.1.** Let \( \phi \) be a mapping from \( N \) into itself and \( f_0 \) be the Radon-Nikodym derivative of the measure \( \lambda\phi^{-1} \) with respect to \( \lambda \). Then \( f_0 = 1 \) if and only if \( \phi \) is the identity.

Proof. Let \( \phi \) be the identity. Then \( \phi(n) = n \) for all \( n \in N \). Hence \( f_0(n) = d\lambda\phi^{-1}(n)/d\lambda n = d\lambda n/d\lambda n = 1 \) for all \( n \in N \).

The converse is proved by induction. We first prove that \( \phi(1) = 1 \). Since \( f_0(n) = d\lambda\phi^{-1}(n)/d\lambda n = 1 \), we get \( \lambda\phi^{-1}(n) = \lambda n \) for all \( n \in N \).

If \( \phi(1) \neq 1 \), let \( \phi(1) = m \) where \( m \neq 1 \). Then \( 1 \in \phi^{-1}(m) \). Hence
\[ \lambda_1 \leq \lambda \phi^{-1}(m) = \frac{1}{m}, \] which is impossible, for \( \lambda \) is a decreasing measure.

Thus \( \phi(1) = 1 \).

Let us suppose that this result is true for 1, 2, ..., \( k \), that is \( \phi(j) = j \) for \( j = 1, 2, ..., k \); we prove \( \phi(k+1) = k + 1 \). If this is not so, then \( \phi(k+1) = m \), where \( m \neq k + 1 \).

**CASE I.** If \( m > k + 1 \), then \( k + 1 \in \phi^{-1}(m) \). Therefore

\[ \lambda_{k+1} \leq \lambda \phi^{-1}(m) = \frac{1}{m}, \] which is a contradiction, since \( \lambda_m < \lambda_{k+1} \).

**CASE II.** If \( m < k + 1 \), then \( \phi(m) = m \). Thus \( \{m, k+1\} \subset \phi^{-1}(m) \).

Hence \( \lambda_m + \lambda_{k+1} \leq \lambda \phi^{-1}(m) = \frac{1}{m} \), which is again a contradiction, since \( \lambda_{k+1} \neq 0 \). Therefore \( \phi(k+1) = k + 1 \), and hence the induction process is complete. Thus the proof of the lemma is finished.

2. Normal and unitary composition operators

For the characterisation of normal composition operators we need a familiarity with the nature of the adjoint of such operators. The computation of the adjoint of a composition operator \( C_\phi \) on the \( L^2 \) of a general measure space is very hard. But in the case of \( L^2_\alpha \), the adjoint \( C^*_\phi \) is computable. The following theorem computes the adjoint of \( C_\phi \).

**THEOREM 2.1.** Let \( C_\phi \in B\left(L^2_\alpha\right) \) and \( C^*_\phi \) be defined by

\[ (C^*_\phi g)(n) = \frac{1}{\lambda} \int g d\lambda \]

for all \( g \in L^2_\alpha \) and \( n \in \mathbb{N} \). Then \( (C_\phi f, g) = (f, C^*_\phi g) \) for all \( f, g \in L^2_\alpha \).

**Proof.** Since
\[ \langle C_\phi f, g \rangle = \int_N (C_\phi f)(m)\overline{g}(m)d\lambda \]
\[ = \sum_{n=1}^{\infty} \int_{\phi^{-1}(n)} (f \circ \phi)(m)\overline{g}(m)d\lambda \]
\[ = \sum_{n=1}^{\infty} \int_{\phi^{-1}(n)} f(n)\overline{g}(m)d\lambda \]
\[ = \sum_{n=1}^{\infty} f(n) \int_{\phi^{-1}(n)} \overline{g}(m)d\lambda \]
\[ = \sum_{n=1}^{\infty} f(n)\lambda_n [C_\phi^* g](n) \]
\[ = \langle f, C_\phi^* g \rangle , \]

\( C_\phi^* \) is the adjoint of \( C_\phi \).

On \( l^2 \), there are plenty of normal composition operators other than the identity operator, as every invertible composition operator in this is normal \([4]\). But strangely enough on \( l^2_a \) there is no non-trivial normal composition operator. This is shown in the following theorem.

**Theorem 2.2.** Let \( C_\phi \in B\left(l^2_a \right) \). Then \( C_\phi \) is normal if and only if \( \phi \) is the identity.

In order to prove this theorem we need the following lemma.

**Lemma 2.3.** Let \( \phi : N \to N \) be a one-to-one and onto mapping. Then \( \phi(m) + \phi^{-1}(m) = 2m \) for all \( m \in N \) implies that \( \phi \) is the identity.

**Proof.** Let \( \phi(1) = n \). Then \( \phi(n) = 2n - 1 \). Let \( \phi^{-1}(1) = m \). Now since \( \phi(1) + \phi^{-1}(1) = 2 \), we have \( n + m = 2 \) which is possible only when \( n = m = 1 \). Thus \( \phi(1) = 1 \). Let us suppose that the result is true for \( 1, 2, \ldots, k \). We prove it for \( k + 1 \). Let \( \phi(k+1) = n \); then \( \phi(n) = 2n - (k+1) \). Let \( \phi^{-1}(k+1) = m \). Then since \( \phi(k+1) + \phi^{-1}(k+1) = 2(k+1) \), \( n + m = 2(k+1) \). Since \( n \) and \( m \) are not less than or equal to \( k \), we conclude \( \phi(k+1) = k + 1 \) and
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\( \phi^{-1}(k+1) = k + 1 \). Hence, by induction, \( \phi(n) = n \) for all \( n \in \mathbb{N} \), which implies that \( \phi \) is the identity.

Proof of Theorem 2.2. The sufficiency is obvious. To prove the necessary part, let \( C_\phi \) be normal and \( e^m \) be the sequence defined by

\[ e^m(p) = \delta_{mp} \] (the Kronecker delta). Then \( \|C_\phi e^m\|^2 = \|C_\phi^m\|^2 \) for all \( m \in \mathbb{N} \). Now

\[ \|C_\phi e^m\|^2 = \sum_{n=1}^{\infty} \lambda_n |C_\phi^m(n)|^2. \]

But since

\[ \left( C_\phi e^m \right)(p) = \frac{1}{\lambda_p} \int_{\phi^{-1}(p)} e^m(p) d\lambda \] and \( m \in \phi^{-1}(p) \) for only one value of \( p \), we get

\[ \left( C_\phi e^m \right)(p) = \frac{1}{\lambda_p} \int_{\{m\}} e^m(m) d\lambda = \frac{\lambda_m}{\lambda_p}. \]

Therefore \( \|C_\phi e^m\|^2 = \frac{\lambda_m^2}{\lambda_p^2} = \frac{\lambda_m^2}{\lambda_p} \), \( m \in \phi^{-1}(p) \). Also

\[ \|C_\phi^m\|^2 = \|C_\phi X_m\|^2 = \sum_{n=1}^{\infty} \lambda_n |C_\phi X_m(n)|^2 = \sum_{n=1}^{\infty} \lambda_n |X_{\phi^{-1}(m)}(n)|^2, \]

where \( X_E \) stands for the characteristic function of the set \( E \). But if \( \phi \) is not onto, \( \phi^{-1}(m) \) is empty for some \( m \in \mathbb{N} \), and hence

\[ X_{\phi^{-1}(m)}(n) = 0 \] for every \( n \in \mathbb{N} \), which implies that \( \|C_\phi e^m\|^2 = 0 \). But

\[ \|C_\phi e^m\|^2 > 0 \] for all \( m \in \mathbb{N} \), so that

\[ \|C_\phi e^m\|^2 \neq \|C_\phi^m\|^2, \]

which is a contradiction to the normality of \( C_\phi \). Hence \( C_\phi \) is normal implies that \( \phi \) is onto. By Corollary 2.3 of Theorem 2.1 of [6], \( C_\phi \) is one-to-one. Since \( C_\phi \) is normal it has dense range. Thus by Corollary 2.6 of Theorem 2.4 of [6], \( \phi \) is one-to-one. Now since \( \phi \) is one-to-one, a simple computation shows that
By normality of \( C_\phi \) and left invertibility of \( \phi \) we have
\[
\lambda \phi^{-1}(m) = \lambda_m^2 / \lambda(m) .
\]
This after further simplication reduces to
\[
\phi(m) + \phi^{-1}(m) = 2m .
\]
Hence by the above lemma, \( \phi \) is the identity.

**COROLLARY 1.** Let \( C_\phi \in B \left( \ell^2_a \right) \). Then \( C_\phi \) is an isometry if and only if \( \phi \) is the identity.

**Proof.** The sufficiency is obvious. To prove the necessary part, suppose \( C_\phi \) is an isometry. Then we have, by \([2]\),
\[
M_{\phi} = C_\phi \phi = I .
\]
From this we conclude that
\[
f_\phi(n) = 1 \text{ for every } n \in N ,
\]
and hence by Lemma 1.1, \( \phi \) is the identity.

**COROLLARY 2.** Let \( C_\phi \in B \left( \ell^2_a \right) \). Then \( C_\phi \) is unitary if and only if \( \phi \) is the identity.

**THEOREM 2.4.** Let \( C_\phi \in B \left( \ell^2_a \right) \). Then \( C_\phi \) is a co-isometry if and only if \( \phi \) is the identity.

**Proof.** The sufficiency is again obvious. To prove the necessary part, let \( C_\phi \) be a co-isometry. Then
\[
\left\| C_\phi e^m \right\|^2 = \left\| e^m \right\| \text{ for all } m \in N .
\]
But
\[
\left\| C_\phi e^m \right\|^2 = \lambda_m^2 / \lambda(p) ,
\]
where \( m \in \phi^{-1}(p) \) and \( \left\| e^m \right\|^2 = \lambda_m \). Therefore we have \( \lambda_m = \lambda(p) \), which implies that \( m = \phi(m) \) for all \( m \in N \).
This shows that $\phi$ is the identity.

3. Invertible composition operators

The invertibility of $\phi$ is a necessary and sufficient condition for the invertibility of $C_\phi$ on $l^2$ [4, Theorem 2.2]. But this is not true in the case of $l^2_\alpha$, as is shown in the next example.

**EXAMPLE.** Let $\phi$ be a mapping from $\mathbb{N}$ into itself defined as

$$
\phi(n) = \begin{cases} 
\frac{n}{3} & \text{when } n = p_n \text{ where } p_n = 3(p_{n-1} + 1) \text{ with } p_0 = 0 \\
\phantom{\frac{n}{3}} +1 & \text{otherwise.}
\end{cases}
$$

Then $\phi$ is invertible. But since

$$
\frac{\|C_\phi x_{\{p_n/3\}}\|^2}{\|x_{\{p_n/3\}}\|^2} = \frac{\|x_{\phi^{-1}\{p_n/3\}}\|^2}{\|x_{\{p_n/3\}}\|^2}
$$

$$
= \frac{\|x_{\{p_n\}}\|^2}{\|x_{\{p_n/3\}}\|^2}
$$

$$
= \frac{\frac{a}{(2/3)p_n}}{\frac{a}{(4/3)p_n}} = \frac{a}{a} = (4/3)p_n,
$$

which goes to zero as $n$ goes to infinity, we have that $C_\phi$ is not bounded away from zero, and consequently $C_\phi$ is not invertible.

It is clear from the above example that characterization of invertibility of $C_\phi$ in terms of invertibility of $\phi$ (and vice versa) is not possible in this case. But the invertibility of $\phi$ together with an extra condition characterizes the invertibility of $C_\phi$, as is shown in the following theorem.

**THEOREM 3.1.** Let $C_\phi \in B[l^2_\alpha]$. Then $C_\phi$ is invertible if and only if $\phi$ is invertible and there exists an integer $k \geq 0$ such that
\[ \phi^{-1}(n) \leq k + n \text{ for all } n \in \mathbb{N}. \]

In order to prove the theorem we need the following lemma.

**Lemma 3.2.** Let \( \phi : \mathbb{N} \rightarrow \mathbb{N} \) be a mapping. Then \( C_\phi \in B \left[ l^2 \right] \) if and only if there exists an integer \( M > 0 \) such that \( \lambda(\phi^{-1}(n)) \leq M\lambda\{n\} \) for all \( n \in \mathbb{N} \).

**Proof.** Proof of this lemma follows from Theorem 1 of [3].

**Proof of Theorem 3.1.** Let \( C_\phi \) be invertible. If \( \phi \) is not one-to-one, then \( \phi(n) = \phi(m) \) for at least two distinct \( m \) and \( n \) in \( \mathbb{N} \), and hence \( g_n = g_m \) for all \( g \) in the range of \( C_\phi \). This shows that \( C_\phi \) is not onto, which is a contradiction. If \( \phi \) is not onto, then there exists a positive integer \( m \) such that \( m \not\mid \phi(N) \). Hence \( C_\phi X\{m\} = X_{\phi^{-1}\{m\}} = 0 \) which shows that \( C_\phi \) is not one-to-one. This is again a contradiction.

Further let there exist no integer \( k \geq 0 \) such that \( \phi^{-1}(n) \leq k + n \) for every \( n \in \mathbb{N} \). This implies that for each integer \( p \geq 0 \) there exists an integer \( n_p \) such that

\[ \phi^{-1}(n_p) > p + n_p \quad [p = 1, 2, 3, \ldots]. \]

Consider the sequence \( \langle X\{n_p\} \rangle \). Then

\[
\frac{\|C_\phi X\{n_p\}\|^2}{\|X\{n_p\}\|^2} = \frac{\|X_{\phi^{-1}\{n_p\}}\|^2}{\|X\{n_p\}\|^2} = \frac{2\phi^{-1}(n_p)}{a} = \frac{2n_p}{a} \leq \frac{2(p + m_p)}{a} < \frac{a}{2} \quad \text{as} \quad p \to \infty.
\]
This implies that $C_\phi$ is not bounded away from zero and hence it is not invertible. This is a contradiction. Hence there exists an integer $k \geq 0$ such that $\phi^{-1}(n) \leq k + n$.

Conversely, suppose $\phi$ is invertible and there exists an integer $k \geq 0$ such that $\phi^{-1}(n) \leq k + n$ for every $n \in \mathbb{N}$. Then there exists a function $\psi$ such that

$$(\phi \circ \psi)(n) = (\psi \circ \phi)(n) = n,$$

and $\phi(n) \geq n - k$ for every $n \in \mathbb{N}$. From this it follows that

$$\lambda \psi^{-1}(n) = \lambda(\phi(n)) = a^{2\phi(n)} \leq a^{-2\lambda}(n) \cdot$$

Hence by Lemma 1.1 we conclude that $C_\phi$ is bounded. Since

$$C_\psi C_\phi = C_{\phi \circ \psi} = I = C_\psi C_\phi,$$

$C_\phi$ is invertible.

This completes the proof of Theorem 3.1.

References


Department of Mathematics,
University of Jammu,
Jammu,
India.