ASYMPTOTIC SOLUTIONS OF EQUATIONS IN BANACH SPACE

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1. Introduction. The equation $Px = y$ in Banach spaces has aroused considerable interest, particularly in view of the various situations in applied analysis which it encompasses, and consequently it has been the topic of numerous investigations **(2; 9; 10; 12)**. Detailed references may be found in **(10)**. The equation is of special interest because of its interpretation as an integral equation; and in turn, many problems related to differential equations can be reformulated as integral equations **(5; 7; 13)**.

Various iterative procedures are available **(10; 11; 12)** by which the existence and uniqueness of a solution x of such an equation can be established, and by which numerical estimates for the solution can be calculated. In any of these procedures, a sequence of elements x_n ($n = 0,1, 2, \ldots$) in the Banach space is constructed recursively, and is proved to converge in the Banach norm to an element x satisfying $Px = y$. The recursive sequences used have been modelled after various familiar ones. In particular, an iterative process modelled after Newton's method of solving real equations has been employed very successfully by Kantorovich **(10)** and others **(2; 16)**. Another recursive sequence, the analogue of that defined by an infinite continued fraction, has been studied recently by McFarland **(12)**. The most widely known iterative procedure is that based on the Liouville-Neumann sequence of successive approximations **(5; 11; 13)**.

The last of these, for example, can be used to prove that a contraction mapping T on a closed, bounded domain in the Banach space has a fixed point x in the domain **(1; 11)**. Therefore, under the assumption that the equation under consideration is equivalent to $Tx = x$ with T a contraction mapping, the existence and uniqueness of the solution follow from the fixed point theorem; and such results will be appropriate to the study of asymptotic properties of the solution.

In an investigation of the asymptotic behaviour of equations, one is interested in the variation of the elements y and the transformations P involved in the equations as a real variable λ (or more general variable) varies over an interval Λ. It is then pertinent to consider mappings (y) of Λ into the Banach space and mappings (P) of Λ into a suitable set of transformations on the space; and furthermore, in an asymptotic investigation, to study the behaviour of

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these mappings as λ approaches a limit point, in general not in Λ. No essential features are lost by the assumptions that Λ is a positive interval $(0, \lambda_0]$ and that 0 is the limit point.

We shall first develop the notion of an asymptotically convergent sequence of mappings (§ 2), and from this, the notions of asymptotic equality and asymptotic summation of series. The main questions to be considered are the following: (1) If the quantities y and P involved in the equation $Px = y$ can be represented by asymptotically convergent series, can the solution be represented by such a series? (2) For a prescribed asymptotically convergent sequence of mappings (P_n), $n = 0, 1, 2, \ldots$, and for prescribed (y_n), does there exist a mapping (x) of Λ into the Banach space so that $\sum(P_n x)$ is asymptotically equal to $\sum(y_n)$? These questions are answered in § 5, in which the appropriate existence, uniqueness, and representation theorems are given. The approach taken here is similar to that employed by van der Corput **(14)** in connection with asymptotic solution of certain numerical equations.

We shall next mention a few examples, to which the subsequent theorems are applicable, obtained when the Banach space is specialized to one of the following: the space of real numbers; the finite dimensional Euclidean space V_n; the space of continuous functions on a closed, bounded interval; and the Lebesgue space L^p $(p \geqslant 1)$ **(13; 15)**.

In the space of real numbers x with norm defined by $||x|| = |x|$, the equation

$$y(\lambda) = \alpha x + \sum_{n=1}^{\infty} \alpha_n(\lambda, x) \qquad (\alpha \neq 0)$$

is to be considered, and the corresponding relation when asymptotic equality replaces equality. A specific example is the problem of finding a real number x with $|x| \leqslant 1$ so that for $|y| < 1$,

$$y \sim x + \sum_{n=1}^{\infty} (n+1)!\,(-\lambda)^n x^{n+1}.$$

In this example, formal substitution will lead to an asymptotic expansion

$$x \sim y + \sum_{n=1}^{\infty} \beta_n(\lambda) y^n$$

for the solution **(14)**. A different discussion of a similar problem has been given by de Bruijn **(6**, p. 25**)**.

As a second illustration, suppose the Banach space is specialized to the finite dimensional Euclidean space V_n. Each element x in V_n is a vector (σ_i) $(i = 1, 2, \ldots, n)$ with norm given by

$$||x|| = \left(\sum_{i=1}^{n} \sigma_i^2\right)^{\frac{1}{2}}.$$

In this context, one considers a system of n non-linear algebraic equations $y = A_0 x + Ex$, where y is a prescribed element of V_n, A_0 is a square matrix

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of order n, and E is a transformation on V_n defined by $Ex = (E_k(\sigma_1, \sigma_2, \ldots, \sigma_n))$ $(k = 1, 2, \ldots, n)$. Under the assumption $\det A_0 \neq 0$, the linear system $y = A_0 x_0$ has a unique solution x_0. Under suitable additional hypotheses, Theorem 3 below guarantees that the non-linear system under consideration possesses a unique solution x such that $||x - x_0|| \to 0$ as $\lambda \to 0$; and Theorem 5 gives an iterative procedure by which an asymptotic expansion can be generated.

We envisage that the most fruitful application will be to non-linear integral equations. The Banach space will be either the space C of all continuous functions over the closed, bounded interval under consideration, or the Lebesgue space L^p $(p \geqslant 1)$. The transformations A_0 and E will be regarded as linear and non-linear integral operators respectively. Consider the integral equation

$$x(s) = y(s) + \int_0^1 K(s, t)x(t)dt + \int_0^1 E_\lambda(s, t; x(t))dt \tag{1.1}$$

with $K(s, t)$ continuous on the closed unit square, and $y \in C(0, 1)$. This integral equation is of the form $x = y + Kx + Ex$, where K is the linear transformation from C into C and E is the non-linear transformation defined by (1.1). The first assumption to be made, of course, is that $(I - K)^{-1}$ exists, where I is the identity transformation, so that the linear integral equation approximating (1.1) for small λ will have a solution. The existence of this inverse transformation is implied by the condition $||K|| < 1$ according to Banach's well-known theorem (**11**; **13**, p. 151). The analogue of the principal hypothesis (4.4) below is the hypothesis that $E_\lambda(s, t; u)$ satisfies a Lipschitz condition in its third argument on a suitable interval, uniformly for (s, t) on the unit square. Theorem 3 guarantees the existence of a unique solution $x = x(s, \lambda)$ of (1.1) with the property that $||x - (I - K)^{-1}y|| \to 0$ as $\lambda \to 0$; and Theorem 5 shows how an asymptotic expansion of the solution can be generated by a recursive process. Similar statements can be made when the space C is replaced by the Hilbert space $L^2(0, 1)$.

2. Asymptotic convergence. A Banach space $\mathfrak{B}$ will be considered, and the Banach norm of an element $x \in \mathfrak{B}$ will be denoted as usual by $||x||$. The following notation will be used throughout: (i) λ denotes a positive real variable on an interval Λ_0: $0 < \lambda \leqslant \lambda_0$; (ii) ϕ denotes a function from Λ_0 into positive numbers; (iii) (x) denotes a mapping $\lambda \to x(\lambda)$ of Λ_0 into $\mathfrak{B}$; (iv) j, k, m, n denote non-negative integers; (v) $\alpha_0, \alpha_1, \ldots, \lambda_0, \lambda_1, \ldots,$ denote *fixed* positive numbers, that is positive numbers independent of λ.

Let ϕ_n $(n = 0, 1, 2, \ldots)$ be a single-valued function from Λ_0 into positive real numbers. The sequence $\{\phi_n\}$ is said to be an *asymptotic sequence* as $\lambda \to 0$ if $\phi_0(\lambda) = 1$ for all $\lambda \in \Lambda_0$, and $\phi_{n+1} = o(\phi_n)$ as $\lambda \to 0$ for each integer n (**8**).

Let $\{\lambda_n\}$ be a non-increasing sequence of positive numbers, and for each integer n let Λ_n denote the interval $0 < \lambda \leqslant \lambda_n$.

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Let $\{x_n(\lambda)\}$ $(n = 0, 1, 2, \ldots)$ be a sequence of elements in $\mathfrak{B}$, with $x_n(\lambda)$ uniquely defined for each $\lambda \in \Lambda_0$, and let (x_n) designate the mapping $\lambda \to x_n(\lambda)$ from Λ_0 into $\mathfrak{B}$. The sequence $\{(x_n)\}$ is said to *converge asymptotically* if there exists a single-valued mapping (x) of Λ_0 into $\mathfrak{B}$, an asymptotic sequence $\{\phi_n\}$, and a sequence of positive numbers α_n so that

$$\|x(\lambda) - x_n(\lambda)\| \leqslant \alpha_n \phi_n(\lambda), \qquad \lambda \in \Lambda_n \tag{2.1}$$

for each integer n. In this event, (x) is referred to as an *asymptotic limit* of the sequence $\{(x_n)\}$. In particular, the sequence is said to converge asymptotically to zero when

$$\|x_n(\lambda)\| \leqslant \alpha_n \phi_n(\lambda), \qquad \lambda \in \Lambda_n, \quad n = 0, 1, \ldots. \tag{2.2}$$

Our terminology follows that used by van der Corput in the asymptotic theory of numerical functions **(14)**.

An asymptotically convergent sequence need not converge in the ordinary sense (in the Banach norm) for any value of λ, as shown by the example $x_n(\lambda) = n!\lambda^n$ in the Banach space of real numbers.

Two mappings (x), (y) defined on Λ_0 are said to be asymptotically equal if for each integer n there exists a positive number α_n so that

$$\|x(\lambda) - y(\lambda)\| \leqslant \alpha_n \phi_n(\lambda) \tag{2.3}$$

whenever $\lambda \in \Lambda_n$. In this event, we write $(x) \leftrightarrow (y)$. The relation $\leftrightarrow$ is evidently reflexive, symmetric, and transitive, and hence it is an equivalence relation among mappings. Each real asymptotic sequence $\{\phi_n\}$ induces such an equivalence relation, the sets of asymptotically equal mappings with respect to $\{\phi_n\}$ forming the equivalence classes.

If $\{(x_n)\}$ is an asymptotically convergent sequence of mappings, then the set of all asymptotic limits of the sequence is characterized by an equivalence class of asymptotically equal mappings. For let (x) be any asymptotic limit. Then if (y) is an asymptotic limit, it follows from (2.1) that

$$\|x(\lambda) - y(\lambda)\| \leqslant \|x(\lambda) - x_n(\lambda)\| + \|y(\lambda) - x_n(\lambda)\| \leqslant \alpha_n \phi_n(\lambda) + \alpha_n' \phi_n(\lambda)$$

whenever $\lambda \in \Lambda_n$. Hence (2.3) holds and $(y) \leftrightarrow (x)$. Conversely, it is easy to see that if $(y) \leftrightarrow (x)$, then (y) is an asymptotic limit of the sequence.

Since $(x) \leftrightarrow (y)$ for any two asymptotic limits (x), (y) we shall say that the asymptotic limit of the sequence is *asymptotically unique*.

A formal series $\sum (x_n)$ is said to have an *asymptotic sum* (x) if (x) is an asymptotic limit of the sequence $\{(x_0 + x_1 + \ldots + x_{n-1})\}$ $(n = 1, 2, \ldots)$. This means that for each n there exists a positive number α_n and an interval Λ_n so that

$$\left\| x(\lambda) - \sum_{j=0}^{n-1} x_j(\lambda) \right\| \leqslant \alpha_n \phi_n(\lambda), \qquad \lambda \in \Lambda_n. \tag{2.4}$$

When the asymptotic sum exists it is not unique, but it follows from the foregoing remarks that it is asymptotically unique. When (x) is an asymptotic

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sum for $\sum(x_n)$, we say that the series is an asymptotic expansion for (x), and write $(x) \sim \sum(x_n)$.

The following theorem may be regarded as the basic theorem concerning asymptotic convergence. It states that an asymptotic sum of $\sum(x_n)$ always exists when $\{(x_n)\}$ converges asymptotically to zero. Results like this for numerical functions have been obtained by various authors **(3**; **4**; **8)**. The present proof is modelled after that of van der Corput **(14)**.

THEOREM 1. *A necessary and sufficient condition for a series of mappings $\sum(x_n)$ to have an asymptotic sum is that the sequence $\{(x_n)\}$ converge asymptotically to zero.*

Proof. If (x) is an asymptotic sum for $\sum(x_n)$, then (2.4) is valid for each integer n, and it is easily established from the Minkowski inequality and the order relation $\phi_{n+1} = o(\phi_n)$ $(\lambda \to 0)$ that (2.2) holds. Hence $\{(x_n)\}$ converges asymptotically to zero.

Conversely, if the sequence converges asymptotically to zero, then (2.2) holds for each integer n. Since $\phi_{n+1} = o(\phi_n)$ as $\lambda \to 0$, it follows that for each n there exist positive numbers α_n, λ_n with $\{\lambda_n\}$ non-increasing, so that

$$||x_{n+1}(\lambda)|| \leqslant \alpha_{n+1}\phi_{n+1}(\lambda) \leqslant \tfrac{1}{2}\alpha_n\phi_n(\lambda)$$

for $0 < \lambda \leqslant \lambda_{n+1}$, that is $\lambda \in \Lambda_{n+1}$. Hence

$$||x_{n+j}(\lambda)|| \leqslant (\tfrac{1}{2})^j\alpha_n\phi_n(\lambda) \qquad (j = 1, 2, \ldots) \tag{2.5}$$

for all $\lambda \in \Lambda_{n+j}$.

If λ_n tends to a positive limit λ^* as $n \to \infty$, it follows from (2.5) that

$$\left\{\sum_{j=0}^{n} x_j(\lambda)\right\}$$

is a Cauchy sequence for each λ satisfying $0 < \lambda \leqslant \lambda^*$. Hence this sequence converges in the Banach norm to an element $x(\lambda)$ because of the completeness of $\mathfrak{B}$. It can then be verified that $x(\lambda)$ satisfies (2.4), and consequently (x): $\lambda \to x(\lambda)$ is an asymptotic sum.

The situation of real interest, however, is that in which $\sum x_n(\lambda)$ does *not* have an ordinary sum for any positive value of λ. Suppose then that $\lambda_n \to 0$ as $n \to \infty$. For each value of λ, let $H = H(\lambda)$ be the largest integer such that $\lambda_H \geqslant \lambda$. Then if $\lambda \in \Lambda_n$, it follows that $H(\lambda) \geqslant n$. We assert that (x) given by

$$(x) = (x_0 + x_1 + \ldots + x_H)$$

is an asymptotic sum for $\sum(x_n)$. In fact, for all $\lambda \in \Lambda_n$, $H(\lambda)$ has been chosen so that $\lambda \in \Lambda_H$, which implies that $\lambda \in \Lambda_{n+j}$ $(j = 0, 1, \ldots, H - n)$. Then by (2.5)

$$\left|\left|x(\lambda) - \sum_{j=0}^{n-1} x_j(\lambda)\right|\right| \leqslant \sum_{j=n}^{H} ||x_j(\lambda)|| \leqslant 2\alpha_n\phi_n(\lambda), \tag{2.6}$$

and hence (x) is an asymptotic sum.

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3. Transformations on the Banach space. A *transformation* E defined on a closed domain $\mathfrak{D}$ in $\mathfrak{B}$ is a single-valued mapping from $\mathfrak{D}$ into $\mathfrak{B}$. Transformations are not necessarily additive, nor are they necessarily even defined on the whole space $\mathfrak{B}$. For each $\lambda \in \Lambda_0$, let $E(\lambda)$ be a uniquely defined transformation on $\mathfrak{D}$, and let (E) be the mapping $\lambda \to E(\lambda)$. We shall say that (E) is in the class Lip $(\mathfrak{D}, \phi)$ whenever there exists a fixed, positive number α and a bounded positive function ϕ on Λ_0 so that

$$||E(\lambda)x - E(\lambda)y|| \leqslant \alpha\phi(\lambda)||x - y|| \tag{3.1}$$

for all pairs of elements x, y in $\mathfrak{D}$, and for all $\lambda \in \Lambda_0$. When $\alpha\phi(\lambda) < 1$, a transformation $E(\lambda)$ from $\mathfrak{D}$ into itself satisfying (3.1) is a contraction mapping, and therefore has a fixed point in $\mathfrak{D}$ **(11)**.

Sums and products of transformations are defined as in the linear case **(13; 15)**. Thus, if E, F are transformations on $\mathfrak{D}$, then $(E + F)x = Ex + Fx$, $x \in \mathfrak{D}$; and $(EF)x = E(Fx)$, $x \in \mathfrak{D} \cap \mathfrak{R}$, where $\mathfrak{R}$ is the range of F.

It is convenient to introduce the symbol $||E||$ to denote the supremum of $||Ex - Ey||/||x - y||$ over all x, y with $x \neq y$. Then (3.1) may be rewritten in the form

$$||E(\lambda)|| \leqslant \alpha\phi(\lambda), \qquad \lambda \in \Lambda_0. \tag{3.2}$$

For each integer n, let (A_n) denote a mapping $\lambda \to A_n(\lambda)$ of a positive interval Λ_n into the set of transformations on $\mathfrak{D}$. The sequence $\{(A_n)\}$ is said to *converge asymptotically* to (A) on $\mathfrak{D}$ whenever there exists an asymptotic sequence $\{\phi_n\}$ so that $(A) \in \mathrm{Lip}(\mathfrak{D}, \phi_0)$ and $(A - A_n) \in \mathrm{Lip}(\mathfrak{D}, \phi_n)$ for each integer n. In this event,

$$||A(\lambda) - A_n(\lambda)|| \leqslant \alpha_n\phi_n(\lambda), \qquad \lambda \in \Lambda_n. \tag{3.3}$$

(A) will be called an *asymptotic limit* of the sequence $\{(A_n)\}$.

A series $\sum (A_n)$ is said to have an *asymptotic sum* (A) if (A) is an asymptotic limit of the sequence $\{(A_0 + A_1 + \ldots + A_{n-1})\}$. This means that there exists a positive number α_n so that

$$\left|\left|A(\lambda) - \sum_{j=0}^{n-1} A_j(\lambda)\right|\right| \leqslant \alpha_n\phi_n(\lambda), \qquad \lambda \in \Lambda_n. \tag{3.4}$$

The following analogue of Theorem 1 is valid for transformations.

THEOREM 2. *A necessary and sufficient condition for the series* $\sum_{n=m}^{\infty}(A_n)$ *to have an asymptotic sum in the class* $\mathrm{Lip}(\mathfrak{D}, \phi_m)$ *is that* $(A_n) \in \mathrm{Lip}(\mathfrak{D}, \phi_n)$ *for each integer* $n \geqslant m$.

The proof parallels that of Theorem 1. To establish the sufficiency, we choose the integer $H(\lambda)$ as in Theorem 1 and define (A) by

$$A(\lambda) = A_m(\lambda) + A_{m+1}(\lambda) + \ldots + A_H(\lambda).$$

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Then the mapping $(A): \lambda \to A(\lambda)$ will be an asymptotic sum; in fact, from $(A_n) \in \mathrm{Lip}(\mathfrak{D}, \phi_n)$ it follows that $\|A_{n+1}(\lambda)\| \leqslant \frac{1}{2}\alpha_n\phi_n(\lambda)$, and hence that

$$\left\| A(\lambda) - \sum_{j=m}^{n-1} A_j(\lambda) \right\| \leqslant 2\alpha_n\phi_n(\lambda), \qquad \lambda \in \Lambda_n$$

for each integer $n \geqslant m + 1$. In particular, $(A) \in \mathrm{Lip}(\mathfrak{D}, \phi_m)$.

4. Equations in Banach space. For a prescribed element $y(\lambda)$ in $\mathfrak{B}$ it will be our purpose to obtain information concerning the solution x of the equation

$$P(\lambda)x = y(\lambda). \tag{4.1}$$

The element $y = y(\lambda)$ is supposed to be uniquely defined for each λ in a positive interval Λ_0. The mapping $(y): \lambda \to y(\lambda)$ is supposed to possess an asymptotic expansion

$$(y) \sim \sum_{n=0}^{\infty} (y_n) \qquad (\lambda \to 0) \tag{4.2}$$

in which y_0 is a fixed element of $\mathfrak{B}$. According to (2.4), this means in particular that $\|y(\lambda) - y_0\| \to 0$ as $\lambda \to 0$.

It will be assumed that the transformation $P(\lambda)$ in (4.1) is uniquely defined on some fixed domain $\mathfrak{D}' \subset \mathfrak{B}$ for each $\lambda \in \Lambda_0$, and that $P(\lambda)$ has the decomposition

$$P(\lambda) = A_0 + E(\lambda), \qquad \lambda \in \Lambda_0 \tag{4.3}$$

valid on the entire domain of definition of $P(\lambda)$. In (4.3) A_0 is a fixed linear transformation with bounded inverse A_0^{-1}, and $E(\lambda)$ is a suitable contraction mapping, to be made precise presently.

For a fixed positive number η, let $\mathfrak{D}$ denote the closed sphere $\{x \in \mathfrak{B}: \|x - A_0^{-1}y_0\| \leqslant \eta\}$. The following assumptions will be made.

(i) $A_0^{-1}y_0 \in \mathfrak{D}'$ *and* η *is chosen small enough so that* $\mathfrak{D}$ *is a subset of* $\mathfrak{D}'$.

(ii) *The mapping* $(E): \lambda \to E(\lambda)$ *has the property that*

$$(E) \in \mathrm{Lip}(\mathfrak{D}, \phi_1) \qquad (\phi_1 = o(1) \quad \textit{as} \quad \lambda \to 0). \tag{4.4}$$

(iii) *For any element* $z \in \mathfrak{D}$

$$\|E(\lambda)z\| = o(1) \qquad \textit{as} \quad \lambda \to 0. \tag{4.5}$$

The assumptions (4.3) and (4.4) together constitute a statement of the approximate linearity of $P(\lambda)$ in the neighbourhood of $\lambda = 0$; there exists a linear transformation A_0 and a positive interval Λ_1 so that $\|P(\lambda) - A_0\| \leqslant \alpha_1\phi_1(\lambda)$ whenever $\lambda \in \Lambda_1$, and $\phi_1(\lambda) \to 0$ as $\lambda \to 0$.

Now we shall establish an existence and uniqueness theorem appropriate to the study of asymptotic properties of the solution, by appealing to the theorem that every contraction mapping on $\mathfrak{D}$ has a fixed point in $\mathfrak{D}$.

THEOREM 3. *Under the assumptions* (4.3), (4.4), (4.5) *there exists a positive interval* Λ *so that the equation* $P(\lambda)x = y(\lambda)$ *has a unique solution* $x(\lambda) \in \mathfrak{D}$ *for each* $\lambda \in \Lambda$. *Furthermore* $||x(\lambda) - A_0^{-1}y(\lambda)|| \to 0$ *as* $\lambda \to 0$.

Proof. On account of (4.3), equation (4.1) is equivalent to

$$x = v(\lambda) + F(\lambda)x, \tag{4.6}$$

where $v(\lambda) = A_0^{-1}y(\lambda)$ and $F(\lambda) = -A_0^{-1}E(\lambda)$. This equation has the form $x = T(\lambda)x$. We shall demonstrate that there exists a positive interval Λ so that $T(\lambda)$ maps $\mathfrak{D}$ into $\mathfrak{D}$ whenever $\lambda \in \Lambda$. In fact, if $x \in \mathfrak{D}$, that is $||x-v_0|| \leqslant \eta$ where $v_0 = A_0^{-1}y_0$, then

$$||T(\lambda)x - v_0|| \leqslant ||v(\lambda) - v_0|| + ||F(\lambda)x||.$$

However,

$$||v(\lambda) - v_0|| \leqslant ||A_0^{-1}|| \, ||y(\lambda) - y_0|| \to 0 \quad \text{as} \quad \lambda \to 0 \text{ by } (4.2),$$

and

$$||F(\lambda)x|| \leqslant ||A_0^{-1}|| \, ||E(\lambda)x|| \to 0 \quad \text{as} \quad \lambda \to 0 \quad \text{by} \quad (4.5).$$

Therefore there exists a positive interval Λ so that $||T(\lambda)x - v_0|| \leqslant \eta$ whenever $\lambda \in \Lambda$, and hence $T(\lambda)x \in \mathfrak{D}$ whenever $\lambda \in \Lambda$.

Clearly $(F) \in \mathrm{Lip}(\mathfrak{D}, \phi_1)$ since $(E) \in \mathrm{Lip}(\mathfrak{D}, \phi_1)$. Then

$$||T(\lambda)x - T(\lambda)y|| = ||F(\lambda)x - F(\lambda)y|| \leqslant \alpha\phi_1(\lambda)||x - y||$$

for all $x, y \in \mathfrak{D}$ and all λ in some positive interval. Since $\phi_1 = o(1)$ as $\lambda \to 0$, there exists a positive interval Λ' so that $||T(\lambda)x - T(\lambda)y|| \leqslant \frac{1}{2}||x - y||$ whenever $\lambda \in \Lambda'$; $x, y \in \mathfrak{D}$. We may assume that $\Lambda' = \Lambda$. Then $T(\lambda)$ is a contraction mapping on $\mathfrak{D}$, a closed sphere in the complete space $\mathfrak{B}$, for all $\lambda \in \Lambda$. Hence $T(\lambda)$ has a fixed point $x = x(\lambda) \in \mathfrak{D}$ **(11)**, that is $x(\lambda)$ satisfies (4.6) and hence (4.1).

Finally,

$$||x(\lambda) - A_0^{-1}y(\lambda)|| = ||x(\lambda) - v(\lambda)|| = ||Fx(\lambda)|| \to 0$$

as $\lambda \to 0$ by (4.5).

For the validity of this theorem, the assumption (4.4) is needed to ensure that the mapping T be a contraction mapping. It is well known that a stronger condition than continuity of the transformation E is required to imply uniqueness of the solution: such a condition is the Lipschitz condition (4.4). Counter-examples can be easily supplied when (4.1) is interpreted as an integral equation (for example, of the type arising from an initial value problem for a differential equation (**5**, chapter I)).

Assumption (4.5) is needed so that T maps $\mathfrak{D}$ into itself. A simple counter-example in the Banach space of real numbers to show that Theorem 3 is false without such an assumption is provided by the real equation $x = 2 + (\lambda x^{\frac{1}{2}} - 3)$ with $Ex = \lambda x^{\frac{1}{2}} - 3$, which does not have a solution in the space.

5. Asymptotic solution of equations. Consider now a mapping (A_n): $\lambda \to A_n(\lambda)$ in the class $\mathrm{Lip}(\mathfrak{D}, \phi_n)$ for each integer $n = 0, 1, 2, \ldots,$ where $\mathfrak{D}$ is the closed sphere defined in the previous section. It will be assumed that A_0 is a fixed linear transformation on $\mathfrak{B}$ with a bounded inverse. We seek a mapping $(x): \lambda \to x(\lambda) \in \mathfrak{D}$ for which a prescribed (y) is an asymptotic sum for the series $\sum(A_n x)$. Such an x will be called an *asymptotic solution* of the relation $\sum(A_n x) \sim (y)$. Next, a theorem will be derived concerning asymptotic solutions, under the following assumptions: (i) $(A_n) \in \mathrm{Lip}(\mathfrak{D}, \phi_n)$ $(n = 1, 2, \ldots)$; (ii) *For any element* $z \in \mathfrak{D}$

$$(5.1) \qquad ||A_n z|| \leqslant \alpha_n \phi_n(\lambda) \qquad (\lambda \in \Lambda_n, \qquad \alpha_n > 0).$$

THEOREM 4. *Under these assumptions, there exists an asymptotically unique solution* $(x): \lambda \to x(\lambda) \in \mathfrak{D}$ *of the relation*

$$\sum_{n=0}^{\infty} (A_n x) \sim (y).$$

Proof. Since $(A_n) \in \mathrm{Lip}(\mathfrak{D}, \phi_n)$ for each integer n, it follows from Theorem 2 that there exists an asymptotic sum (P) of

$$\sum_{n=0}^{\infty} (A_n)$$

defined on $\mathfrak{D}$; and in fact (P) is given by

$$P(\lambda) = \sum_{n=0}^{H(\lambda)} A_n(\lambda),$$

where $H(\lambda)$ is a suitable integer depending on λ. It follows in particular that the mapping $(E) = (P - A_0)$ is in the class $\mathrm{Lip}(\mathfrak{D}, \phi_1)$, which is Assumption (4.4) of Theorem 3. Since the sequence $\{(A_n x)\}$ converges asymptotically to zero for any $x \in \mathfrak{D}$ by (5.1) it follows from Theorem 1 that (Px) is an asymptotic sum for $\sum(A_n x)$, $x \in \mathfrak{D}$. In particular, $||E(\lambda)x|| = ||[P(\lambda) - A_0]x|| \leqslant \alpha_1 \phi_1(\lambda)$ $(\lambda \in \Lambda_1)$, which is the content of Assumption (4.5) of Theorem 3. Therefore the assumptions of the present theorem imply those of Theorem 3, and there exists an element $x(\lambda) \in \mathfrak{D}$ satisfying $P(\lambda)x(\lambda) = y(\lambda)$ $(\lambda \in \Lambda)$. Then for $\lambda \in \Lambda_n$, we obtain from (5.1)

$$(5.2) \qquad \left|\left| y(\lambda) - \sum_{j=0}^{n-1} A_j(\lambda)x(\lambda) \right|\right| \leqslant \sum_{j=n}^{H} ||A_j(\lambda)x(\lambda)|| \leqslant 2\alpha_n \phi_n(\lambda)$$

by the same reasoning which led to (2.5) and (2.6), and hence (y) is an asymptotic sum for $\sum(A_n x)$.

To show that (x) is asymptotically unique, let (u) be any other asymptotic solution. Then for each integer n,

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$$||A_0(x-u)|| - \left|\left|\sum_{j=1}^{n-1} (A_jx - A_ju)\right|\right| \leqslant \left|\left|\sum_{j=0}^{n-1} A_jx - A_ju\right|\right|$$
$$\leqslant \left|\left|y - \sum_{j=0}^{n-1} A_jx\right|\right| + \left|\left|y - \sum_{j=0}^{n-1} A_ju\right|\right|,$$

and according to (5.2) there is a positive number β_n $(n = 1, 2, \ldots)$ so that

$$(5.3) \qquad ||A_0(x-u)|| - \left|\left|\sum_{j=1}^{n} (A_jx - A_ju)\right|\right| \leqslant \beta_n\phi_n(\lambda), \qquad \lambda \in \Lambda_n.$$

By hypothesis, there exists a positive number α so that $||A_0^{-1}|| \leqslant \alpha$, and hence

$$(5.4) \qquad ||x-u|| \leqslant ||A_0^{-1}||\,||A_0(x-u)|| \leqslant \alpha||A_0(x-u)||.$$

Since $(A_n) \in \mathrm{Lip}(\mathfrak{D}, \phi_n)$ and $\{\phi_n\}$ is an asymptotic sequence it follows that there exists a sequence of positive intervals Λ_n' so that

$$\left|\left|\sum_{j=1}^{n-1} A_jx - A_ju\right|\right| \leqslant \sum_{j=1}^{n-1} \alpha_j\phi_j(\lambda)||x-u|| \leqslant 2\alpha_1\phi_1(\lambda)||x-u||$$

whenever $\lambda \in \Lambda_n'$. Since $\phi_1(\lambda) \to 0$ as $\lambda \to 0$, there is a positive interval Λ so that $2\alpha_1\phi_1(\lambda) \leqslant \frac{1}{2}\alpha^{-1}$ whenever $\lambda \in \Lambda$. We may assume that $\Lambda_n' \subseteq \Lambda$, and hence

$$(5.5) \qquad \left|\left|\sum_{j=1}^{n} A_jx - A_ju\right|\right| \leqslant \tfrac{1}{2}\alpha^{-1}||x-u||, \qquad \lambda \in \Lambda_n'.$$

Then (5.3), (5.4), and (5.5) together establish that

$$\tfrac{1}{2}\alpha^{-1}||x-u|| \leqslant \beta_n\phi_n(\lambda)$$

whenever λ is in the smaller of the positive intervals Λ_n', Λ_n. Hence (x) is asymptotically unique.

In our final theorem, we shall derive an asymptotic expansion for the asymptotically unique solution (x) of $\sum(A_jx) \sim (y)$ given in Theorem 4. Suppose that (y) has the asymptotic expansion (4.2). Suppose also that (P) is an asymptotic sum for the series $\sum(A_n)$, as in Theorem 4. Then the mappings (v) and (F) defined in (4.6) will be asymptotic sums for corresponding series $\sum(v_n)$, $\sum(F_n)$, that is

$$(5.6) \qquad \begin{array}{lll} (v) \sim \sum (v_n) & \text{where} & v_n = A_0^{-1}y_n \\ (F) \sim \sum (F_n) & \text{where} & F_n = -A_0^{-1}A_n. \end{array}$$

For the solution $(x): \lambda \to x(\lambda)$, it follows from Theorem 4 that $x(\lambda)$ satisfies equation (4.1), and hence satisfies (4.6). An asymptotic expansion for (x) will be obtained in a natural way from a sequence of successive approximations to the solution of (4.6), defined in terms of the quantities v_n, F_n.

The additional hypothesis will be made that the sequence $\{\phi_n\}$ has the multiplicative property

$$(5.7) \qquad \sum_{j=1}^{n} \phi_j(\lambda)\phi_{n-j}(\lambda) \leqslant \gamma_n \phi_n(\lambda) \qquad (\lambda \in \Lambda_0; n = 1, 2, \ldots)$$

where γ_n is a fixed positive number for each integer n.

THEOREM 5. *Under the hypotheses of Theorem 4, the asymptotically unique solution (x) of the relation $\sum(A_n x) \sim (y)$ is an asymptotic limit of the sequence $\{(x_n)\}$ defined by*

$$(5.8) \qquad x_0 = v_0; \qquad x_n = \sum_{j=0}^{n} v_j + \sum_{j=1}^{n} F_j x_{n-j} \qquad (n = 1, 2, \ldots).$$

An equivalent conclusion is that (x) has the asymptotic expansion

$$\sum(x_n - x_{n-1}) \qquad (\text{with} \quad x_{-1} = 0).$$

Proof. It is enough to show that there exists a sequence of positive numbers β_n and a sequence of intervals Λ_n so that

$$(5.9) \quad ||x(\lambda) - x_{n-1}(\lambda)|| \leqslant \beta_n \phi_n(\lambda) \qquad (\lambda \in \Lambda_n; \qquad n = 1, 2, \ldots).$$

This will be proved by mathematical induction on n. First, it is easily seen from (5.2) and (5.6) that the proposition is true for $n = 1$. Under the hypothesis that it is true for all integers $j \leqslant n - 1$, we shall show that it is true for n. Since $x = v + Fx$, it follows that

$$(5.10) \quad ||x - x_n|| \leqslant \left|\left| v - \sum_{j=0}^{n} v_j \right|\right| + \left|\left| Fx - \sum_{j=1}^{n} F_j x \right|\right| + \left|\left| \sum_{j=1}^{n} F_j x - F_j x_{n-j} \right|\right| .$$

On account of the hypotheses (5.6) there exists a positive number α_{n+1} so that each of the first two terms on the right side of (5.10) is bounded above by $\alpha_{n+1}\phi_{n+1}(\lambda)$ for all λ in a positive interval Λ_{n+1}. The inductive proof of (5.9) will then be finished if it can be shown that the third term also is of order ϕ_{n+1}. To see this, observe that

$$\begin{aligned} \left|\left| \sum_{j=1}^{n} F_j x - F_j x_{n-j} \right|\right| &\leqslant \sum_{j=1}^{n} ||F_j||\, ||x - x_{n-j}|| \\ &\leqslant ||A_0^{-1}|| \sum_{j=1}^{n} ||A_j||\, ||x - x_{n-j}|| \\ &\leqslant \alpha \sum_{j=1}^{n} \alpha_j \phi_j(\lambda) \beta_{n-j+1} \phi_{n-j+1}(\lambda), \end{aligned}$$

where use has been made of the inductive hypothesis (5.9) and the hypothesis $(A_j) \in \text{Lip}(\mathfrak{D}, \phi_j)$ at the last step. Let

$$\delta_n = \max_j \alpha_j \beta_{n-j+1} \qquad (1 \leqslant j \leqslant n).$$

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Then, since $\{\phi_n\}$ has the multiplicative property (5.7),

$$\left|\left|\sum_{j=1}^{n} F_j x - F_j x_{n-j}\right|\right| \leqslant \alpha\delta_n\gamma_{n+1}\phi_{n+1}(\lambda).$$

Hence (5.9) is valid for each integer n, and the theorem is proved.

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