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VALUE DISTRIBUTION OF CERTAIN MONOMIALS OF ALGEBROID FUNCTIONS

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Abstract

Hayman has shown that if f is a transcendental meromorphic function and $n \geqslant 3$, then $f^n f'$ assumes all finite values except possibly zero infinitely often. We extend his result in three directions by considering an algebroid function w, its monomial $w^{n_0} w'^{n_1}$, and by estimating the growth of the number of a-points of the monomial.

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1. Introduction and results

As Hayman noted in [2, p. 34], the problem of possible Picard values of derivatives of a meromorphic function having no zeros reduces to the problem of whether certain differential polynomials of an entire function necessarily have zeros. In this connection he proved the following theorem:

THEOREM A. (i) *If $f(z)$ is a transcendental entire function and $n \geqslant 2$, then $f(z)^n f'(z)$ assumes all values except possibly zero infinitely often.*

(ii) *If $f(z)$ is a transcendental meromorphic function and $n \geqslant 3$, then $f(z)^n f'(z)$ assumes all finite values except possibly zero infinitely often.*

Clunie [1] showed that Theorem A(i) is also true for $n = 1$. Moreover, Mues [6] showed that Theorem A(ii) is also true for $n = 2$.

It is assumed that the reader is familiar with the notation of standard Nevanlinna theory [3] and its algebroid counterpart [7, 8, 9]. In [5], Theorem A(ii) was generalized

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as follows:

THEOREM B. *Let $w(z)$ be a ν-valued transcendental algebroid function and set $\phi(z) := w(z)^n w'(z)$, where $n \in \mathbb{N}$. Then if $n \geqslant 4\nu - 1$, we have*

$$\overline{N}\left(r, \frac{1}{\phi - b}\right) \neq S(r, w)$$

for each $b \in \mathbb{C} \setminus \{0\}$.

In this paper, we will prove the following generalization of Theorem A(ii) and Theorem B:

THEOREM. *Let $w(z)$ be a ν-valued transcendental algebroid function and set*

$$\psi(z) := w(z)^{n_0} w'(z)^{n_1}, \quad n_0,\ n_1 \in \mathbb{N}.$$

Then if $n_0 \geqslant 4\nu - 1 + 2(\nu - 1)(n_1 - 1)$, we have for each $a \in \mathbb{C} \setminus \{0\}$,

$$\overline{N}\left(r, \frac{1}{\psi - a}\right) \geqslant pT(r, w) - S(r, w),$$

where $p := n_0 - 4\nu + 2 - 2(\nu - 1)(n_1 - 1) \geqslant 1$.

COROLLARY. *With the same hypotheses we have*

$$\Theta(a, \psi) := 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, 1/(\psi - a))}{T(r, \psi)} \leqslant 1 - \frac{p}{n_0 + 2\nu n_1}$$

for each $a \in \mathbb{C} \setminus \{0\}$.

2. Proof of the theorem and its corollary

To prove the theorem, we write $u(z) := 1/w(z)$ and get

$$\psi(z) = w^{n_0} w'^{n_1} = (-u')^{n_1} / u^{2n_1 + n_0}.$$

In what follows, we regard ψ, u'/u, etcetera as functions on the Riemann surface of u; see [5, Section 2].

If ψ is a constant function, then

$$(2n_1 + n_0)T(r, u) = T(r, u^{2n_1 + n_0}) = T\left(r, \frac{(-u')^{n_1}}{\psi}\right) = n_1 T(r, u') + O(1)$$
$$\leqslant 2\nu n_1 T(r, u) + S(r, u).$$

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But this is impossible since $n_0 > 2(\nu - 1)n_1$. Hence ψ is non-constant.

Let $a \in \mathbb{C} \setminus \{0\}$. The second main theorem (see [4, p. 18]) then yields

$$(1)\quad T(r,\psi) \leqslant \overline{N}(r,\psi) + \overline{N}\left(r, \frac{1}{\psi}\right) + \overline{N}\left(r, \frac{1}{\psi - a}\right) + N_3(r,\psi) + S(r,\psi),$$

where $N_3(r,\psi)$ is the integrated counting function of branch points of ψ. On the other hand, we have

$$\begin{aligned} T(r,\psi) &\leqslant n_1 T(r,u') + (2n_1 + n_0)T(r,u) + O(1) \\ &\leqslant \big(2(\nu+1)n_1 + n_0\big)T(r,u) + S(r,u), \end{aligned}$$

so that we may replace $S(r,\psi)$ in (1) by $S(r,u)$. By [5, Lemma], we also have $N_3(r,\psi) \leqslant N_3(r,u)$. Thus (1) becomes

$$(2)\quad T(r,\psi) \leqslant \overline{N}(r,\psi) + \overline{N}\left(r, \frac{1}{\psi}\right) + \overline{N}\left(r, \frac{1}{\psi - a}\right) + N_3(r,u) + S(r,u).$$

In what follows, we denote by τ' the orders of the zeros of u' and write

$$\begin{aligned} n\left(r, \frac{1}{u'}\right) &= \sum_{\substack{u'=0\\ u\neq 0}} \tau' + \sum_{\substack{u'=0\\ u=0\\ \psi=\infty}} \tau' + \sum_{\substack{u'=0\\ u=0\\ \psi=0}} \tau' + \sum_{\substack{u'=0\\ u=0\\ \psi\neq 0,\infty}} \tau' \\ &=: n_1(r) + n_{0\infty}(r) + n_{00}(r) + n_{01}(r). \end{aligned}$$

First, let us estimate the term $\overline{N}(r,\psi)$ in (2). Let u have expansions of the general form

$$(3)\quad u(z) = u(z_0) + b_\tau (z - z_0)^{\tau/\lambda} + \cdots, \qquad \text{or}$$

$$(4)\quad u(z) = b_{-\tau}(z - z_0)^{-\tau/\lambda} + \cdots,$$

where $\tau, \lambda \in \mathbb{N}$ and $\lambda \leqslant \nu$. We denote by $n_{3_1}(r,u)$ the counting function of branch points of u such that $\tau < \lambda$ in (3). Then we have

$$(5)\quad (2n_1 + n_0)\overline{n}(r,\psi) \leqslant n(r,\psi) + n_1 n_{0\infty}(r) + (n_1 + n_0)n_{3_1}(r,u).$$

In fact, poles of ψ arise from zeros of u and poles of u'. Suppose that z_0 is a pole of ψ, a zero of u of order $\tau \geqslant 1$ and a zero of u' of order $\tau' \geqslant 0$. Then z_0 is a pole of ψ of order $(2n_1 + n_0)\tau - n_1\tau'$. So, the point z_0 contributes $(2n_1 + n_0)\tau - n_1\tau'$ to $n(r,\psi)$ and $n_1\tau'$ to $n_1 n_{0\infty}(r)$. Thus the contribution to $n(r,\psi) + n_1 n_{0\infty}(r)$ is $(2n_1 + n_0)\tau \geqslant 2n_1 + n_0$. On the other hand, a pole of u as in (4) is a zero of ψ of order

$$(6)\quad (2n_1 + n_0)\tau - n_1\tau - n_1\lambda = (n_1 + n_0)\tau - n_1\lambda \geqslant n_1 + n_0 - n_1\nu > 0$$

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as soon as $n_0 > n_1(\nu - 1)$. Thus the remaining poles of ψ, that is, poles of u', are branch points of u such that $\tau < \lambda$ in (3). But the contribution of such a point to $n(r, \psi) + (n_1 + n_0)n_{3_1}(r, u)$ is at least $2n_1 + n_0$. The estimate (5) now follows.

Integrating (5) logarithmically we obtain

$$\overline{N}(r, \psi) \leqslant \frac{1}{2n_1 + n_0} N(r, \psi) + \frac{n_1}{2n_1 + n_0} N_{0\infty}(r) + \frac{n_1 + n_0}{2n_1 + n_0} N_{3_1}(r, u), \tag{7}$$

where $N_{0\infty}(r)$ and $N_{3_1}(r, u)$ are defined similarly to $N(r, u)$.

Secondly, let us estimate the term $\overline{N}(r, 1/\psi)$ in (2). We denote by $n_{3_2}(r, u)$ the counting function of those branch points of u that are also poles of u. Then we have

$$n_0 \overline{n}\left(r, \frac{1}{\psi}\right) \leqslant n\left(r, \frac{1}{\psi}\right) + (n_0 - n_1)n_1(r) + \frac{n_0 n_1}{2n_1 + n_0} n_{00}(r) + n_1 n_{3_2}(r, u). \tag{8}$$

In fact, zeros of ψ arise from poles of u and zeros of u'. By (6), a pole of u as in (4) contributes $(n_1 + n_0)\tau - n_1\lambda + n_1(\lambda - 1) = (n_1 + n_0)\tau - n_1 \geqslant n_0$ to $n(r, 1/\psi) + n_1 n_{3_2}(r, u)$. A zero of u' of order τ' that is not a zero of u contributes $n_1\tau' + (n_0 - n_1)\tau' = n_0\tau' \geqslant n_0$ to $n(r, 1/\psi) + (n_0 - n_1)n_1(r)$. Finally, a zero of u' that is also a zero of u and ψ must be of order greater than $(2n_1 + n_0)/n_1$ and so contributes more than n_0 to $(n_0 n_1/(2n_1 + n_0))n_{00}(r)$. The estimate (8) now follows. Integrating (8) logarithmically we obtain

$$\overline{N}\left(r, \frac{1}{\psi}\right) \leqslant \frac{1}{n_0} N\left(r, \frac{1}{\psi}\right) + \frac{n_0 - n_1}{n_0} N_1(r) + \frac{n_1}{2n_1 + n_0} N_{00}(r) + \frac{n_1}{n_0} N_{3_2}(r, u), \tag{9}$$

where $N_1(r)$ and $N_{00}(r)$ are defined similarly to $N(r, u)$.

Combining (2), (7) and (9), and noticing $N(r, \psi) \leqslant T(r, \psi)$ and $N(r, 1/\psi) \leqslant T(r, \psi) + O(1)$, we now get

$$\begin{aligned}\left(1 - \frac{1}{2n_1 + n_0} - \frac{1}{n_0}\right) T(r, \psi) \leqslant{}& \frac{n_0 - n_1}{n_0} N_1(r) + \frac{n_1}{2n_1 + n_0}\big(N_{0\infty}(r) + N_{00}(r)\big) \\ &+ \frac{n_1 + n_0}{2n_1 + n_0} N_{3_1}(r, u) + \frac{n_1}{n_0} N_{3_2}(r, u) \\ &+ \overline{N}\left(r, \frac{1}{\psi - a}\right) + N_3(r, u) + S(r, u).\end{aligned} \tag{10}$$

On the other hand, we have

$$N(r, 1/\psi) \geqslant n_0 N(r, u) + n_1 N_1(r) - n_1 N_{3_2}(r, u). \tag{11}$$

In fact, at a pole of u as in (4), ψ has a zero of order $(n_1+n_0)\tau - n_1\lambda \geqslant n_0\tau - n_1\lambda + n_1 = n_0\tau - n_1(\lambda-1)$. Also, at a zero of u' of order τ' that is not a zero of u, the function ψ has a zero of order $n_1\tau'$. Thus (11) holds.

Furthermore, we have

$$\begin{aligned}(2n_1+n_0)m(r,u) = m\left(r, (-u')^{n_1}/\psi\right) &\leqslant n_1 m(r,u') + m\left(r, 1/\psi\right) \\ &\leqslant n_1 m(r,u) + n_1 m\left(r, u'/u\right) + m\left(r, 1/\psi\right),\end{aligned}$$

and so

$$(n_1+n_0)m(r,u) \leqslant m\left(r, 1/\psi\right) + S(r,u).$$

Combining this with (11) we obtain

$$n_0 T(r,u) \leqslant T(r,\psi) - n_1 N_1(r) + n_1 N_{3_2}(r,u) + S(r,u).$$

Using this and (10) we get

$$\begin{aligned}&n_0\left(1 - \frac{1}{2n_1+n_0} - \frac{1}{n_0}\right) T(r,u) \\ &\quad\leqslant n_1\left(\frac{1}{n_0} + \frac{1}{2n_1+n_0} - 1\right) N_1(r) + \frac{n_0-n_1}{n_0} N_1(r) \\ &\quad\quad + \frac{n_1}{2n_1+n_0}\left(N_{0\infty}(r) + N_{00}(r)\right) + \frac{n_1+n_0}{2n_1+n_0} N_{3_1}(r,u) \\ &\quad\quad + \frac{n_1}{n_0} N_{3_2}(r,u) + n_1\left(1 - \frac{1}{2n_1+n_0} - \frac{1}{n_0}\right) N_{3_2}(r,u) \\ &\quad\quad + \overline{N}\left(r, \frac{1}{\psi - a}\right) + N_3(r,u) + S(r,u), \qquad \text{or}\end{aligned}$$

$$\begin{aligned}&\left(n_0 - \frac{n_0}{2n_1+n_0} - 1\right) T(r,u) \\ &\quad\leqslant (1-n_1)N_1(r) + \frac{n_1}{2n_1+n_0}\left(N_1(r) + N_{0\infty}(r) + N_{00}(r)\right) \\ &\quad\quad + \frac{n_1+n_0}{2n_1+n_0} N_{3_1}(r,u) + \frac{n_1(2n_1+n_0-1)}{2n_1+n_0} N_{3_2}(r,u) \\ &\quad\quad + \overline{N}\left(r, \frac{1}{\psi - a}\right) + N_3(r,u) + S(r,u).\end{aligned}$$

Since

$$N_1(r) + N_{0\infty}(r) + N_{00}(r) \leqslant N\left(r, \frac{1}{u'}\right) \leqslant T(r,u') + O(1) \leqslant 2\nu T(r,u) + S(r,u)$$

and

$$N_{3_1}(r,u)+N_{3_2}(r,u)\leqslant N_3(r,u)\leqslant(2\nu-2)T(r,u)+O(1),$$

we get

(12)
$$\begin{aligned}\left(n_0+\frac{2n_1}{2n_1+n_0}-2\right)&T(r,u)\\ &\leqslant\left(\frac{2\nu n_1}{2n_1+n_0}+(2\nu-2)\left(\frac{n_1(2n_1+n_0-1)}{2n_1+n_0}+1\right)\right)T(r,u)\\ &\quad+\overline{N}\left(r,\frac{1}{\psi-a}\right)+S(r,u)\\ &=\left(2\nu n_1+2\nu-2n_1+\frac{2n_1}{2n_1+n_0}-2\right)T(r,u)+\overline{N}\left(r,\frac{1}{\psi-a}\right)+S(r,u).\end{aligned}$$

Set $p:=n_0-2\nu n_1-2\nu+2n_1=n_0-4\nu+2-2(\nu-1)(n_1-1)$. By the hypothesis we then have $p\geqslant 1$ and (12) yields

$$pT(r,w)=pT(r,u)+O(1)\leqslant\overline{N}\left(r,\frac{1}{\psi-a}\right)+S(r,w).$$

This completes the proof.

PROOF OF THE COROLLARY. Firstly, we have

(13) $$T(r,\psi)=T(r,w^{n_0}w'^{n_1})\leqslant(n_0+2\nu n_1)T(r,w)+S(r,w).$$

Let $a\in\mathbb{C}\setminus\{0\}$. By (13) and the theorem, there exists a set E of finite linear measure such that

$$\varliminf_{r\notin E}\frac{\overline{N}(r,1/(\psi-a))}{T(r,\psi)}\geqslant\varliminf_{r\notin E}\frac{(p-o(1))T(r,w)}{(n_0+2\nu n_1+o(1))T(r,w)}=\frac{p}{n_0+2\nu n_1}.$$

The corollary is proved.

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