OPERATORS THAT ARE NUCLEAR WHENEVER THEY ARE NUCLEAR FOR A LARGER RANGE SPACE

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1. Introduction

Let $X$ and $Y$ be Banach spaces. A bounded linear operator $T \in \mathcal{L}(X,Y)$ is said to be nuclear if there exist $x^*_n \in X^*$ and $y_n \in Y$ such that $\sum_{n=1}^{\infty} \|x^*_n\| \|y_n\| < \infty$ and $Tx = \sum_{n=1}^{\infty} x^*_n(x)y_n$ for all $x \in X$. In this case, one writes $T = \sum_{n=1}^{\infty} x^*_n \otimes y_n$ and calls the latter sum a nuclear representation of $T$. Let us denote by $\mathcal{N}(X,Y)$ the collection of all nuclear operators from $X$ to $Y$.

Every operator $T \in \mathcal{L}(X,Y)$ may be viewed as an operator from $X$ to $Y^{**}$ considering the operator $j_Y T$, where $j_Y : Y \to Y^{**}$ denotes the canonical embedding. Grothendieck proved, in his famous memoir [7, Chapter I, pp. 85, 86], that

$$j_Y T \in \mathcal{N}(X,Y^{**}) \Rightarrow T \in \mathcal{N}(X,Y) \quad (1.1)$$

whenever $X^*$ has the approximation property. He also affirmed (see [6, p. 17] and [7, Chapter I, p. 86]) that implication (1.1) holds whenever the second dual space $Y^{**}$ has the approximation property. A counterexample to this affirmation of Grothendieck was given by Oja and Reinov (see [24] or [25]). On the other hand, they proved [25] that implication (1.1) is true whenever the third dual space $Y^{***}$ has the approximation property. The latter assumption seems to be rather confusing because one does not know
much about the third duals of non-reflexive Banach spaces, even of so-called classical ones (for instance, let us quote Diestel [1, p. 35] here: ‘ba = c_0^{**} is the best I can do’).

The purpose of the present article is to extend the above-described Grothendieck–Oja–Reinov theorem to a more general natural setting in such a way that the assumption about the approximation property would solely concern the first dual spaces. In particular, this clarifies the appearance of the third dual $Y^{***}$ in the Oja–Reinov result.

To state the main result of this article, Theorem 1.1 below, we need the notion of an extension operator. Let $Y$ be a closed subspace of a Banach space $Z$. An operator $\Phi \in L(Y^*, Z^*)$ is called an extension operator if $(\Phi y^*)(y) = y^*(y)$ for all $y^* \in Y^*$ and all $y \in Y$. Remark that the existence of an extension operator is equivalent to the annihilator of $Y$ being complemented in $Z^*$.

Let us also recall that the nuclear norm $\|T\|_N$ of a nuclear operator $T \in N(X, Y)$ is defined by the equality

$$\|T\|_N = \inf \left\{ \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| : T = \sum_{n=1}^{\infty} x_n^* \otimes y_n \right\},$$

where the infimum is taken over all possible nuclear representations of $T$. It is straightforward to verify that if there are two more bounded linear operators $A$, acting to $X$, and $B$, acting from $Y$, then $BTA$ is a nuclear operator and $\|BTA\|_N \leq \|B\| \|T\|_N \|A\|$.\n
**Theorem 1.1.** Let $X$ be a Banach space. Let $Y$ be a closed subspace of a Banach space $Z$ and let $j : Y \rightarrow Z$ denote the identity embedding. Assume that there is an extension operator $\Phi \in L(Y^*, Z^*)$. If $X^*$ or $Z^*$ has the approximation property, then, for every operator $T \in L(X, Y)$, the following implication holds:

$$jT \in N(X, Z) \Rightarrow T \in N(X, Y). \quad (1.2)$$

Moreover,

$$\frac{1}{\|\Phi\|} \|T\|_N \leq \|jT\|_N \leq \|T\|_N.$$

Pairs of Banach spaces $Z$ and their closed subspaces $Y$ for which there exists an extension operator $\Phi \in L(Y^*, Z^*)$ were systematically studied by Fakhoury [3] and Kalton [11]. The existence of $\Phi$ with $\|\Phi\| = 1$ means, according to the terminology of Godefroy, Kalton and Saphar [5], that $Y$ is an ideal in $Z$. Different subclasses of ideals have been extensively studied by many authors (for references see [22, §4]).

Theorem 1.1 obviously contains the Grothendieck–Oja–Reinov result as a special case when $Z = Y^{**}$ and $\Phi = j_{Y^{**}}$. Other applications of Theorem 1.1 will be discussed in §5 of this article.

Sections 2 and 3 contain preliminaries and preparatory results needed for the proof of Theorem 1.1. For instance, we prove (see Proposition 2.1) that the approximation property of a Banach space $Z$ is inherited by its closed subspace $Y$ whenever there exists an extension operator $\Phi \in L(Y^*, Z^*)$. This result is essentially due to Kalton [11] and Lima [12], but we give here a direct proof that does not rely on the principle of local reflexivity. We also prove (see Theorems 3.3 and 3.4) that the existence of an extension
operator (respectively, a norm-preserving extension operator) is sufficient and necessary for projective tensor products to respect their subspace structure isomorphically (respectively, isometrically). This improves a result due to Grothendieck [7, Chapter I, p. 40] where $Y$ was assumed to be complemented in its bidual $Y^{**}$ (see Remark 3.5).

Section 4 contains the proof of Theorem 1.1. We have tried to give a self-contained proof with as few prerequisites as possible. It turns out (see Remark 4.1) that our proof is simpler than the existing proofs for the above-mentioned special case when $Z = Y^{**}$.

The notation we use is standard. We consider Banach spaces over the same, either real or complex, field. Let $X$ and $Y$ be Banach spaces. We denote by $L(X,Y)$ the Banach space of bounded linear operators from $X$ to $Y$ and by $F(X,Y)$ its linear subspace of finite-rank operators. If $A \in L(X,Y)$ is an into isomorphism, then its injection modulus $i(A)$ is defined by

$$i(A) = \sup\{\tau > 0 : \|Ax\| \geq \tau \|x\| \forall x \in X\}.$$ 

The identity operator on $X$ is denoted by $I_X$. We shall always consider $X$ as a subspace of $X^{**}$, identifying the canonical embedding $j_X : X \to X^{**}$ with the identity embedding.

2. Approximation property

A Banach space $X$ is said to have the approximation property if the identity operator $I_X$ on $X$ can be uniformly approximated on compact subsets of $X$ by bounded linear operators of finite rank. In other words, this means that $I_X$ belongs to the closure of $F(X,X)$ in the topology $\tau$ of uniform convergence on compact sets in $X$. Using the description (due to Grothendieck [7]) of the linear functionals on $L(X,X)$ which are continuous in $\tau$ (see, for example, [20, p. 31]), it is easy to show (see, for example, [20, p. 32]) that $X$ has the approximation property if and only if the following condition holds.

(AP). For all sequences $(x_n) \subset X$ and $(x_n^*) \subset X^*$ such that $\sum_{n=1}^{\infty} \|x_n^*\| \|x_n\| < \infty$ and $\sum_{n=1}^{\infty} x_n^*(x)x_n = 0$, whenever $x \in X$, one has $\sum_{n=1}^{\infty} x_n^*(x_n) = 0$.

Criterion (AP) is one of the most well known from the eight criteria of the approximation property established by Grothendieck in his memoir [7, Chapter I, p. 165], called by him the ‘condition de biunivocité’. Several recent criteria of the approximation property may be found in [13], [15], [14] and [23].

By now, it is well known that the approximation property is generally not inherited by subspaces. For instance, the spaces $\ell_p$, $p \neq 2$, and $c_0$ are saturated with subspaces which do not have the approximation property: every infinite-dimensional closed subspace of them contains a closed subspace without the approximation property (see, for example, [20, pp. 53 and 90] and [21, p. 107]). Relying on condition (AP), we shall prove the following positive result which is essentially known (see Remark 2.2 below).

**Proposition 2.1.** Let $Y$ be a closed subspace of a Banach space $Z$. Assume that there is an extension operator $\Phi \in L(Y^*, Z^*)$. If $Z$ has the approximation property, then $Y$ also has the approximation property.
Proof. Suppose that $Z$ has the approximation property. We shall use condition (AP) to show that $Y$ has the approximation property.

Let $(y_n) \subset Y$ and $(y_n^\ast) \subset Y^\ast$ satisfy $\sum_{n=1}^{\infty} \|y_n^\ast\| \|y_n\| < \infty$ and let $\sum_{n=1}^{\infty} y_n^\ast(y)y_n = 0$ for all $y \in Y$. Then $\sum_{n=1}^{\infty} y_n^\ast(y_n)y_n^\ast = 0$ for all $y^\ast \in Y^\ast$, the series being absolutely converging in $Y^\ast$. Therefore,

$$\sum_{n=1}^{\infty} y_n^\ast(y_n)\Phi y_n^\ast = 0 \quad \forall y^\ast \in Y^\ast,$$

meaning that

$$y^\ast\left(\sum_{n=1}^{\infty} (\Phi y_n^\ast)(z)y_n\right) = 0 \quad \forall y^\ast \in Y^\ast, \; \forall z \in Z.$$

Hence

$$\sum_{n=1}^{\infty} (\Phi y_n^\ast)(z)y_n = 0 \quad \forall z \in Z.$$

Since $\Phi$ is an extension operator and $Z$ has the approximation property, we have, using condition (AP) for $Z$,

$$\sum_{n=1}^{\infty} y_n^\ast(y_n) = \sum_{n=1}^{\infty} (\Phi y_n^\ast)(y_n) = 0.$$

Consequently, $Y$ has the approximation property. \square

Remark 2.2. Kalton [11, Theorem 5.1] and Lima [12, Corollary 2] proved Proposition 2.1 for a locally complemented subspace $Y$ of $Z$. A closed subspace $Y$ of a Banach space $Z$ is called locally complemented in $Z$ if there exists a constant $\lambda \geq 1$ such that whenever $F$ is a finite-dimensional subspace of $Z$ and $\varepsilon > 0$, there is a linear operator $T : F \to Y$ with $Tx = x$ for all $x \in F \cap Y$ and $\|T\| \leq \lambda + \varepsilon$. Kalton considered the separable case and the bounded approximation property, but his argument clearly works also for the general (non-separable) case and for the approximation property, and it was essentially applied by Lima. Let us mention that if $Y$ is locally complemented in $Z$, then, using a compactness argument due to Lindenstrauss [16], one can prove (see [3, Theorem 2.14] or [11, Theorem 3.5]) that there exists an extension operator $\Phi \in \mathcal{L}(Y^\ast, Z^\ast)$. On the other hand, by the principle of local reflexivity, the converse also holds true (see [3, Theorem 2.14] or [11, Theorem 3.5]).

3. Projective tensor products and nuclear operators

Let $X$ and $Y$ be Banach spaces. For the sake of readers who are not acquainted with the theory of tensor products of Banach spaces, let us recall that any element $u = \sum_{n=1}^{m} x_n \otimes y_n$ of the algebraic tensor product $X \otimes Y$ can be algebraically identified with the finite-rank operator

$$\sum_{n=1}^{m} x_n \otimes y_n : x^\ast \mapsto \sum_{n=1}^{m} x^\ast(x_n)y_n$$

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from $X^*$ to $Y$. Thus $X \otimes Y$ may always be viewed as a linear subspace of $\mathcal{F}(X^*, Y)$. In particular, $X^* \otimes Y = \mathcal{F}(X, Y^*)$.

The class of nuclear operators is closely related to the so-called projective tensor products of Banach spaces. Let us recall that the projective tensor product $X \hat{\otimes} Y$ of Banach spaces $X$ and $Y$ is the completion of the algebraic tensor product $X \otimes Y$ in the (projective or $\pi$-) norm $\| \cdot \|_\pi$ defined as

$$
\| u \|_\pi = \inf \left\{ \sum_{n=1}^{m} \| x_n \| \| y_n \| : u = \sum_{n=1}^{m} x_n \otimes y_n, \quad u \in X \otimes Y, \right\},
$$

where the infimum is taken over all possible representations of $u$.

The projective tensor product $X \hat{\otimes} Y$ has a simple description due to Grothendieck [7] (for a proof, we refer, for example, to [2, p. 227]): every $u \in X \hat{\otimes} Y$ has a representation

$$
u = \sum_{n=1}^{\infty} x_n \otimes y_n \quad \text{with} \quad \sum_{n=1}^{\infty} \| x_n \| \| y_n \| < \infty \quad (3.1)
$$

(the series is (absolutely) converging for the $\pi$-norm). Moreover,

$$
\| u \|_\pi = \inf \left\{ \sum_{n=1}^{\infty} \| x_n \| \| y_n \| : u = \sum_{n=1}^{\infty} x_n \otimes y_n, \quad u \in X \hat{\otimes} Y, \right\},
$$

where the infimum is taken over all representations of $u$ of the form (3.1).

The dual space of a projective tensor product is even easier to describe than the projective tensor product itself. A simple straightforward verification (see, for example, [2, pp. 229, 230]) shows that the dual space of $X \hat{\otimes} Y$ can be identified with $L(X, Y^*)$ or with $L(Y, X^*)$ under the duality

$$
\langle A, \sum_{n=1}^{\infty} x_n \otimes y_n \rangle = \sum_{n=1}^{\infty} (Ax_n)(y_n)
$$
or, respectively, under the duality

$$
\langle B, \sum_{n=1}^{\infty} x_n \otimes y_n \rangle = \sum_{n=1}^{\infty} (Bx_n)(y_n).
$$

This identification is, in fact, a linear isometry. Therefore, one writes

$$(X \hat{\otimes} Y)^* = L(X, Y^*) \quad \text{or} \quad (X \hat{\otimes} Y)^* = L(Y, X^*).$$

The above description was known already to Schatten [31].

Let $X$ be a Banach space and let $Y$ be a closed subspace of a Banach space $Z$. Denote by $j : Y \to Z$ the identity embedding and consider the identity embedding

$$I_X \otimes j : X \otimes Y \to X \hat{\otimes} Z.$$
It clearly satisfies
\[ \| (I_X \otimes j)u \|_\pi \leq \| u \|_\pi. \]

Let us denote its (unique) bounded extension to \( X \hat{\otimes} Y \) also by \( I_X \otimes j \) and call \( I_X \otimes j : X \hat{\otimes} Y \to X \hat{\otimes} Z \) the natural inclusion.

It is well known (see, for example, [2, pp. 230, 231]) that the natural inclusion \( I_X \otimes j \) need not be isometric: the projective tensor products do not respect the subspace structure. The natural inclusion need not even be isomorphic (see [7, Chapter I, p. 40]; for a stronger result in this direction, see [33, Theorem V.1]). However, as can be seen from the next result, the natural inclusion is an into isomorphism whenever an extension operator exists.

**Theorem 3.1.** Let \( X \) be a Banach space. Let \( Y \) be a closed subspace of a Banach space \( Z \) and let \( j : Y \to Z \) denote the identity embedding.

(a) If there exists an extension operator \( \Phi \in \mathcal{L}(Y^*, Z^*) \), then the natural inclusion \( I_X \otimes j : X \hat{\otimes} Y \to X \hat{\otimes} Z \) is an into isomorphism satisfying the inequalities
\[ \frac{1}{\| \Phi \|} \| u \|_\pi \leq \| (I_X \otimes j)u \|_\pi \leq \| u \|_\pi \quad \forall u \in X \hat{\otimes} Y. \]

(b) If the natural inclusion \( I_{Y^*} \otimes j : Y^* \hat{\otimes} Y \to Y^* \hat{\otimes} Z \) is an into isomorphism, then there exists an extension operator \( \Phi \in \mathcal{L}(Y^*, Z^*) \) with \( \| \Phi \| = 1/i(I_{Y^*} \otimes j) \).

**Proof.** (a) Consider any \( u = \sum_{n=1}^{\infty} x_n \otimes y_n \in X \hat{\otimes} Y \). One need only verify that
\[ \| u \|_\pi = \sum_{n=1}^{\infty} \langle A, (I_X \otimes j)u \rangle. \]

Since \( (X \hat{\otimes} Y)^* = \mathcal{L}(X,Y^*) \), there exists an operator \( A \in \mathcal{L}(X,Y^*) \) with \( \| A \| = 1 \) so that
\[ \| u \|_\pi = \langle A, u \rangle = \sum_{n=1}^{\infty} \langle Ax_n, y_n \rangle = \sum_{n=1}^{\infty} \langle \Phi(Ax_n), y_n \rangle. \]

On the other hand, since \( \Phi A \in \mathcal{L}(X,Z^*) = (X \hat{\otimes} Z)^* \), we have
\[ \sum_{n=1}^{\infty} \langle \Phi(Ax_n), y_n \rangle = \langle \Phi A, (I_X \otimes j)u \rangle \leq \| \Phi A \| \| (I_X \otimes j)u \|_\pi \]
\[ \leq \| \Phi \| \| (I_X \otimes j)u \|_\pi. \]

This yields the desired inequality.

(b) Using the description of duals of projective tensor products, let us consider
\[ (I_{Y^*} \otimes j)^*: \mathcal{L}(Y^*, Z^*) \to \mathcal{L}(Y^*, Y^*). \]

Since \( I_{Y^*} \otimes j \) is an into isomorphism, there exists a \( \Phi \in \mathcal{L}(Y^*, Z^*) \) so that \( (I_{Y^*} \otimes j)^* \Phi = I_{Y^*} \) and
\[ \| \Phi \| \leq \frac{1}{i(I_{Y^*} \otimes j)} \| I_{Y^*} \| = \frac{1}{i(I_{Y^*} \otimes j)}. \]
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(for a proof of this ‘folkloristic’ fact, see, for example, [8, Lemma 2.2]). Since, for all \( y^* \in Y^* \) and \( y \in Y \),

\[
(\Phi y^*)(y) = \langle \Phi, y^* \otimes y \rangle = (I_{Y^*} \otimes j)^* \Phi, y^* \otimes y \rangle = \langle I_{Y^*}, y^* \otimes y \rangle = y^*(y),
\]

\( \Phi \) is an extension operator. By the already-proved part (a), we also have \( i(I_{Y^*} \otimes j) \geq 1/\|\Phi\| \).

\[ \square \]

Remark 3.2. If \( \|\Phi\| = 1 \) in Theorem 3.1 (a), that is, if \( \Phi \) is a norm-preserving extension operator, then \( I_X \otimes j \) is an isometry and \( X \otimes Y \) is a subspace of \( X \otimes Z \) (this result was proved by Randrianantoanina [27] and Rao [28]). In particular, if \( Z = Y^{**} \) and \( \Phi = j_{Y^*} \), one has the well-known result due to Grothendieck [7] that \( X \hat{\otimes} Y \) is a subspace of \( X \hat{\otimes} Y^{**} \). The special case of Theorem 3.1 (a), when \( Y \) is complemented in \( Z \) (observe that if \( P \in \mathcal{L}(Z, Y) \) is a projection onto \( Y \), then \( P^* \) is clearly an extension operator), is well known (see [7, Chapter I, p. 40] or, for example, [30, p. 18]).

The following characterizations of the situations when the projective tensor products respect the subspace structure isomorphically or isometrically are immediate from Theorem 3.1.

**Theorem 3.3.** Let \( Y \) be a closed subspace of a Banach space \( Z \). Then the following assertions are equivalent.

(a) The natural inclusion from \( X \hat{\otimes} Y \) to \( X \hat{\otimes} Z \) is an into isomorphism for all Banach spaces \( X \).

(b) The natural inclusion from \( Y^* \hat{\otimes} Y \) to \( Y^* \hat{\otimes} Z \) is an into isomorphism.

(c) There exists an extension operator \( \Phi \in \mathcal{L}(Y^*, Z^*) \).

Moreover, in this case, \( i(I_X \otimes j) \geq i(I_{Y^*} \otimes j) \), for all Banach spaces \( X \), and

\[
i(I_{Y^*} \otimes j) = \max \{1/\|\Phi\| : \Phi \in \mathcal{L}(Y^*, Z^*) \text{ is an extension operator} \},
\]

where \( j \) denotes the identity embedding from \( Y \) to \( Z \).

**Theorem 3.4.** Let \( Y \) be a closed subspace of a Banach space \( Z \). Then the following assertions are equivalent.

(a) \( X \hat{\otimes} Y \) is a closed linear subspace of \( X \hat{\otimes} Z \) (under the natural inclusion) for all Banach spaces \( X \).

(b) \( Y^* \hat{\otimes} Y \) is a closed linear subspace of \( Y^* \hat{\otimes} Z \) (under the natural inclusion).

(c) There exists a linear norm-preserving extension operator from \( Y^* \) to \( Z^* \).

**Remark 3.5.** Theorem 3.3, together with Theorem 3.4, improves the following result due to Grothendieck [7, Chapter I, p. 40].

Let \( Y \) be a closed subspace of a Banach space \( Z \) such that \( Y \) is complemented in its bidual \( Y^{**} \). Then the natural inclusion from \( Y^* \hat{\otimes} Y \) to \( Y^* \hat{\otimes} Z \) is an into isomorphism if and only if \( Y \) is complemented in \( Z \).

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To see that Theorem 3.3 contains Grothendieck's result as a particular case, let us
observe that if \( P \in \mathcal{L}(Y^{**}, Y) \) is a projection onto \( Y \) and if \( \Phi \in \mathcal{L}(Y^*, Z^*) \) is an extension
operator, then, clearly, \( \Phi|_Y = I_Y \) and therefore \( P\Phi|_Z \) is a projection from \( Z \) onto \( Y \).
On the other hand, if \( Q \in \mathcal{L}(Z, Y) \) is a projection onto \( Y \), then \( Q^* \) is clearly an extension
operator.

Let \( X \) and \( Y \) be Banach spaces. Then there is a natural linear surjection from \( X^* 
\otimes Y \) onto \( \mathcal{N}(X, Y) \). It assigns to any \( u \in X^* \otimes Y \), having a representation
\[
u = \sum_{n=1}^{\infty} x_n^* \otimes y_n \quad \text{with} \quad \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty,
\]
the nuclear operator
\[
u = \sum_{n=1}^{\infty} x_n^* \otimes y_n : x \mapsto \sum_{n=1}^{\infty} x_n^*(x) y_n.
\]
If now \( X^* \) or \( Y \) has the approximation property, then using the description of \((X^* \otimes Y)^* \)
as \( \mathcal{L}(Y, X^{**}) \) or, respectively, \( \mathcal{L}(X^*, Y^*) \) together with the Hahn–Banach theorem,
and relying on condition (AP), it is rather straightforward to verify that the natural surjection
above is also injective (cf. [7, Chapter I, p. 167]). In fact, it is a linear isometry between
\( X^* \otimes Y \) and \( \mathcal{N}(X, Y) \) (this is obvious from the definitions of the norms \( \|\cdot\|_{\pi} \) and \( \|\cdot\|_{\mathcal{N}} \)).
Therefore, one writes
\[
X^* \otimes Y = \mathcal{N}(X, Y)
\]
whenever \( X^* \) or \( Y \) has the approximation property, identifying \( X^* \otimes Y \) and \( \mathcal{N}(X, Y) \) as
Banach spaces.

For the proof of our main Theorem 1.1, we shall need the following corollary of Propo-
sition 2.1 and Theorem 3.1 (a).

**Proposition 3.6.** Let \( X \) be a Banach space. Let \( Y \) be a closed subspace of a Banach
space \( Z \) and let \( j : Y \rightarrow Z \) denote the identity embedding. Assume that there is an
extension operator \( \Phi \in \mathcal{L}(Y^*, Z^*) \). If \( X^* \) or \( Z \) has the approximation property, then, for
every nuclear operator \( T \in \mathcal{N}(X, Y) \), the following inequalities hold:
\[
\frac{1}{\|\Phi\|} \|T\|_{\mathcal{N}} \leq \|jT\|_{\mathcal{N}} \leq \|T\|_{\mathcal{N}}.
\]

**Proof.** First of all, notice that \( Y \) also has the approximation property (see Proposition
2.1). Let a nuclear representation of \( T \) be given by
\[
u = \sum_{n=1}^{\infty} x_n^* \otimes y_n, \quad \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty.
\]
Then a nuclear representation of \( jT \) is given by \((I_{X^*} \otimes j)u = \sum_{n=1}^{\infty} x_n^* \otimes jy_n \in X^* \otimes Z \).
Since \( X^* \) or \( Y \) has the approximation property, \( \|T\|_{\mathcal{N}} = \|u\|_{\pi} \). Since \( X^* \) or \( Z \) has
the approximation property, \( \|jT\|_{\mathcal{N}} = \|(I_{X^*} \otimes j)u\|_{\pi} \). And the required inequalities are
obvious from Theorem 3.1 (a). \( \square \)
4. Proof of Theorem 1.1

First of all, notice that \( Z \) also has the approximation property if \( Z^* \) does (this is a well-known result due to Grothendieck [7] which is obvious from condition (AP)). Therefore, relying on Proposition 3.6, one need only prove that \( T \in \mathcal{N}(X, Y) \) whenever \( T \in \mathcal{L}(X, Y) \) and \( jT \in \mathcal{N}(X, Z) \). In other words, one need prove that \( jT \) belongs to the subspace \( \{ jS : S \in \mathcal{N}(X, Y) \} \) of \( \mathcal{N}(X, Z) \). This subspace, being isomorphic to \( \mathcal{N}(X, Y) \) (by Proposition 3.6), is a closed subspace of \( \mathcal{N}(X, Z) \). Therefore, it suffices to show that every continuous linear functional on \( \mathcal{N}(X, Z) \) that vanishes on \( \{ jS : S = x^* \otimes y : x^* \in X^*, \ y \in Y \} \), also vanishes on \( jT \). Let \( jT = \sum_{n=1}^{\infty} x_n^* \otimes z_n \) with \( x_n^* \in X^*, \ z_n \in Z \), and \( \sum_{n=1}^{\infty} \|x_n^*\| \|z_n\| < \infty \).

(1) Assume that \( X^* \) has the approximation property and use the canonical identifications

\[
\mathcal{N}(X, Z)^* = (X^* \hat{\otimes} Z)^* = \mathcal{L}(Z, X^{**}).
\]

Suppose that \( A \in \mathcal{L}(Z, X^{**}) \) satisfies

\[
\langle A, j(x^* \otimes y) \rangle = 0 \quad \forall x^* \in X^*, \ \forall y \in Y.
\]

This means that \( (Ax^*)(y^*) = 0 \) for all \( x^* \in X^* \) and all \( y \in Y \), or \( A j = 0 \) as an operator from \( Y \) to \( X^{**} \).

The desired equality

\[
\langle A, jT \rangle = \sum_{n=1}^{\infty} (Az_n)(x_n^*) = 0
\]

follows from condition (AP) for \( X^* \), because, for all \( x^* \in X^* \) and \( x \in X \),

\[
\left( \sum_{n=1}^{\infty} (Az_n)(x^*) x_n \right)(x) = \left( \sum_{n=1}^{\infty} x_n^*(x) Az_n \right)(x^*) = ((AjT)x)(x^*) = ((0T)x)(x^*) = 0.
\]

(2) Assume that \( Z^* \) has the approximation property and use the canonical identifications

\[
\mathcal{N}(X, Z)^* = (X^* \hat{\otimes} Z)^* = \mathcal{L}(X^*, Z^*).
\]

Suppose that \( A \in \mathcal{L}(X^*, Z^*) \) satisfies

\[
\langle A, j(x^* \otimes y) \rangle = 0 \quad \forall x^* \in X^*, \ \forall y \in Y.
\]

This means that \( (Ax^*)(jy) = (j^*Ax^*)(y) = 0 \) for all \( x^* \in X^* \) and \( y \in Y \), or \( j^*A = 0 \) as an operator from \( X^* \) to \( Y^* \).

To establish the desired equality

\[
\langle A, jT \rangle = \sum_{n=1}^{\infty} (Ax_n^*)(z_n) = 0,
\]
let us observe that, for all \( n \),

\[
(j^{**}\Phi^* z_n)(Ax_n^*) = (\Phi^* z_n)(j^* Ax_n^*) = (\Phi^* z_n)(0) = 0.
\]

Consequently,

\[
\langle A, j^T \rangle = \sum_{n=1}^{\infty} (z_n - j^{**}\Phi^* z_n)(Ax_n^*).
\]

Therefore, the equality \( \langle A, j^T \rangle = 0 \) follows from condition \((\text{AP})\) for \( Z^* \), because, for all \( z^* \in Z^* \), we have

\[
\sum_{n=1}^{\infty} (z_n - j^{**}\Phi^* z_n)(z^*)Ax_n^* = A\left(\sum_{n=1}^{\infty} z_n(z^*)x_n^* - \sum_{n=1}^{\infty} z_n(\Phi j^* z^*)x_n^*\right)
\]

\[
= A((j^T)^*(z^* - \Phi j^* z^*)) = (AT^*)(j^* z^* - j^* \Phi j^* z^*)
\]

\[
= (AT^*)(j^* z^* - j^* z^*) = 0.
\]

\( \square \)

**Remark 4.1.** If one carried out the proof of Theorem 1.1 in the well-known particular case (due to Grothendieck [7]) when \( Z = Y^{**} \) with \( \Phi = j_Y^* \), and \( X^* \) has the approximation property, then one would observe that our argument is simpler and shorter than the traditional ones (cf. [7, Chapter I, pp. 85, 86], [2, p. 243] or [30, p. 77]). If one carried out our proof in the other known particular case (see [24] or [25]) when \( Z = Y^{**} \) with \( \Phi = j_Y^* \), and \( Y^{***} \) has the approximation property, then one would observe that it is much simpler and shorter than the proof in [25]. The main difference is that the above-mentioned proofs rely on derivations of condition \((\text{AP})\), whereas our proof makes a direct use of this condition. We have also been trying to reduce to a minimum the use of tensor products machinery.

5. Comments and applications

The notation will be as above. In particular, \( X \) is a Banach space, \( Y \) is a closed subspace of a Banach space \( Z \), and \( j : Y \rightarrow Z \) denotes the identity embedding.

Theorem 1.1 shows that if \( Y \) is sufficiently well placed in \( Z \)—more precisely, if there exists an extension operator \( \Phi \in \mathcal{L}(Y^*, Z^*) \)—and if \( X^* \) or \( Z^* \) has the approximation property, then

\[
T \in \mathcal{L}(X, Y) \text{ and } j^T \in \mathcal{N}(X, Z) \Rightarrow T \in \mathcal{N}(X, Y).
\]

(2')

5.1.

As was already mentioned in §1, Theorem 1.1 contains the Grothendieck–Oja–Reinov result as an immediate particular case with \( Z = Y^{**} \) and \( \Phi = j_Y^* \). In this case, and therefore also in Theorem 1.1, the assumptions about the approximation properties of \( X^* \) and \( Z^* \) are essential and cannot be weakened to the approximation properties of \( X \) or/and \( Z \) (or even to the existence of bases). In fact, as it was shown by Oja and Reinov
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(2) holds for arbitrary Banach spaces X whenever Y is a complemented subspace of Z. Indeed, let P ∈ L(Z, Y) be a projection onto Y. If T ∈ L(X, Y) and jT ∈ N(X, Z), then T = PjT ∈ N(X, Y) (and, moreover, ∥T∥_{\mathcal{N}} ≤ ∥P∥∥jT∥_{\mathcal{N}}).

On the other hand, Y is complemented in Z whenever Y is complemented in its bidual Y^{**}, Y^{**} has the approximation property, and J : N(Y, Y) → N(Y, Z) is an into isomorphism. (This result, which is due to Grothendieck, easily follows from Proposition 5.1 (see Remark 3.5).)

Therefore, considering uncomplemented subspaces Y of Z, one can give numerous examples of situations when (2) does not hold despite the fact that X, Y, Z and all their higher duals have the approximation property. For instance, (2) does not hold for X = Y being an isomorphic copy of ℓ_2 in Z = ℓ_∞. Similar examples may be obtained using, for example, uncomplemented subspaces isomorphic to ℓ_1 in C[0, 1] (classical Banach–Mazur example) or to ℓ_2 in L_{4/3}[0, 1] (example due to Rosenthal [29, p. 52]).
Stegall and Retherford [33, p. 475] essentially showed that (2′) fails for Rosenthal’s example, and also if $X = \ell_2$ and $Y$ is a sufficiently Euclidean subspace, with a basis, of $Z = \ell_1$. In the last case, they used results established in [33] and a simple well-known fact (see, for example, [35, p. 126]) (that can also be used for the examples of the latter paragraph): if $X^*$ has the approximation property, then the natural embedding $J : \mathcal{N}(X,Y) \to \mathcal{N}(X,Z)$ is an into isomorphism if and only if the restriction mapping from $\mathcal{L}(Z,X^{**})$ to $\mathcal{L}(Y,X^{**})$ given by $A \mapsto A|_Y$ (which actually coincides with $J^*$) is surjective.

5.4.

In contrast with 5.3 above, let us observe the following result, which is essentially due to Stegall and Retherford [33]. For the definition and basic properties of $L_\infty$-spaces (or, more generally, $L_p$-spaces), we refer to [18] and [19].

**Theorem 5.2.** Implication (2′) holds for arbitrary Banach spaces $Z$ and their closed subspaces $Y$ if and only if $X$ is an $L_\infty$-space.

**Proof.** By [33, Theorem III.3], $X$ is an $L_\infty$-space if and only if for any Banach space $Z$ and for any $T \in \mathcal{N}(X,Z)$, the astriction $T_a : X \to \overline{\text{ran}} T$ is a nuclear operator. From this, our claim is straightforward.

5.5.

There are many important situations that are different, in general, from the classical one considered in 5.1, when an extension operator $\Phi \in \mathcal{L}(Y^*, Z^*)$ exists and Theorem 1.1 applies.

5.5.1.

Recall that a Banach space is said to be a $P_\lambda$-space, for some $\lambda \geq 1$, if it is complemented, by a projection whose norm does not exceed $\lambda$, in any Banach space containing it (as an isometrically isomorphic subspace). Recall also that, for every set $\Gamma$, the space $\ell_\infty(\Gamma)$ is a $P_1$-space (see, for example, [20, p. 105]).

The next result is due to Fakhoury [3, Corollary 3.3]. Fakhoury’s proof relies on Lindenstrauss’s memoir [16] and his own results established in [3]. We present a simple direct proof.

**Proposition 5.3.** Let $Y$ be a closed subspace of a Banach space $Z$. If $Y^{**}$ is a $P_\lambda$-space, then there exists an extension operator $\Phi \in \mathcal{L}(Y^*, Z^*)$ with $\|\Phi\| \leq \lambda$.

**Proof.** Since $j^{**} : Y^{**} \to Z^{**}$ is an into isometry, there exists an operator $P \in \mathcal{L}(Z^{**}, Y^{**})$ with $\|P\| \leq \lambda$ such that $Pj^{**}y^{**} = y^{**}$ for all $y^{**} \in Y^{**}$. Put $\Phi = j_Z P^* j_Y^*$. Since, for all $y^* \in Y^*$ and $y \in Y$,

$$(\Phi y^*)(jy) = (j_Y \cdot y^*)(Pj^{**}y) = (j_Y\cdot y^*)(Pj^{**}j_Y y) = (j_Y\cdot y^*)(j_Y y) = y^*(y),$$

$\Phi$ is an extension operator. □
Proposition 5.3 gives access to an immediate application of Theorem 1.1. However, in this case, it is possible to avoid the approximation conditions imposed on $X^*$ or $Z^*$.

**Corollary 5.4.** Let $Y^{**}$ be a $P_\lambda$-space. If $T \in \mathcal{L}(X,Y)$ and $jT \in \mathcal{N}(X,Z)$, then $T \in \mathcal{N}(X,Y)$ and

$$\frac{1}{\lambda} \|T\|_\mathcal{N} \leq \|jT\|_\mathcal{N} \leq \|T\|_\mathcal{N}.$$ 

**Proof.** Let $\Gamma$ be a set such that there exists an into isometry $i : Z \to \ell_\infty(\Gamma)$. Since $jT$ is nuclear, we have that $ijT \in \mathcal{N}(X,\ell_\infty(\Gamma))$. Because of Proposition 5.3, and since $\ell_\infty(\Gamma)^*$ has the (metric) approximation property, Theorem 1.1 applies. Hence, $T \in \mathcal{N}(X,Y)$ and

$$\frac{1}{\lambda} \|T\|_\mathcal{N} \leq \|ijT\|_\mathcal{N} \leq \|jT\|_\mathcal{N} \leq \|T\|_\mathcal{N}. \quad \square$$

Recall that the Banach–Mazur distance of two isomorphic Banach spaces $X$ and $Y$ is defined as $d(X,Y) = \inf \{\|\varphi\| \|\varphi^{-1}\| : \varphi$ is an isomorphism from $X$ onto $Y\}$.

**Corollary 5.5.** Let a subspace $Y$ of a Banach space $Z$ be isomorphic to $c_0(\Gamma)$, for some set $\Gamma$. If $T \in \mathcal{L}(X,Y)$ and $jT \in \mathcal{N}(X,Z)$, then $T \in \mathcal{N}(X,Y)$ and

$$\frac{1}{d(Y,c_0(\Gamma))} \|T\|_\mathcal{N} \leq \|jT\|_\mathcal{N} \leq \|T\|_\mathcal{N}.$$ 

**Proof.** Using that $c_0(\Gamma)^{**} = \ell_\infty(\Gamma)$ is a $P_1$-space, it is straightforward to verify that $Y^{**}$ is a $P_{d(Y,c_0(\Gamma))}$-space. And Corollary 5.4 applies. \quad \square

Concerning Corollary 5.5, let us mention that $d(Y,c_0) = 1$ implies that $Y$ is isometrically isomorphic to $c_0$ (this is proved in [9, p. 120], using $M$-ideal arguments).

It is well known that if $Y$ is an $\mathcal{L}_\infty$-space, then $Y^{**}$ is a $P_\lambda$-space for some $\lambda \geq 1$. (This clearly follows from Lindenstrauss’s memoir [16]. (In fact, if $Y$ is an $\mathcal{L}_\infty$-space, then, by definition, $Y$ is an $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda \geq 1$, easily implying that $Y$ is an $\mathcal{N}_\lambda$-space (defined in [16]). It is proved in [16] that $Y^{**}$ is a $P_\lambda$-space whenever $Y$ is an $\mathcal{N}_\lambda$-space.) Alternatively, this is also proved in [19, p. 335], relying on [18].) Therefore, Corollary 5.4 immediately yields the following.

**Corollary 5.6.** If $Y$ is an $\mathcal{L}_\infty$-space, then (2') holds for arbitrary Banach spaces $X$ and $Z$ whenever $Z$ contains $Y$.

The special case of Corollary 5.6 with $Z = \ell_\infty(\Gamma)$ may be restated in terms of quasi-nuclear operators [26] as follows. *All quasi-nuclear operators from any Banach space $X$ to any $\mathcal{L}_\infty$-space $Y$ are nuclear.* This result is due to Stegall and Retherford (see the theorem on p. 480 of [33]).
Let $E$ and $F$ be Banach spaces and let $\mathcal{K}(E, F)$ denote the subspace of compact operators of $\mathcal{L}(E, F)$. Assume that $E^*$ or $F$ has the $\lambda$-bounded approximation property for some $\lambda \geq 1$. (Recall that this means that the finite-rank operators in the definition of the approximation property can be chosen so that their norms do not exceed $\lambda$. Recall also that most of the common Banach spaces have the 1-bounded (or metric) approximation property.) In this case, an extension operator $\Phi \in \mathcal{L}(\mathcal{K}(E, F)^*, \mathcal{L}(E, F)^*)$ with $\|\Phi\| \leq \lambda$ can be constructed more or less explicitly (as it was by Johnson in the proof of Lemma 1 in [10]). And the following is immediate from Theorem 1.1.

**Corollary 5.7.** Let $X^*$ have the approximation property and let $E^*$ or $F$ have the $\lambda$-bounded approximation property. If $T \in \mathcal{L}(X, \mathcal{K}(E, F))$ and $jT \in \mathcal{N}(X, \mathcal{L}(E, F))$, then $T \in \mathcal{N}(X, \mathcal{K}(E, F))$ and

$$\frac{1}{\lambda} \|T\|_{\mathcal{N}} \leq \|jT\|_{\mathcal{N}} \leq \|T\|_{\mathcal{N}}.$$ 

We did not dare spell out the version of Corollary 5.7 under the hypothesis that $\mathcal{L}(E, F)^*$ had the approximation property since, for instance, by a well-known result of Szankowski [34], $\mathcal{L}(\ell_2, \ell_2)$ already fails to have the approximation property.

5.5.3.

Sims and Yost [32] have proved that every separable subspace of a (non-separable) Banach space $Z$ is contained in a separable closed subspace $Y$ which admits a norm-preserving extension operator $\Phi \in \mathcal{L}(Y^*, Z^*)$. From this and Theorem 1.1, the following is immediate.

**Corollary 5.8.** Let $X^*$ or $Z^*$ have the approximation property. Then every separable subspace of $Z$ is contained in a closed separable subspace $Y$ of $Z$ with the following property: if $T \in \mathcal{L}(X, Y)$ and $jT \in \mathcal{N}(X, Z)$, then $T \in \mathcal{N}(X, Y)$ and $\|T\|_{\mathcal{N}} = \|jT\|_{\mathcal{N}}$.

5.5.4.

It is well known and easy to verify that an extension operator $\Phi \in \mathcal{L}(Y^*, Z^*)$ exists if and only if the annihilator $Y^\perp$ coincides with the kernel of a bounded linear projection $P$ on $Z^*$. In this case, $\|\Phi\| = \|P\|$. If, moreover, $\|Pz^*\| + \|z^* - Pz^*\| = \|z^*\|$ for all $z^* \in Z^*$, then $Y$ is called an M-ideal. There is an extensive literature dealing with M-ideals. In particular, M-ideals have been described in many important classes of Banach spaces (e.g. in the space $A(K)$ of real-valued affine continuous functions on a compact convex set; in the space $C_0(S)$ of continuous functions on a locally compact Hausdorff space $S$ vanishing at infinity; in the disc algebra; in function algebras on a compact space $K$; in unital commutative Banach algebras; in $C^*$-algebras; in injective tensor products) (see the monograph [9] for results and references). Theorem 1.1 applies in all those situations. Let us conclude the paper with a couple of such applications.
Corollary 5.9. Let $C(K)$ denote the space of continuous functions on a compact space $K$. Let $D$ be a closed subset of $K$ and let $Y = \{ x \in C(K) : x(t) = 0 \text{ for all } t \in D \}$. If $T \in \mathcal{L}(X,Y)$ and $jT \in \mathcal{N}(X,C(K))$, then $T \in \mathcal{N}(X,Y)$ and $\|T\|_\mathcal{N} = \|jT\|_\mathcal{N}$.

In Corollary 5.9, we used the well-known fact that $C(K)^*$ (actually any higher dual of $C(K)$) has the (metric) approximation property.

Corollary 5.10. Let $Z$ be a $C^*$-algebra and let $Y$ be a closed two-sided ideal of $Z$. Assume that $X^*$ or $Z^*$ has the approximation property. If $T \in \mathcal{L}(X,Y)$ and $jT \in \mathcal{N}(X,Z)$, then $T \in \mathcal{N}(X,Y)$ and $\|T\|_\mathcal{N} = \|jT\|_\mathcal{N}$.

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References


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