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Closed and Exact Functions in the Context of Ginzburg–Landau Models

Dedicated to Professor George A. Elliott on the occasion of his sixtieth birthday

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Abstract. For a general vector field we exhibit two Hilbert spaces, namely the space of so called *closed functions* and the space of *exact functions* and we calculate the codimension of the space of exact functions inside the larger space of closed functions. In particular we provide a new approach for the known cases: the Glauber field and the second-order Ginzburg–Landau field and for the case of the fourth-order Ginzburg–Landau field.

1 Introduction

Statistical physics has developed a variety of interacting particle systems that capture some aspects of the movement of particles on the microscopic scale. An interacting particle system is usually a complex Markov process with a finite or infinite state space. By taking an appropriate scaling limit of an interacting particle system, we expect to derive the evolution of the system on the macroscopic scale, in general a nonlinear partial differential equation. The transition from microscopic scale to macroscopic scale is fairly well understood, at least for some systems, and in this note we take this step for granted.

The most interesting microscopic models constructed so far lack the so-called gradient condition. This condition corresponds to Fick's law of fluid dynamics according to which the instantaneous current w of particles over a bond is the gradient $\tau h - h$ of some local function h. Since the work of Varadhan [7], Quastel [4], and Varadhan and Yau [8] on nongradient systems, new ideas have been introduced in the field. The main idea is that a nongradient system has a generalized version of Fick's law, also called the fluctuation-dissipation equation, of the form $w \approx \hat{a}(m)(\tau h - h) + Lg$, where $\hat{a}$ is the transport coefficient depending on the particle density m in a microscopic cube, h is some local function, and L is the generator of the microscopic dynamics. The Lg part of the approximate equation above is negligible on the macroscopic scale, and is called the fluctuation part of the equation.

One of the main difficulties in finding the scaling limit of a nongradient system is to make rigorous sense of the fluctuation-dissipation equation. As has been shown in [3, 4, 7, 8], the current w, the gradient $\tau h - h$, and the fluctuations Lg are elements of the Hilbert space of *closed functions* and the fluctuation-dissipation equation is

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a consequence of a direct-sum decomposition of this Hilbert space. The gradient part $\tau h - h$ of the current w that survives after taking the scaling limit of the model is just the projection of w onto a one-dimensional subspace of the Hilbert space of closed functions. The remaining negligible fluctuations Lg are vectors of the Hilbert subspace of *exact functions*.

The purpose of the present paper is not to show how the Hilbert spaces of closed functions and exact functions arise in the context of interacting particle systems, but rather to motivate the direct-sum decomposition of the Hilbert space of closed functions and to find the codimension of the space of exact functions inside the space of closed functions. We calculate this codimension for an arbitrarily chosen vector field. The three continuum models known as the Glauber system, the second-order Ginzburg–Landau system and the continuum solid-on-solid model, also called the fourth-order Ginzburg–Landau system, are covered by our general result. Our approach to establishing the direct-sum decomposition of the Hilbert space of closed functions is new and differs from the approach used before to study the first two models (see [7]). We have followed a different path based on Fourier analysis that has allowed us to handle a general vector field.

2 The Decomposition Theorem

In this section we introduce some terminology and state the main result.

The Hermite polynomials provide an orthogonal basis for the Hilbert space of functions defined on the real axis that are square integrable with respect to the Gaussian probability measure $\frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})dx$. The i-th Hermite polynomial is defined through

$$H_i(x) = \frac{(-1)^i}{i!}\exp\Big(\frac{x^2}{2}\Big)\Big(\frac{d^i}{dx^i}\exp\Big(-\frac{x^2}{2}\Big)\Big), \quad i \in \mathbb{N}.$$

We stress that H_i is not normalized to have L^2 norm 1 with respect to the probability measure $\frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2}))dx$, but rather $\frac{1}{\sqrt{i!}}$.

There is an extension of Hermite polynomials to more variables. A multi-index is a double-sided infinite sequence $I = \{i_n\}_{n\in\mathbb{Z}}$ of positive integers, with at most finitely many non-zero entries. The degree of a multi-index is $|I| = \sum_{n\in\mathbb{Z}} i_n$. Call $\mathcal{I}$ the set of multi-indices and $\mathcal{I}_N$ the set of multi-indices of fixed degree N. The multidimensional Hermite polynomials are

$$H_I(x) = \Pi_{n\in\mathbb{Z}} H_{i_n}(x_n),\ I \in \mathcal{I}.$$

We assume the convention that if a multi-index I has some strictly negative entries, then $H_I = 0$. Together, the multidimensional Hermite polynomials $\{H_I\}_{I\in\mathcal{I}}$ form an orthogonal basis for the Hilbert space of functions defined on $\mathbb{R}^{\mathbb{Z}}$ that are square integrable with respect to the probability measure

$$d\nu_0^{gc} = \bigotimes_{i\in\mathbb{Z}} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x_i^2}{2}\Big)\,dx_i.$$

It is interesting to note that this Hilbert space is a model for the symmetric Fock space over the space of square summable, double-sided sequences $l^2(\mathbb{Z})$, and decomposes as a direct sum of the degree N subspaces $\mathcal{H}_N = \{H_I \mid |I| = N\}^c$. The superscript on the line above means that we take the closed linear span of the set.

The shift τ acts on configurations as $(\tau(x))_n = x_{n+1}$ and on functions as $(\tau f)(x) = f(\tau x)$, and τ^n stands for the n-fold composition $\tau \circ \cdots \circ \tau$. If a multi-index $I = (i_n)_{n\in\mathbb{Z}}$ has $i_n = 0$ for all $n < 0$, we shall say that the multi-index is supported on the set of positive integers. We shall use the notation δ_n for the multi-index that corresponds to the configuration with a single particle at the site n. Two multi-indices can be added and the addition is point-wise.

The action of the annihilation, creation, and shift operators on the multidimensional Hermite polynomial H_I is very simple:

$$\partial_n H_I(x) = H_{I-\delta_n}(x), \quad (x_n - \partial_n)H_I(x) = H_{I+\delta_n}(x), \quad \tau H_I = H_{\tau^{-1}I}.$$

Above, ∂_n stands for the partial derivative with respect to the n-th coordinate.

Given a double-sided sequence of real numbers $(a_k)_{k\in\mathbb{Z}}$, that are all but finitely many zero, consider the vector field $D_0 = \sum_{k\in\mathbb{Z}} a_k\partial_k$ with constant coefficients. Translating a's to the left or to the right produces a new sequence that defines the vector field $D_n = \sum_{k\in\mathbb{Z}} a_k\partial_{k+n}$, $n \in \mathbb{Z}$. Now we have the setup needed to introduce the closed and exact functions.

Definition 2.1 We shall say that a function $\xi \in L^2(\mathbb{R}^{\mathbb{Z}}, d\nu_0^{gc})$ is closed (or more precisely, D_0-closed) if it satisfies the condition $D_n(\tau^m\xi) = D_m(\tau^n\xi)$ in the weak sense, for all integers m and n. Let $\mathcal{C}_D$ denote the space of all D_0-closed functions.

Definition 2.2 We shall say that a function $\xi^g \in L^2(\mathbb{R}^{\mathbb{Z}}, d\nu_0^{gc})$ is exact (or more precisely D_0-exact) if there is a local function g, a function that depends on finitely many co-ordinates, such that $\xi^g = D_0\big(\sum_{k\in\mathbb{Z}} \tau^k g\big) = \sum_{k\in\mathbb{Z}} D_0(\tau^k g)$. Let $\mathcal{E}_D$ denote the closed linear span of the set of D_0-exact functions.

Although the infinite sum $\sum_{k\in\mathbb{Z}} \tau^k g$ does not make sense, after applying the differential operator D_0, we get a meaningful expression. Since g is a local function, the vector field D_0 kills all but finitely many terms of the infinite formal sum.

The terminologies of exact and closed functions are not arbitrarily chosen. We can define formally the form $w = \sum_{n\in\mathbb{Z}} \tau^n\xi \; dx_n$ and the boundary operator $df = \sum_{n\in\mathbb{Z}} D_n(f)\, dx_n$. With these new definitions, it is not hard to see that the form w is closed ($dw = 0$) in the vector calculus sense if and only if $D_n(\tau^m\xi) = D_m(\tau^n\xi)$, *i.e.*, if and only if ξ is a closed function.

Knowing that any exact function is closed, a natural question to ask is about the codimension of the space of exact functions inside the space of closed functions. In this paper we provide the answer for this question.

***Theorem 2.1* (Decomposition Theorem)** *Let $D_0 = \sum_{k\in\mathbb{Z}} a_k\partial_k$ be a vector field with constant real coefficients. All but finitely many numbers in the sequence $(a_k)_{k\in\mathbb{Z}}$ are zero. The following decomposition results hold.*

(i) *If the sum of the coefficients of the vector field D_0 is not equal to zero, then* $\mathcal{C}_D = \mathcal{E}_D$.

(ii) *If the sum of the coefficients of the vector field D_0 is equal to zero, then* $\mathcal{C}_D = \mathbb{R}\mathbf{1} \oplus \mathcal{E}_\mathbf{D}$.

2.1 Idea of the Proof for Theorem 2.1

We outline the main ideas used to prove the Decomposition Theorem. We shall show later that a function ξ is D_0-closed if and only if the projections $\mathrm{Proj}_{\mathcal{H}_N}\xi$, $N \geq 0$ are D_0-closed. Degree 0 subspace is easy to analyze since it is one dimensional. Any constant function is always D_0-closed, but is exact if and only if the sum of the coefficients of D_0 is not equal to zero. If the sum of the coefficients of D_0 is equal to zero, then any D_0-closed function is orthogonal on the degree 0 subspace. Therefore, the result of the theorem holds if we can prove that given a D_0-closed function ξ in $\mathcal{H}_N$, $N \geq 1$, the function ξ can be approximated with D_0-exact functions.

We shall investigate the properties of the Fourier coefficients of closed and exact functions, and we shall rather establish that the Fourier coefficients of a closed function can be approximated in an appropriate sense with Fourier coefficients of exact functions. The ideas will be elaborated in the following sections.

Note In two cases relevant for statistical physics questions, namely the second-order Ginzburg–Landau vector field $Y_0 = \partial_1 - \partial_0$ and the fourth-order Ginzburg–Landau vector field $X_0 = \partial_1 - 2\partial_0 + \partial_{-1}$, the decomposition result of Theorem 2.1 is equivalent to the fluctuation-dissipation equation mentioned in the introduction.

Note To get a flavour of the result stated in Theorem 2.1, we give some examples of exact and closed functions in the case of the fourth-order Ginzburg–Landau field, $X_0 = \partial_1 - 2\partial_0 + \partial_{-1}$: the functions $x_n + x_{-n} - 2x_0$ are X_0-exact, and $\mathbf{1}$, x_0, and $x_n + x_{-n}$ are examples of X_0-closed, but not X_0-exact functions. A strange phenomenon appears, for besides the function $\mathbf{1}$, there exists another function that is X_0-closed and not X_0-exact, namely x_0. Therefore, one might expect that the codimension of the space of exact functions is two. This is not the case and x_0 can in fact be approximated with exact functions.

3 The Set of Multi-Indices

A multi-index $I = \{i_n\}_{n\in\mathbb{Z}}$ can be thought of as a configuration of particles sitting on the sites of the lattice $\mathbb{Z}$. On top of the site n sit i_n particles. Rather than saying how many particles are at each site, we give the positions of the particles. This way we obtain a vector

$$z_I = (\underbrace{n_1 \dots n_1}_{i_{n_1}}, \dots, \underbrace{n_k \dots n_k}_{i_{n_k}}). \tag{3.1}$$

that lists, in increasing order, all occupied sites of I repeated according to the number of particles that occupy the site. We assume the only non-zero entries of the multi-index I are $i_{n_1}, \dots i_{n_k}$. Note that the dimension of the vector z_I is the degree of the multi-index I. If the multi-index has zero degree, then z_I is just a point. We say that z_I is a new coding of the multi-index I. This correspondence shows that the set $\mathcal{I}_N$ is bijective with the set of vectors of $\mathbb{Z}^N$ with entries in increasing order or is in bijection with the quotient space $\mathbb{Z}^N/S_N$, where S_N is the group of permutations of N letters.

For the results that follow we need to say more about the set of multi-indices. We partition the set of multi-indices into orbits with the help of the group action

$$\mathbb{Z} \times \mathbb{Z}^{\mathbb{Z}}, \longrightarrow \mathbb{Z}^{\mathbb{Z}} \quad (n, I) \longmapsto n \cdot I := \tau^n(I - \delta_n + \delta_0). \tag{3.2}$$

When restricted to $\mathbb{Z} \times \mathcal{I}$, the map (3.2) is not an action any more since the multi-indices that enumerate the basis of the L^2 space are constrained to have positive entries.

The orbits of the action (3.2) provide a partition of the set of multi-indices $\mathbb{Z}^{\mathbb{Z}}$. For each multi-index $I \in \mathcal{I}$ we define $o(I)$ to be the shadow of the orbit of I on the set $\mathcal{I}$, *i.e.*, $o(I) = \{J \mid J = n \cdot In \in \mathbb{Z}\} \cap \mathcal{I} = \{J \mid J = n \cdot In \in s(I)\}$. Here, $s(I) = \{n \in \mathbb{Z} \mid i_n \neq 0\}$ is the finite set of occupied positions of I. From now on we will refer to $o(I)$ as the orbit of I, although this is just a part of the actual orbit of the action. It has the advantage of being finite since the multi-index I has all but finitely many entries zero and there are just finitely many n's that after acting on I give rise to a multi-index with positive entries. All the multi-indices in the same orbit have the same degree. The orbits partition $\mathcal{I}$ and $\mathcal{I}_N$. Denote the set of orbits by $\mathcal{O}$, and the set of orbits containing multi-indices of degree N by $\mathcal{O}_N$.

It is worth mentioning that inside each orbit $o(I)$ there exists a unique representative supported on the positive integers. Denote this multi-index by $R(o(I))$. To see that this is true, let us assume that I has some particles in some negative position, *i.e.*, that there exists some $n < 0$ such that $i_n \geq 1$. If k is the leftmost occupied position of I and $k < 0$, then $k \cdot I \in o(I)$ is supported on the positive integers. Let us call $\mathcal{R}$ the set of all representatives, and $\mathcal{R}_N$ the set of degree-N representatives.

So far the orbit space $\{o(I)\}_{I \in \mathcal{I}}$ is an abstract object. Fortunately, we are able to give a concrete description of the orbit space. For this purpose it is very useful to know that any orbit o has a unique representative $R(o)$ supported on the positive semi-axis. The vector $z_{R(o)}$ is a point in the positive cone

$$\mathcal{C}_N^+ = \{z \in \mathbb{Z}^N \mid z = (z_1, \dots, z_N), 0 \leq z_1 \leq \dots \leq z_N\}.$$

Therefore the set of representatives, and in particular the set of orbits $\mathcal{O}_N$, are in bijective correspondence with the cone $\mathcal{C}_N^+$. Since there is only one multi-index with zero degree, namely $\mathbf{0} = (0)_{n \in \mathbb{Z}}$, the sets $\mathcal{I}_0$, $\mathcal{O}_0$ and $\mathcal{R}_0$ each contain just a single element. By convention, $\mathcal{C}_0^+$ is just the one-point set.

We can say even more about this picture. The cone $\mathcal{C}_N^+$ itself is an orbit space which we shall describe below.

Let us define the following transformations that act on the lattice $\mathbb{Z}^N$. For any $1 \leq i, j \leq N$,

$$\begin{aligned} \sigma_{i,j} \colon \mathbb{Z}^N \to \mathbb{Z}^N, &\quad \sigma_{i,j}(z_1, \dots, z_i, \dots, z_j \dots, z_N) = (z_1, \dots, z_j, \dots, z_i, \dots, z_N), \\ \gamma_1 \colon \mathbb{Z}^N \to \mathbb{Z}^N, &\quad \gamma_1(z_1, z_2, \dots, z_N) = (-z_1, z_2 - z_1 \dots, z_N - z_1). \end{aligned}$$

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The smallest group generated by $\sigma_{i,j}$, $1 \le i, j \le N$ and γ_1 will be denoted by $\widetilde{S}_N$. To see that $\widetilde{S}_N$ is isomorphic with the group of permutations of N letters, we write down the basic relations among the generating transformations: $(\gamma_1\sigma_{1,2})^3 = \mathbf{id}$ and $\gamma_1\sigma_{i,i+1} = \sigma_{i+1,i}\gamma_1$, $1 \le i \le N-1$. The group $\widetilde{S}_N$ has S_N, the group of permutations of N letters, as a subgroup, and $\widetilde{S}_N$ decomposes into left cosets with respect to S_N, as $\widetilde{S}_N = S_N \cup \gamma_1 S_N \cdots \cup \gamma_N S_N$, where the transformations γ_i are

$$\gamma_i \colon \mathbb{Z}^N \to \mathbb{Z}^N, \quad \gamma_i(z_1, z_2, \dots, z_N) = (-z_i, z_2 - z_i \dots, z_N - z_i).$$

It is interesting to note that $\mathbb{Z}^N/\widetilde{S}_N$ is in bijective correspondence with the cone $\mathcal{C}_N^+$, as the next argument proves. Any orbit of $\mathbb{Z}^N/\widetilde{S}_N$ contains at least one vector, let us say z, with components in increasing order. If this vector does not have positive coordinates, it means that $z_1 < 0$. But $(-z_1, z_2 - z_1, \dots, z_N - z_1)$ is still a point in the orbit of z under the action of $\widetilde{S}_N$. We can rearrange the coordinates of the new vector to be in increasing order and hence the orbit of z under the action of $\widetilde{S}_N$ contains at least one vector of the cone $\mathcal{C}_N^+$. To see that the orbit of z does not contain more than one vector of $\mathcal{C}_N^+$, we use the coset decomposition of $\widetilde{S}_N$. If z is in $\mathcal{C}_N^+$, then rearranging the coordinates of z we obtain either the vector z or some vector outside the cone $\mathcal{C}_N^+$. If we act on z or some other vector obtained from z by changing the places of the coordinates, with either of the transformations $\gamma_1, \dots, \gamma_N$ we get a vector that has at least one negative coordinate, and so does not belong to $\mathcal{C}_N^+$.

Now we can say that the set of orbits $\mathcal{O}_N$ is in bijective correspondence with the cone $\mathcal{C}_N^+$, and hence with the quotient space $\mathbb{Z}^N/\widetilde{S}_N$. The bijection is $o \in \mathcal{O}_N \mapsto z_{R(o)} \in \mathcal{C}_N^+$.

In addition, if I and J are two multi-indices in the same orbit of the action (3.2), then z_I and z_J are in the same orbit of the action of $\widetilde{S}_N$ on $\mathbb{Z}^N$. Assume that $J = n_j \cdot I$ with $I = \sum_{i=1}^k a_i\delta_{n_i}$, where $a_i \neq 0$ and $n_1 \le \dots \le n_k$. Then

$$J = \sum_{i=1,\dots,k, i\neq j} a_i\delta_{n_i - n_j} + (a_j - 1)\delta_0 + \delta_{-n_j},$$

and so

$$z_I = (\underbrace{n_1, \dots, n_1}_{a_1}, \dots, \underbrace{n_k, \dots, n_k}_{a_k}),$$

$$z_J = (-n_j, \underbrace{n_1 - n_j, \dots, n_1 - n_j}_{a_1}, \dots, \underbrace{0, \dots, 0}_{a_j - 1}, \dots, \underbrace{n_k - n_j, \dots, n_k - n_j}_{a_k}).$$

It follows that z_J is the image of z_I under some element of $\widetilde{S}_N$.

Let us denote by $z \overset{S_N}{\sim} z'$ and $z \overset{\widetilde{S}_N}{\sim} z'$ two lattice points z and z' that have the same image in the quotient space $\mathbb{Z}^N/S_N$ and $\mathbb{Z}^N/\widetilde{S}_N$, respectively.

Before we leave this section it is important to notice the following crucial facts. Let $N \ge 1$. Since $\mathcal{I}_N$ is identified with $\mathbb{Z}^N/S_N$, we can think of any function $\hat{\xi} \colon \mathcal{I}_N \to \mathbb{R}$

as being an S_N-invariant function $\hat{\xi}\colon \mathbb{Z}^N \to \mathbb{R}$, where $\hat{\xi}(z) = \hat{\xi}(I)$ if $z \overset{S_N}{\sim} z_I$. Similarly, since $\mathcal{O}_N$ is identified with $\mathbb{Z}^N/\widetilde{S}_N$ we can think of any function $c\colon \mathcal{O}_N \to \mathbb{R}$ as being a $\widetilde{S}_N$-invariant function $\widetilde{c}\colon \mathbb{Z}^N \to \mathbb{R}$, where $\widetilde{c}(z) = c(o)$ if there exists a multi-index $I \in o$ such that $z \overset{S_N}{\sim} z_I$.

4 Properties of Closed Functions and of Exact Functions

This section contains a detailed study of closed and exact functions.

4.1 Closed Functions

We start with a very simple but important property of closed functions.

Lemma 4.1 *Let D_0 be a vector field with constant coefficients, $D_0 = \sum_{k\in\mathbb{Z}} a_k \partial_k$. Assume that all but finitely many coefficients of the vector field D_0 are zero. A function $\xi \in L^2(\mathbb{R}^{\mathbb{Z}}, d\nu_0^{gc})$ is D_0-closed if and only if the projection* $\mathrm{Proj}_{\mathcal{H}_N}\xi$ *onto the degree N subspace $\mathcal{H}_N$ is D_0-closed for any $N \geq 0$.*

Proof By ∂_j is meant the differential operator with respect to the j-th coordinate. We shall also consider the adjoint operator $\partial_j^* = -\partial_j + x_j$ the adjoint operator of ∂_j. The adjoint is taken with respect to the inner product $\langle\,,\rangle$ of $L^2(\mathbb{R}^{\mathbb{Z}}, d\nu_0^{gc})$. The operators ∂_j and ∂_j^* are bounded operators when restricted to a degree subspace, although they are unbounded on the whole space $L^2(\mathbb{R}^{\mathbb{Z}}, d\nu_0^{gc})$.

If $\xi \in \mathcal{H}_N$, with Fourier series $\xi = \sum_{I\in\mathcal{I}_N} \hat{\xi}_I H_I$, then the image of ξ under the operator ∂_j is $\partial_j(\xi) = \sum_{I\in\mathcal{I}_N} \hat{\xi}_I H_{I-\delta_j}$, with the convention that if the multi-index $I-\delta_j$ has some negative entries, then $H_{I-\delta_j} = 0$. For a function $f \in L^2(\mathbb{R}^{\mathbb{Z}}, d\nu_0^{gc})$ denote by $\|f\| = \sqrt{\langle f, f\rangle}$ the L^2 norm of f.

The operators ∂_j and ∂_j^* act on the degree N subspaces as follows:

$$\partial_j(\mathcal{H}_N) \subseteq \mathcal{H}_{N-1},\ N \geq 1, \quad \partial_j^*(\mathcal{H}_N) \subseteq \mathcal{H}_{N+1},\ N \geq 0.$$

The boundedness of these operators follows from the observation that

$$\frac{1}{(N!)^N} \leq \inf_{I\in\mathcal{I}_N} \|H_I\|^2 \leq \sup_{I\in\mathcal{I}_N} \|H_I\|^2 \leq 1,$$

and from the existence of two strictly positive constants, C_1^N, C_2^N, that depend just on N, such that

$$C_1^N \sum_{I\in\mathcal{I}_N} \hat{\xi}_I^2 \leq \|\xi\|^2 \leq C_2^N \sum_{I\in\mathcal{I}_N} \hat{\xi}_I^2, \quad C_1^N \sum_{I\in\mathcal{I}_N} \hat{\xi}_I^2 \leq \|\partial_j(\xi)\|^2 \leq C_2^N \sum_{I\in\mathcal{I}_N} \hat{\xi}_I^2. \tag{4.1}$$

Indeed,

$$\|\partial_j\xi\|^2 = \sum_{I\in\mathcal{I}_N} \hat{\xi}_I^2 \|H_{I-\delta_j}\|^2 \leq \sum_{I\in\mathcal{I}_N} \hat{\xi}_I^2 \leq (N!)^N \sum_{I\in\mathcal{I}_N} \hat{\xi}_I^2 \|H_I\|^2 \leq (N!)^N \|\xi\|^2,$$

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and hence the norm of the operator $\partial_j \colon \mathcal{H}_N \to \mathcal{H}_{N-1}$ is bounded above by $(N!)^N$.

The vector field D_0 with constant coefficients has similar properties:

$$D_0(\mathcal{H}_N) \subseteq \mathcal{H}_{N-1},\ N \geq 1, \quad D_0^*(\mathcal{H}_N) \subseteq \mathcal{H}_{N+1},\ N \geq 0.$$

For any function $\xi \in L^2(\mathbb{R}^{\mathbb{Z}}, d\nu_0^{gc})$ and any test function $\phi \in \mathcal{H}_{N-1}$ we have

$$\begin{aligned} \langle D_n(\tau^m \xi), \phi \rangle &= \xi, \tau^{-m}(D_n^* \phi) \rangle = \langle \operatorname{Proj}_{\mathcal{H}_N} \xi, \tau^{-m}(D_n^* \phi) \rangle \\ &= \langle D_n(\tau^m \operatorname{Proj}_{\mathcal{H}_N} \xi), \phi \rangle, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \langle D_m(\tau^n \xi), \phi \rangle &= \langle \xi, \tau^{-n}(D_m^* \phi) \rangle = \langle \operatorname{Proj}_{\mathcal{H}_N} \xi, \tau^{-n}(D_m^* \phi) \rangle \\ &= \langle D_m(\tau^n \operatorname{Proj}_{\mathcal{H}_N} \xi), \phi \rangle. \end{aligned} \tag{4.3}$$

It follows that $D_n(\tau^m \xi) = D_m(\tau^n \xi)$ in the weak sense if and only if

$$D_n(\tau^m \operatorname{Proj}_{\mathcal{H}_N} \xi) = D_m(\tau^n \operatorname{Proj}_{\mathcal{H}_N} \xi)$$

in the strong sense for all $N \geq 0$.

We recall that a function ξ is closed if and only if $D_n(\tau^m \xi) = D_m(\tau^n \xi)$ for all $m, n \in \mathbb{Z}$, which, by the previous equalities (4.2) and (4.3), is equivalent to

$$D_n(\tau^m \operatorname{Proj}_{\mathcal{H}_N} \xi) = D_m(\tau^n \operatorname{Proj}_{\mathcal{H}_N} \xi) \quad m, n \in \mathbb{Z}, \quad N \geq 0.$$

Therefore, a function ξ is closed if and only if $\operatorname{Proj}_{\mathcal{H}_N} \xi$ is closed for all $N \geq 0$. ■

Note If $\xi = \sum_{I \in \mathcal{I}_N} \hat{\xi}_I H_I$ is a function inside the space $\mathcal{H}_N$, two norms can be defined for ξ: the L^2 norm $\|\xi\|$ and the sum of squared Fourier coefficients $\sum_{I \in \mathcal{I}_N} \hat{\xi}_N^2$. It is important to note the inequality (4.1) implies that these two norms define the same topology on the space $\mathcal{H}_N$.

Note Assume that $\xi \in \mathcal{H}_N$ is a D_0-closed function, with Fourier series expansion $\xi = \sum_{I \in \mathcal{I}} \hat{\xi}_I H_I$. We calculate,

$$D_n \xi = \sum_{I \in \mathcal{I}} \Big[\sum_{k \in \mathbb{Z}} a_k \hat{\xi}_{I + \delta_{(n+k)}} \Big] H_I, \quad D_0(\tau^n \xi) = \sum_{I \in \mathcal{I}} \Big[\sum_{k \in \mathbb{Z}} a_k \hat{\xi}_{\tau^n(I + \delta_k)} \Big] H_I.$$

Therefore, a function is closed if and only if its Fourier coefficients satisfy the relations

$$\sum_{k \in \mathbb{Z}} a_k \hat{\xi}_{I + \delta_{(n+k)}} = \sum_{k \in \mathbb{Z}} a_k \hat{\xi}_{\tau^n(I + \delta_k)} \quad n \in \mathbb{Z},\ I \in \mathcal{I}. \tag{4.4}$$

4.2 Construction of Exact Functions

It is important to have some examples of functions that are exact. The functions which will be constructed next will be used in the proof of Theorem 2.1, to approximate closed functions with exact ones.

Lemma 4.2 *Let c be a function defined on the set of orbits with finite support,* i.e., *$c(o) = 0$ except for finitely many orbits o. The function*

$$\xi = \sum_{o\in\mathcal{O}} c(o) D_0 \Big[\sum_{n\in\mathbb{Z}} \tau^n H_{R(o)+\delta_0} \Big]$$

is D_0-exact and the Fourier coefficients of ξ are

$$\hat{\xi}_I = \sum_{k\in\mathbb{Z}} a_k c(o(\tau^{-k} I)).$$

Proof The function ξ which has been introduced is well defined since the sum is over a finite set, and it is exact, as a sum of exact functions. To obtain the conclusion of the lemma we need to calculate the Fourier coefficients of ξ. We have

$$\begin{aligned} \text{(4.5)} \quad \xi &= \sum_{o\in\mathcal{O}} c(o) D_0 \Big[\sum_{n\in\mathbb{Z}} H_{\tau^{-n}(R(o)+\delta_0)} \Big] = \sum_{o\in\mathcal{O}, n\in\mathbb{Z}} c(o) \sum_{k\in\mathbb{Z}} a_k H_{\tau^{-n}(R(o)+\delta_0-\delta_{-n+k})} \\ &= \sum_{o\in\mathcal{O}, n\in\mathbb{Z}} c(o) \sum_{k\in\mathbb{Z}} a_k H_{\tau^{-k}[(-n+k)\cdot R(o)]} = \sum_{I\in\mathcal{I}} \sum_{k\in\mathbb{Z}} a_k c(o(\tau^k I)) H_I. \end{aligned}$$

To justify the integration by parts in (4.5), we make the following observation. For any multi-index $I \in \mathcal{I}$, there exists a unique orbit $o \in \mathcal{O}$ and a unique integer $n \in \mathbb{Z}$ such that $I = \tau^{-k}[(-n+k)\cdot R(o)]$. This is a consequence of the freeness of the action (3.2). Moreover, the orbit o is the same as $o(\tau^k I)$. We stress again that the sums in (4.5) are over finite sets as c has finite support. Actually all computations that we carried out to prove this lemma are valid because c is a function with finite support and the sums are finite, although this was not emphasized each time we used it. Also we have made use of the convention that $H_I = 0$ if I is a multi-index with negative entries. ■

Lemma 4.3 *Let $N \geq 1$ be a natural number and $e = (1, \dots, 1) \in \mathbb{Z}^N$. In addition if $\widetilde{c}$ is a real-valued function defined on $\mathbb{Z}^N$ with finite support and $\widetilde{S}_N$-invariant, then the function*

$$\text{(4.6)} \qquad \xi_{\widetilde{c}} = \sum_{I\in\mathcal{I}_N} \Big(\sum_{k\in\mathbb{Z}} a_k \widetilde{c}(z_I - ke) \Big) H_I$$

is a well-defined D_0-exact function in the degree N subspace $\mathcal{H}_N$.

Proof This lemma follows from Lemma 4.2. Since $\widetilde{c}\colon \mathbb{Z}^N \to \mathbb{R}$ is $\widetilde{S}_N$-invariant, it makes sense to introduce the function $c\colon \mathcal{O} \to \mathbb{R}$, where $c(o) = \widetilde{c}(z_I)$ if I is a multi-index in the orbit o of degree N, and $c(o) = 0$ otherwise. We should note that if I is a multi-index in the orbit o, then $z_I + ke = z_{\tau^{-k}I}$ and $\widetilde{c}(z_I - ke) = c(o(\tau^k I))$. Hence the Fourier coefficients of the function $\xi_{\widetilde{c}}$ are of the form $\sum_{k\in\mathbb{Z}} a_k c(o(\tau^k I))$, and the function $\xi_{\widetilde{c}}$ is D_0-exact. ∎

In the previous lemma an operator has come out in a natural way in our construction of exact functions. Below we provide the exact definition of this operator.

Definition 4.4 Let $D_0 = \sum_{k\in\mathbb{Z}} a_k\partial_k$ be a vector field with constant coefficients, all the coefficients being zero except finitely many. The vector field D_0 defines an operator T_{D_0} that acts on a function $c\colon \mathbb{Z}^N \to \mathbb{R}$ to produce a function $T_{D_0}c\colon \mathbb{Z}^N \to \mathbb{R}$, where

$$(T_{D_0}c)(z) = \sum_{k\in\mathbb{Z}} a_k c(z - ke), \quad z \in \mathbb{Z}^N.$$

Here e is the vector $(1,\dots,1)$ of the lattice $\mathbb{Z}^N$.

5 Proof of the Decomposition Theorem 2.1

We start by listing two important properties of the operator T_{D_0} introduced at the end of the previous section.

Lemma 5.1 *Let c be a real-valued function defined on the lattice $\mathbb{Z}^N$, $N \geq 1$. Assume that the function c is square-summable and $\widetilde{S}_N$- invariant. Then there exists a sequence $(c_n)_{n\geq 1}$ of real-valued, finitely supported, $\widetilde{S}_N$-invariant functions such that $T_{D_0}c_n \to T_{D_0}c$ as $n \to \infty$ and the convergence is in the Hilbert space topology of $L^2(\mathbb{Z}^N)$.*

Proof We define a sequence of $\widetilde{S}_N$-invariant regions of the lattice $\mathbb{Z}^N$, namely,

$$P_i = \bigcup_{\gamma\in\widetilde{S}_N} \gamma\{z = (z_1,\dots,z_N) \in \mathbb{Z}^N | 0 \leq z_1 \leq \cdots \leq z_N \leq i-1\}, \quad i \geq 1.$$

For the readers convenience we add two pictures of the region P_i in dimension $N = 1$, respectively, $N = 2$. In dimension $N = 1$ the region P_i contains the lattice points inside the segment $[-i+1, i-1]$ (Figure 1), whereas in dimension $N = 2$ the region P_i contains the lattice points inside the hexagon shown in Figure 2

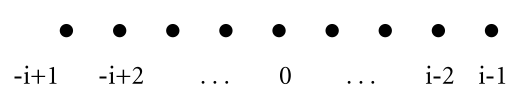

Figure 1: The region P_i in dimension $N = 1$.

Besides being $\widetilde{S}_N$-invariant, the sequence of regions $(P_i)_{i\geq 1}$ defined above grows to cover the entire lattice $\mathbb{Z}^N$ as $i \to \infty$. Define c_n to be $c\mathbf{1}_{P_n}$, for $n \geq 1$. Since $\mathbf{1}_{P_n}$ is the characteristic function of the region P_n, we have immediately that c_n is a finitely

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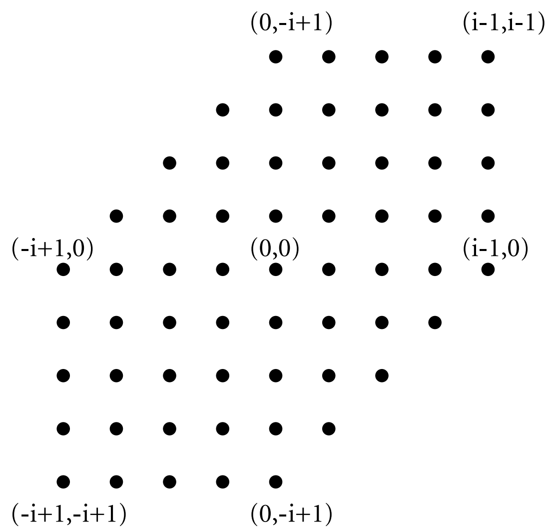

Figure 2: The region p_i in dimension $n = 2$.

supported, $\widetilde{S}_N$-invariant function. Square-summability of c implies that $c_n \to c$ as $n \to \infty$ in the topology of $L^2(\mathbb{Z}^N)$ (the norm $\|c - c_n\|^2 = \sum_{z \notin P_n} c^2(z)$ involves only the values of c outside the region P_n, and these values decay to zero as $n \to \infty$ since c is square-summable). Then, obviously, $c_n \to c$ and $Tc_n \to Tc$ as $n \to \infty$ in the topology of $L^2(\mathbb{Z}^N)$. ■

Below we discuss certain facts about the Fourier transform of functions defined on the lattice $\mathbb{Z}^N$. The Fourier transform of a function $c\colon \mathbb{Z}^N \to \mathbb{C}$ is formally defined to be

$$\mathcal{F}c\colon [-\pi, \pi)^N \to \mathbb{C}, \quad \mathcal{F}c(\alpha) = \frac{1}{\sqrt{2\pi}} \sum_{z \in \mathbb{Z}^N} c(z) e^{iz\alpha}.$$

In the exponent above $z\alpha$ stands for the dot product $z_1\alpha_1 + \cdots + z_N\alpha_N$. The reader may consult Rudin [5] for an extended treatment of the Fourier transform of functions defined on a lattice. We remind the reader that $\mathcal{F}$ is an isometry between the spaces $L^2(\mathbb{Z}^N)$ and $L^2([-\pi, \pi)^N)$. The space $L^2([-\pi, \pi)^N)$ is considered with respect to Lebesgue measure on $[-\pi, \pi)^N$. Also if c is invariant under a certain group of transformations, then $\mathcal{F}c$ is invariant, as well. Note though that the symmetry group of $\mathcal{F}c$ might not coincide with the symmetry group of c. Indeed, if c is symmetric, or S_N-invariant, then $\mathcal{F}c$ is symmetric. Now suppose that c is $\widetilde{S}_N$-invariant. Then $\mathcal{F}c$ is invariant under the action of the group $\widetilde{\Sigma}_N$ generated by the transformations

$$s_{ii+1}\colon [-\pi, \pi)^N \to [-\pi, \pi)^N, \quad 1 \le i \le N-1,$$

$$s_{ii+1}(\alpha_1, \ldots, \alpha_N) = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_N),$$

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and

$$g\colon [-\pi,\pi)^N \to [-\pi,\pi)^N,$$
$$g(\alpha_1,\dots,\alpha_N) = (\operatorname{mod}_{2\pi}(-\alpha_1 - \cdots - \alpha_N), \alpha_2,\dots,\alpha_N).$$

On the line above we used the notation $\operatorname{mod}_{2\pi}(t)$. Any real number t can be written uniquely as $2\pi a + b$, where a is an integer number and b is a real number in the interval $[-\pi,\pi)$. By $\operatorname{mod}_{2\pi}(t)$ we denote the remainder b. It is also true that if the Fourier transform $\mathcal{F}c$ is $\widetilde{\Sigma}_N$-invariant, then c is $\widetilde{S}_N$-invariant.

In Section 4 it was established that a function $\xi = \sum_{I\in\mathcal{I}_N} \hat{\xi}_I H_I$ is D_0-closed if and only if the following holds:

$$\sum_{k\in\mathbb{Z}} a_k \hat{\xi}_{I+\delta_{(n+k)}} = \sum_{k\in\mathbb{Z}} a_k \hat{\xi}_{\tau^n(I+\delta_k)} \quad n\in\mathbb{Z},\ I\in\mathcal{I}. \tag{5.1}$$

Obviously we can use the Fourier coefficients of ξ to construct an S_N-invariant function $\hat{\xi}\colon \mathbb{Z}^N \to \mathbb{R}$, $\hat{\xi}(z) = \hat{\xi}_I$ if $z \overset{S_N}{\sim} z_I$. The relations (5.1) force our function ξ to satisfy

$$\sum_{k\in\mathbb{Z}} a_k \hat{\xi}(z + ke_1) = \sum_{k\in\mathbb{Z}} a_k \hat{\xi}(z - z_1 e - (z_1 + k)e_1), \quad z = (z_1,\dots,z_N)\in\mathbb{Z}^N. \tag{5.2}$$

The vectors e and e_1 of the lattice $\mathbb{Z}^N$ are $(1,\dots,1)$ and $(1,0,\dots,0)$, respectively. After applying the Fourier transform in both sides of equation (5.2), we find that $\hat{\xi}$ satisfies

$$p(e^{-i\alpha_1})(\mathcal{F}\hat{\xi})(\alpha) = p(e^{i(\alpha_1+\cdots+\alpha_N)})(\mathcal{F}\hat{\xi})(g(\alpha)), \qquad \alpha = (\alpha_1,\dots,\alpha_N)\in[-\pi,\pi)^N, \tag{5.3}$$

where p is the rational function $p(x) = \sum_{k\in\mathbb{Z}} a_k x^k$. To wrap up our argument, we can say that the D_0-closedness condition implies property (5.3).

Next we shall establish a crucial fact about functions that satisfy relation (5.3).

Lemma 5.2 *Let $\hat{\xi}$ be a real-valued, S_N-invariant function defined on the lattice $\mathbb{Z}^N$, $N \geq 1$ that satisfies* (5.3). *Then there exists a sequence $(c_n)_{n\geq 1}$ of real-valued, square-summable, $\widetilde{S}_N$-invariant functions defined on the lattice $\mathbb{Z}^N$ such that $T_{D_0}c_n \to \hat{\xi}$ as $n\to\infty$ in the topology of $L^2(\mathbb{Z}^N)$.*

Proof Examples of functions $\hat{\xi}$ satisfying the properties listed in the hypothesis of this lemma, are the functions constructed, as explained before in this section, from the Fourier coefficients of closed functions.

The first step towards establishing our result is to solve, at least on a formal level, the equation $T_{D_0}c = \hat{\xi}$. The Fourier transform for functions defined on the lattice will help us to make our guess.

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After applying the Fourier transform in each side of the equation $T_{D_0}c = \hat{\xi}$, we get

$$p(e^{i(\alpha_1+\cdots+\alpha_N)})(\mathcal{F}c)(\alpha) = (\mathcal{F}\hat{\xi})(\alpha), \quad \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}^N,$$

where p is the rational function $\sum_{j\in\mathbb{Z}} a_j x^j$ canonically associated to T_{D_0}.

After multiplying the equation $p(x) = 0$ with a high enough power of x, we see that any solution x of $p(x) = 0$ has to be a root of a certain polynomial. Since the number of roots of any polynomial is finite, the number of solutions of $p(x) = 0$ is finite as well. If the equation $p(x) = 0$ has no solutions on the unit circle, then the equation $T_{D_0}c = \hat{\xi}$ can be solved in the space $L^2(\mathbb{Z}^N)$. Indeed the unique, square-summable solution of $T_{D_0}c = \hat{\xi}$ is

$$c = \mathcal{F}^{-1}\Big(\frac{1}{p(e^{i(\alpha_1+\cdots+\alpha_N)})}(\mathcal{F}\hat{\xi})\Big).$$

That $\hat{\xi}$ has been built out of the Fourier coefficients of a closed function implies that c is $\widetilde{S}_N$-invariant. Hence the lemma is proved in this case. We can choose $c_n = c$, for any $n \geq 1$.

A more involved case is when the equation $p(x) = 0$ has solutions on the unit circle. If the sum of the coefficients of p is equal to 0, then the number 1 is a solution of $p(x) = 0$. The cases arising from interacting particle models are of this kind. The difficulty in this case arises because the equation $Tc = \hat{\xi}$ cannot be solved in $L^2(\mathbb{Z}^N)$. To get around this problem, we shall consider a slightly modified equation $\mathcal{F}(T_{D_0}c) = (\mathcal{F}\hat{\xi})1_{A_n}$. The function $\mathcal{F}\hat{\xi}$ is multiplied with the characteristic function of a set A_n that is $\widetilde{\Sigma}_N$-invariant and carefully chosen to avoid the unit roots of p. More precisely,

$$A_n = \bigcap_{\gamma\in\widetilde{\Sigma}_N} \Big\{\gamma(\alpha) \;\Big|\; \alpha = (\alpha_1, \ldots, \alpha_N) \in [-\pi, \pi]^N,$$
$$|\operatorname{mod}_{2\pi}(\alpha_1 + \cdots + \alpha_N) - r_k| > \frac{1}{n}, e^{ir_k} \text{ unit root of } p\Big\}.$$

The next table contains the roots of the rational function $p(x)$ in four particular cases.

Vector field D_0	Rational function $p(x)$	Solutions of $p(x) = 0$
∂_0	1	none
$\partial_1 - \partial_0$	$x - 1$	1
$\partial_1 - 2\partial_0 + \partial_{-1}$	$x - 2 + x^{-1}$	$1, 1$
$\partial_3 - \partial_0$	$x^3 - 1$	$1, \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}$

If the vector field D_0 is $\partial_1 - \partial_0$ or $\partial_1 - 2\partial_0 + \partial_{-1}$, then the region A_n in dimension $N = 1$ is just $A_n = \{\alpha \in [-\pi, \pi) \mid |\alpha| > \frac{1}{n}\}$ (Figure 3).

If the vector field D_0 is $\partial_1 - \partial_0$ or $\partial_1 - 2\partial_0 + \partial_{-1}$ then the region A_n in dimension $N = 2$ is the subset of the square $[-\pi, \pi]^2$, enclosed by the polygonal lines depicted in Figure 4.

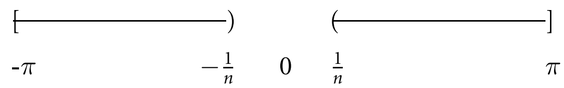

Figure 3: The region A_n in dimension $N = 1$.

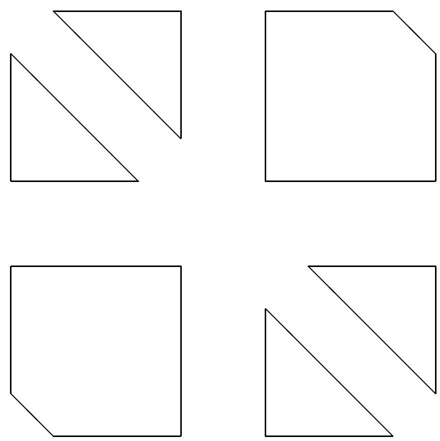

Figure 4: The region A_n in dimension $N = 2$, as a subset of the square $[-\pi, \pi]^2$.

Let c_n be the unique $L^2(\mathbb{Z}^N)$ solution of the equation $\mathcal{F}(T_{D_0} c_n) = (\mathcal{F}\hat{\xi})1_{A_n}$, $n \geq 1$. The solution c_n is defined through

$$c_n = \mathcal{F}^{-1}\Big(\frac{1}{p(e^{i(\alpha_1+\cdots+\alpha_N)})}(\mathcal{F}\hat{\xi})(\alpha)1_{A_n}(\alpha)\Big).$$

The $\widetilde{\Sigma}_N$-invariance of the set A_n and the fact that $\hat{\xi}$ is constructed from the Fourier coefficients of a closed function implies that c_n is $\widetilde{S}_N$-invariant, $n \geq 1$.

Obviously we have the convergence $\mathcal{F}(T_{D_0} c_n) \to \mathcal{F}\hat{\xi}$ as $n \to \infty$ in the topology of $L^2([-\pi, \pi]^N)$, hence $T_{D_0} c_n \to \hat{\xi}$ as $n \to \infty$ in the topology of $L^2(\mathbb{Z}^N)$. ■

Proof of Theorem 2.1 Let $\xi \in L^2(\mathbb{R}^{\mathbb{Z}}, d\nu_0^{gc})$ be a D_0-closed function. The decomposition $\xi = \operatorname{Proj}_{\mathcal{H}_0} \xi + \sum_{N\geq 1} \operatorname{Proj}_{\mathcal{H}_N} \xi$ is obvious. From Lemma 4.1 we know that $\operatorname{Proj}_{\mathcal{H}_N} \xi$ is D_0-closed, for any $N \geq 0$. Since $\mathcal{H}_0$ is generated by the constant function $\mathbf{1}$, $\operatorname{Proj}_{\mathcal{H}_0} \xi$ is a constant function equal to $\langle \xi, \mathbf{1}\rangle \mathbf{1}$.

If the sum of the coefficients of D_0 is not equal to zero, then the constant function $\sum_{j\in\mathbb{Z}} a_j = \sum_{j\in\mathbb{Z}} D_0(x_j)$ is a D_0-exact function and hence, the constant function $\operatorname{Proj}_{\mathcal{H}_0} \xi$ is D_0-exact, as well. In this case we see that ξ is D_0-closed if any of the projections $\mathrm{Proj}_{\mathcal{H}_N}\xi$, $N \geq 1$ belong to the space $\mathcal{E}_D$, or if any of the projections $\mathrm{Proj}_{\mathcal{H}_N}\xi$, $N \geq 1$ can be approximated by D_0-exact functions.

When the coefficients of D_0 sum up to zero, as the computation below shows, the constant function $\mathbf{1}$ and hence, $\mathrm{Proj}_{\mathcal{H}_0}\xi$ are orthogonal to any D_0-exact function.

$$\Big\langle \sum_{j\in\mathbb{Z}} D_0(\tau^j g), \mathbf{1} \Big\rangle = \sum_{k\in\mathbb{Z}} a_k \sum_{j\in\mathbb{Z}} \langle \partial_k(\tau^j g), \mathbf{1}\rangle = \sum_{k\in\mathbb{Z}} a_k \sum_{j\in\mathbb{Z}} \langle \partial_0(\tau^j g), \mathbf{1}\rangle$$
$$= \sum_{j\in\mathbb{Z}} \Big\langle \partial_0(\tau^j g), \Big(\sum_{k\in\mathbb{Z}} a_k\Big)\mathbf{1} \Big\rangle = 0.$$

Again, if we can prove that any of the projections $\mathrm{Proj}_{\mathcal{H}_N}$, $N \geq 1$ belong to the space $\mathcal{E}_D$, or that if any of the projections $\mathrm{Proj}_{\mathcal{H}_N}$, $N \geq 1$ can be approximated by D_0-exact functions, the decomposition theorem follows in this case, as well.

Therefore, in any possible case, the decomposition theorem follows as long as we establish that any D_0-closed function $\xi \in \mathcal{H}_N$, $N \geq 1$, can be approximated by D_0-exact functions.

Assume that $\xi = \sum_{I\in\mathcal{I}_N} \hat{\xi}_I H_I \in \mathcal{H}_N$, $N \geq 1$. Define the S_N-invariant function $\hat{\xi}\colon \mathbb{Z}^N \to \mathbb{R}$ through $\hat{\xi}(z) = \hat{\xi}_I$ if $z \overset{S_N}{\sim} z_I$; see (3.1). We use Lemmas 5.1 and 5.2 to find a sequence of finitely supported $\widetilde{S}_N$-invariant functions $(c_n)_{n\geq 1}$ such that $T_{D_0}c_n \to \hat{\xi}$ as $n \to \infty$ in the topology of $L^2(\mathbb{Z}^N)$. But Lemmas 4.2 and 4.3 tell us that each of $T_{D_0}c_n$, $n \geq 1$, defines a D_0-exact function ξ_{c_n}; see (4.6). At the end of Lemma 4.1 we noticed that the topology of $\mathcal{H}_N$ and $L^2(\mathbb{Z}^N)$ are equivalent, and hence we can claim that $\xi_{c_n} \to \xi$ as $n \to \infty$ in the Hilbert space topology of $\mathcal{H}_N$, or $L^2(\mathbb{R}^{\mathbb{Z}}, d\nu_0^{gc})$. ■

6 Second-Order Ginzburg–Landau Field and Algebraic Topology

We conclude with some remarks about the second-order Ginzburg–Landau field $Y_0 = \partial_1 - \partial_0$ which has been studied in the work of S. R. S. Varadhan [7]. Our approach places Varadhan's result in a new light by depicting a topological aspect, to be explained below.

In Section 3 we presented an extensive study of the set of multi-indices $\mathcal{I}$. There we partitioned the set of multi-indices $\mathcal{I}$ into disjoint orbits, and we denoted by $\mathcal{O}$ the space of orbits. Below we exhibit a procedure to construct a directed graph that has as vertices the orbits of the set of multi-indices.

A directed graph is a pair (V, E) of two sets, where V is the set of vertices of the graph and E is the set of directed edges. A directed edge is a pair of two vertices (v_1, v_2) where the first vertex indicates the starting point of the edge and the second vertex indicates the tip of the edge. We choose V to be the set of orbits $\mathcal{O}$. We also say that we have a directed edge (o_1, o_2) if there exists a multi-index $I \in o_1$ such that $\tau I \in o_2$. Notice that if there exists an edge between two orbits, then the orbits contain multi-indices with identical degrees. Hence our graph will have at least one connected component for each degree $N \geq 0$. We shall show that there exists precisely one connected component for each degree $N \geq 0$.

We would like to have a concrete or geometric presentation of the graph. For this purpose we use the identification of the set of orbits $\mathcal{O}_N$ containing the multi-indices of degree N, with the cone $\mathcal{C}_N^+$ of the lattice $\mathbb{Z}^N$, $N \geq 0$.

If $N = 0$, then $\mathcal{O}_N$ contains a single orbit, the orbit of the multi-index 0, and this orbit contains a single multi-index. Since the multi-index 0 has the property that $0 = \tau 0$, we will have a directed edge going out of and returning to 0; in other words, we have a loop at 0.

Assume that $N \geq 1$. It can be shown that a directed edge links $z \in \mathcal{C}_N^+$ to $z' \in \mathcal{C}_N^+$ if and only if either $z' = z - e_1 - \cdots - e_N$ or $z' = z + e_i$ for some $1 \leq i \leq N$. Here e_i is the lattice vector $(0, \dots, 1, \dots, 0)$ with the i-th coordinate 1. We shall indicate below how this presentation of the graph can be obtained.

Let $o \in \mathcal{O}_N$ be some orbit and $R(o) = \sum_{i=1}^{k} a_i \delta_{n_i}$, with $a_i \geq 1$ and $0 \leq n_1 < n_2 < \cdots < n_k$ be the representative of the orbit o. Given our rule, the orbit o is connected by an edge going out of o to each of the orbits $o(\tau R(o))$, $o(\tau(n_1 \cdot R(o)))$, …, $o(\tau(n_k \cdot R(o)))$. For each of the orbits in this list we can calculate the representatives and the corresponding point in the cone $\mathcal{C}_N^+$.

For example, the representative of the orbit o is $R(o) = \sum_{i=1}^{k} a_i \delta_{n_i}$ and the cone point is

$$z_{R(o)} = (\underbrace{n_1, \dots, n_1}_{a_1}, \dots, \underbrace{n_k, \dots, n_k}_{a_k}).$$

Assume that $n_1 \geq 1$. The representative of the orbit of $\tau R(o)$ is $\tau R(o)$ and the corresponding cone point is

$$z_{\tau R(o)} = (\underbrace{n_1 - 1, \dots, n_1 - 1}_{a_1}, \dots, \underbrace{n_k - 1, \dots, n_k - 1}_{a_k}).$$

We notice that $z_{\tau R(o)} = z_{R(o)} - e_1 - \cdots - e_N$. Also, if $n_1 = 0$, the representative of the orbit of $\tau R(o)$ is $(-1) \cdot \tau R(o)$ and the corresponding cone point is

$$z_{(-1)\cdot \tau R(o)} = (\underbrace{0, \dots, 0}_{a_1 - 1}, 1, \dots, \underbrace{n_k - 1, \dots, n_k - 1}_{a_k}),$$

and $z_{(-1)\cdot\tau R(o)} = z_{R(o)} + e_{a_1}$. Similarly, we can analyze the other orbits connected with o.

In particular our discussion proves that for any two given orbits o_1 and o_2, if there exists a multi-index $I \in o_1$ and $\tau I \in o_2$, then this multi-index is unique. We will see later that this observation allows us to assign in a unique way a multi-index to any directed edge of our graph.

We include three pictures (Figures 5–7) of the connected components of the directed graph for $N = 0$, $N = 1$ and $N = 2$.

Suppose we are given a function $\xi \in L^2(\mathbb{R}^{\mathbb{Z}}, d\nu_0^{gc})$ with Fourier expansion $\xi = \sum_{I \in \mathcal{I}} \hat{\xi}_I H_I$. We can actually turn our directed graph into a weighted graph by assigning to each directed edge (o_1, o_2) the Fourier coefficient $\hat{\xi}_I$ corresponding to the unique multi-index I such that $I \in o_1$ and $\tau I \in o_2$. Note that each Fourier coefficient will be assigned to one and only one edge and each edge will have assigned one and only one Fourier coefficient, since there is a one-to-one correspondence between the edges of our graph and the set $\mathcal{I}$ of multi-indices. For example the edge $(o(I), o(\tau I))$ will have attached the weight $\hat{\xi}_I$, Figure 8.

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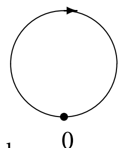

Figure 5: The connected component of the graph for $N = 0$.

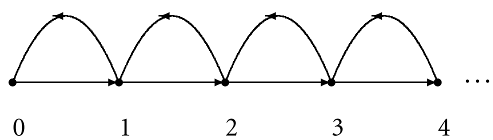

Figure 6: The connected component of the graph for $N = 1$.

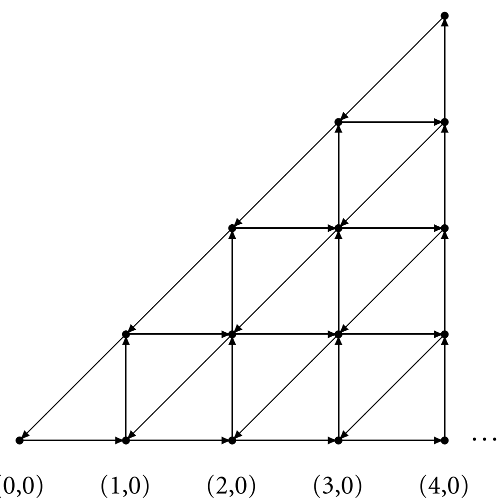

Figure 7: The connected component of the graph for $N = 2$.

It is interesting to note that if ξ is Y_0-closed, then the weights of the graph discussed above sum up to zero along any directed cycle of the graph except the loop of the connected component corresponding to $N = 0$. Indeed, the closedness condition of ξ imposes no restriction on the coefficient $\hat{\xi}_0$. Also note that the Y_0-closedness condition (4.4)

$$\hat{\xi}_{I+\delta_{(n+1)}} - \hat{\xi}_{I+\delta_n} = \hat{\xi}_{\tau^n(I+\delta_1)} - \hat{\xi}_{\tau^n(I+\delta_0)} \quad n \in \mathbb{Z},\ I \in \mathfrak{I},$$

plus the square-integrability of ξ are equivalent to the property that the weights of the graph associated to ξ sum up to zero around any directed cycle of any connected component corresponding to $N \geq 1$. However, if ξ is Y_0-exact then $\hat{\xi}_0 = 0$ and hence the weights of the graph sum up to zero around any directed cycle of the graph. If ξ is

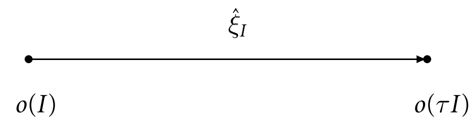

Figure 8: A directed edge and its attached weight.

Y_0-exact and is constructed as in Lemma 4.2, then we can say that in any connected component of the graph all but finitely many weights are zero.

The above can be explained from a topological point of view. We can turn our directed graph into a 2-dimensional Δ-complex (see [2]) by attaching enough discs to cycles of the graph that each of the connected components $N \geq 1$ can be retracted to a single point. We do not attach a disc onto the loop of the connected component $N = 0$. After the attaching process, the 2-dimensional Δ-complex can be retracted to the disjoint union of a circle with a countable number of points. The Fourier coefficients of a Y_0-exact function form a coboundary of our 2-dimensional Δ-complex and the Fourier coefficients of a Y_0-closed function form a cocycle for our 2-dimensional Δ-complex. Since the cohomology group $H^1(C, \mathbb{R})$ of a circle C is one-dimensional, we expect the space of Y_0-exact functions to have codimension one inside the space of Y_0-closed functions.

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