SORGE'S LAW OF DENSIFICATION OF SNOW ON HIGH POLAR GLACIERS

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Abstract. Ernst Sorge, a member of Wegener's Greenland Expedition 1930–31, discovered an important law on the densification of snow in high polar glaciers. It is here given a simple mathematical form and some of its implications are formulated and discussed. Apart from its significance to glaciology and snow mechanics, Sorge's law should also be useful in the study of consolidation of accumulating fine-grained lacustrian and oceanic sediments.

Zusammenfassung. Ernst Sorge, Mitglied der Wegener Grönlandexpedition 1930–31, entdeckte ein wichtiges Gesetz über die Schneeverdichtung auf polaren Gletschern. Das Gesetz wird hier einfach mathematisch formuliert und diskutiert. Neben seiner Bedeutung für die Gletscherkunde und der Schneemechanik sollte Sorges Gesetz auch in der Untersuchung der Konsolidation von akkumulierenden, feinkörnigen, lakustrischen und ozeanischen Sedimenten Anwendung finden.

The following is a theoretical development inspired by Ernest Sorge's report[1] on work done at "Eismitte" in Greenland in 1930–31.

Sorge pointed out that in central Greenland, where there is no melting in the summer, the density of snow at a given depth below the surface does not change with time. A given small volume of snow will increase in density continuously as time passes and as it gets buried deeper and deeper by accumulating snowfall. If the conditions of accumulation remain constant, this implies that the density at a given depth remains constant, as the snow at this depth will always be of the same age. In a later paper[2] Sorge used this concept to deduce a method for determining the annual accumulation where the annual layers cannot be distinguished. Unfortunately he used a method of finite increments which did not permit him to shape his theory in adequate mathematical form. The matter is here further developed by the application of calculus.

Sorge's law is applicable whenever the annual climatic cycle in the accumulation zone does not appreciably change in the course of as many years as are under consideration. The second major premise is that snow melting be insignificant, which is only the case in the higher reaches of polar glaciers and in some very high *névé* fields of mountains at lower latitudes. The law, as pointed out by Sorge himself, is purely deductive from its premises, and is therefore necessarily valid to the degree of their validity.

Sorge's law can be formulated as follows:

If, at a given locality, h is the depth below the snow surface and γ is the snow density, then the function

$$\gamma = f(h) \qquad (1)$$

is invariant with time.

Attributes of a Small Volume of Snow

h = depth below snow surface

γ_h = density in terms of equal volume of water

t_h = time since deposition on surface

σ_h = weight of the overlying snow in terms of thickness of an equivalent layer of water

V_h = vertical velocity downward in relation to snow surface

v_h = specific velocity of densification, velocity of approach of two particles at unit distance, one above the other.

A = accumulation of snow in unit time in terms of the thickness of an equivalent layer of water.

We can now formulate a number of interesting relations:

(*a*) The load of the overlying snow at any depth is equal to the density integrated over the depth.

$$\sigma_h = \int_0^h \gamma dh = \int_0^h f(h) dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(*b*) The time required for a snow particle to reach any depth is equal to the load divided by the accumulation in unit time.

$$t_h = \frac{\sigma_h}{A} = \frac{1}{A} \int_0^h f(h) dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(*c*) A load increment $d\sigma$ accumulates on the surface during a time increment dt. During this time a small volume of snow at any depth sinks by an increment dh and increases in density by an increment $d\gamma$.

$$d\sigma = A dt = \frac{\gamma + (\gamma + d\gamma)}{2} \, . \, dh = \gamma dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus the vertical velocity in relation to the snow surface is

$$V_h = \frac{dh}{dt} = \frac{A}{\gamma_h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(*d*) Differentiating equation (5) we obtain

$$dV = -\frac{A}{\gamma^2} \, . \, d\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The specific velocity of densification is

$$v = \frac{dV}{dh} = -\frac{A}{\gamma^2} \frac{d\gamma}{dh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$\frac{d\gamma}{dh}$ is of course the slope of the density-depth curve.

The function $\gamma = f(h)$ has to be continuous and simple in order to be useful. Strictly speaking, this would require that snow falls all the time at a constant rate and constant initial density. Fortunately for the polar traveler, this is not the case. Yet the above relations are well applicable if one takes the zig-zag line graph of actual density-depth measurements and draws a mean continuous curve resulting in minimum deviation of the computed values for σ_h from the actual values.

Since extrapolation is not admissible, the formulation of $\gamma = f(h)$ from the graph is a matter of taste and depends on how much labor one is willing to expend to obtain a close fit. Unless the calculated curve fits very well indeed, it should not be differentiated to obtain slope values. Drawing tangents on the graph will then be more accurate.

The relations formulated above are theoretically and practically useful. For instance, if it is not possible to distinguish annual layers, one can nevertheless make at least a fair estimate of mean annual accumulation, as already pointed out by Sorge.[2] It is then necessary to measure the rate with which two points approach each other, making a series of measurements at different depths and taking the mean. The rate of approach of the two points (1 and 2) is the difference of velocity $V_1 - V_2$, and using equation (5):

$$V_1 = \frac{A}{\gamma_1} \quad \text{and} \quad V_2 = \frac{A}{\gamma_2}$$

we then obtain the facts, astonishing at first sight, that the accumulation is

$$A = \frac{V_1 - V_2}{\gamma_2 - \gamma_1} \gamma_1 \gamma_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The rate of densification of the snow is of great interest in snow mechanics.

It is known that the rate of densification by plastic deformation is a product of two nonlinear functions:

$$v = F_1(\gamma, T, S)\, F_2(\sigma)$$

where T is the temperature
and S a number of parameters characterizing snow type.

Using equation (7) we obtain:

$$F_1(\gamma, T, S) \,.\, F_2(\sigma) = -\frac{A}{\gamma^2}\frac{d\gamma}{dh}$$

and substituting for σ from equation (2):

$$F_1(\gamma, T, S) = -\frac{A}{\gamma^2}\frac{d\gamma}{dh}\frac{1}{F_2\left(\int_0^h \gamma dh\right)}$$

It is a laboratory task to determine $F_1(\gamma, T)$ and $F_2(\sigma)$ on different snow types. This, together with careful field work, measuring a number of other properties of snow, should facilitate the choice of snow-type parameters (grain size, grain shape, tensile strength, shear strength, crushing strength, air permeability, etc.), most useful as a basis for predicting plastic deformational behavior of snow under different conditions of stress.

Outside the field of glaciology, the theory developed above should be applicable to the interpretation of depth-density values obtained from cores of fine-grained oceanic sediments. Consolidation of sediments by the squeezing out of water then corresponds to densification of snow. The analogue to the condition that "snow falls all the time at a constant rate and constant initial density" is likely to be well fulfilled in the quiet depths of the ocean.

Conditions at Eismitte (Long. $71\frac{3}{4}$° N., Lat. $40\frac{3}{4}$° W.)

Fig. 1 (p. 323) shows the mean density of summer (S) and winter (W) snow layers measured by Sorge at Eismitte, Greenland. There is an error due to densification during the several months when Sorge was excavating his pit and measuring densities, but it is not very large. The error is corrected for (by Sorge) in the smoothed-out curve also shown on the graph. The salient feature is that the summer layers are less dense than the winter layers. Thus the identifiable snow layers constitute a most remarkable and valuable record of seasonal accumulation. The summer season lasts from the middle of April to the middle of September.

The depth-density curve is fairly well rendered by the following empirical function

$$\gamma = \mathrm{a} + \mathrm{b}h + \mathrm{c}h^2 + \mathrm{d}h^3 + \mathrm{e}h^4 \quad \ldots\ldots\ldots \quad (9)$$

If h is in meters, then

$$\mathrm{a} = +0{\cdot}33800 \qquad \mathrm{d} = -3{\cdot}29092 \times 10^{-4}$$
$$\mathrm{b} = +0{\cdot}01958 \qquad \mathrm{e} = +1{\cdot}15327 \times 10^{-5}$$
$$\mathrm{c} = +2{\cdot}02274 \times 10^{-3}$$

Curve slopes were obtained from tangents drawn on the graph. The accumulation is 0·314 meters of water equivalent per year (mean of 22 years).

Table I lists conditions at Eismitte. The values in the third column are calculated from (9). The load is calculated from (2), integrating (9), t from (3), V from (5) and v from (7). In (3), (5) and (7), Sorge's value for the 22-year mean annual accumulation (A=0·314 meters) was used.

The table shows that the theory is well applicable. Compare, for instance, the measured with the calculated values of v, and also the calculated values of t with the depth of the layers of corresponding age on the graph.

The annual climatic cycle with its difference of initial density of summer and winter snow, its difference of summer and winter accumulation, and its temperature cycle, has not been considered in the theory. It appears that in central Greenland the climatic cycle is sufficiently constant over a period of many years to warrant incorporation of its effects into the theory. This would

21

require considerable mathematical labor. Such factors as thermal conductivity of the snow as a function of depth (or density) and the magnitude and course of the annual temperature fluctuation as a function of depth have been the object of a beautiful harmonic analysis by Sorge in his report of 1935. Sufficient quantitative data are yet lacking, however, on the influence of temperature on the specific velocity of densification, which is known to be very sensitive to temperature changes. Summer and winter snow accumulation will furthermore have to be treated separately because they go through different paths in their early stages of metamorphism. The new depth-density curve will then consist of alternating segments of two slowly converging curves, one for summer and one for winter snow. These curves should no longer be of empirical form, but must be formulated in terms of the pertinent facts of snow mechanics.

Within recent years the use of aircraft has given new impetus to glaciological studies in the polar regions, warranting a re-examination of older publications. Sorge's painstaking work, done under the most adverse conditions imaginable, as vividly described in Georgi's diaries, is of outstanding scientific value. It is a fitting tribute to Ernst Sorge that the law he discovered shall henceforth bear his name.

TABLE I. CONDITIONS AT EISMITTE, GREENLAND

h	γ		σ	t	V	v		Temperatures °C 1930–31	
Meters	*Sorge*	*Calculated*	*Meters of water calculated*	*Years calculated*	*Meters per year calculated*	*Measured by Sorge*	*Calculated years* [1]	*Middle of June*	*Middle of February*
0	0·338	0·338	0	0	0·929	—	0·0621	— 9·7	—42·4
1	0·361	0·359	0·348	1·11	0·870	—	0·0545	—20·6	—38·3
2	0·384	0·383	0·719	2·29	0·818	—	0·0481	—26·7	—34·4
3	0·407	0·407	1·111	3·54	0·771	—	0·0429	—29·4	—31·6
4	0·429	0·431	1·552	4·94	0·732	—	0·0386	—30·3	—29·6
5	0·451	0·453	1·975	6·29	0·696	0·0324	0·0349	—30·2	—28·5
6	0·472	0·472	2·437	7·76	0·665	0·0252	0·0265	—29·8	—27·9
7	0·489	0·489	2·918	9·29	0·642	0·0180	0·0190	—29·3	—27·8
8	0·501	0·503	3·414	10·87	0·627	0·0144	0·0139	—28·9	—27·8
9	0·511	0·514	3·923	12·49	0·614	0·0120	0·0113	—28·6	—27·9
10	0·520	0·522	4·441	14·14	0·604	0·0108	0·0102	—28·4	—28·1
11	0·529	0·529	4·967	15·81	0·594	0·0096	0·0095	—28·3	—28·2
12	0·537	0·534	5·499	17·51	0·585	0·0096	0·0087	—28·3	—28·3
13	0·545	0·541	6·036	19·22	0·576	0·0096	0·0085	—28·3	—28·3
14	0·553	0·550	6·581	20·95	0·568	0·0084	0·0075	—28·3	—28·3
15	0·560	0·560	7·135	22·75	0·561		0·0073	—28·3	—28·3

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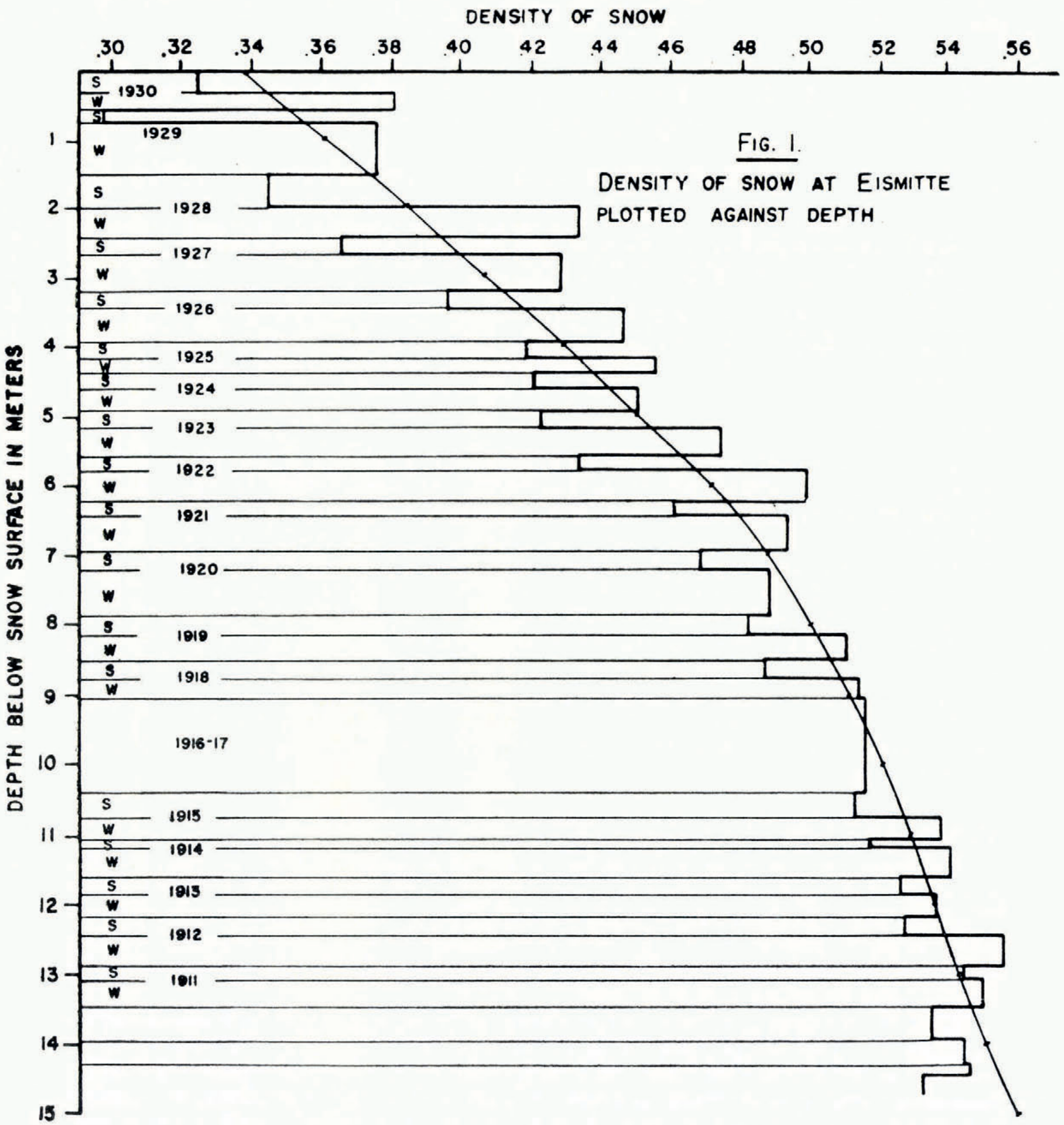

DENSITY OF SNOW
.30 .32 .34 .36 .38 .40 .42 .44 .46 .48 .50 .52 .54 .56
DEPTH BELOW SNOW SURFACE IN METERS
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
1930
1929
1928
1927
1926
1925
1924
1923
1922
1921
1920
1919
1918
1916-17
1915
1914
1913
1912
1911
S
W
FIG. 1.
DENSITY OF SNOW AT EISMITTE PLOTTED AGAINST DEPTH