Can. J. Math., Vol. XXVIII, No. 5, 1976, pp. 968–976

# THE ALGEBRAIC INDEPENDENCE OF CERTAIN EXPONENTIAL FUNCTIONS

W. DALE BROWNAWELL

In 1897 E. Borel proved a general theorem which implied as a special case the following result equivalent to his celebrated generalization of Picard's theorem [**2**]: *If* $f_1, \ldots, f_m$ *are entire functions such that for each* $1 \leqq i < j \leqq m, f_i - f_j \notin \mathbf{C}$, *then the functions* $\exp f_1, \ldots, \exp f_m$ *are linearly independent over* $\mathbf{C}$. In 1929 R. Nevanlinna [**6**] extended Borel's theorem to consider arbitrary $\mathbf{C}$-linearly independent meromorphic functions $\varphi_1, \ldots, \varphi_m$ satisfying $\varphi_1 + \ldots + \varphi_m = 1$.

Recently R. Narasimhan [**5**] has applied Nevalinna's basic approach to show that *if* $f_1(\mathbf{z}), \ldots, f_m(\mathbf{z})$ *are entire functions such that for each* $1 \leqq i < j \leqq m$, $f_i - f_j \notin \mathbf{C}$, *then* $\exp f_1, \ldots, \exp f_m$ *are linearly independent over* $\mathbf{C}[\mathbf{z}]$. Here $\mathbf{z}$ will denote the $n$-tuple $(z_1, \ldots, z_n)$, $\mathbf{C}[\mathbf{z}]$ will denote the polynomial ring in $z_1, \ldots, z_n$, and $\mathbf{z}_0$ will denote the $(n + 1)$-tuple $(z_0, z_1, \ldots, z_n)$. Narasimhan took complex lines through the origin to reduce the general $n$ variable case to $n = 1$.

More recently P. Bundschuh [**3**] has suggested that *if* $f_1(\mathbf{z}), \ldots, f_m(\mathbf{z})$ *are entire and for each* $1 \leqq i < j \leqq m$, $f_i - f_j \notin \mathbf{C}[\mathbf{z}]$, *then* $\exp f_1, \ldots, \exp f_m$ *are linearly independent over the ring of functions of finite order*. Unfortunately his proof for $n > 1$ is inconclusive. As usual, we say that the entire function $F(z)$ has order

$$\rho(F) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r, F)}{\log r}$$

where

$$M(r, F) = \max_{|z_\nu| \leqq r} |F(z)| \quad \text{and} \quad \log^+ x = \begin{cases} \log x & \text{if } x \geqq 1 \\ 0 & \text{if } 0 < x < 1. \end{cases}$$

The purpose of this note is threefold: First of all, we will establish a theorem which includes the above results by explicitly relating the degree of $f_i - f_j$ to the order of the functions which are coefficients in a dependence relation. Secondly, we will prove a general theorem which can cope with dependence relations having as coefficients certain entire functions which are not of finite order with respect to a distinguished variable. Finally we note that the $f_i$ can be meromorphic functions.

For $\rho_1, \ldots, \rho_n \in \mathbf{R}_{\geqq 0}$, let $R_{\rho_1, \ldots, \rho_n}$ be the ring of entire functions $F(z)$ such

Received August 6, 1975 and in revised form, May 7, 1976. This research was supported in part by the National Science Foundation.

that for all fixed $z_1, \ldots, \hat{z}_i, \ldots, z_n \in \mathbf{C}$,

$$\limsup_{r\to\infty} \frac{\log^+ \log^+ M(r, F)}{\log r} \leqq \rho_i, \quad 1 \leqq i \leqq n.$$

THEOREM 1. *Let* $f_1(\mathbf{z}), \ldots, f_m(\mathbf{z})$ *be meromorphic functions. Let* $\rho_1, \ldots, \rho_n \in \mathbf{R}_{\geqq 0}$. *Then* $\exp(f_1), \ldots, \exp(f_m)$ *are linearly dependent over* $R_{\rho_1,\ldots,\rho_n}$ *if and only if for some* $1 \leqq k < l \leqq m$, $f_k(\mathbf{z}) - f_l(\mathbf{z}) \in \mathbf{C}[\mathbf{z}]$ *with* $\deg_{z_i} f_k(\mathbf{z}) - f_l(\mathbf{z}) \leqq \rho_i$, *for all* $i$, $1 \leqq i \leqq n$.

Narasimhan's result follows on taking $\rho_1 = \ldots = \rho_n = 0$; Bundschuh's on considering the ring $R = \bigcup_{\rho=1}^{\infty} R_{\rho,\ldots,\rho}$.

It may be somewhat unsatisfying that the coordinates chosen should play such a special role in Theorem 1. However one can deduce as a corollary the following result which does not depend on the selection of any particular coordinates.

COROLLARY. *Let* $f_1(\mathbf{z}), \ldots, f_m(\mathbf{z})$ *be meromorphic functions. Then* $\exp f_1, \ldots, \exp f_m$ *are linearly dependent over the ring of entire functions of order* $\leqq \rho$ *if and only if, for some* $1 \leqq k < l \leqq m$, $f_k(\mathbf{z}) - f_l(\mathbf{z}) \in \mathbf{C}[\mathbf{z}]$ *with total degree at most* $\rho$.

To see this, we introduce a linear change of variables by $z_1 = z_1'$, $z_2 = z_2' + \lambda_2 z'_1, \ldots, z_n = z'_n + \lambda_n z'_1$, with each $0 < |\lambda_i| \leqq 1$. By the Weierstrass Preparation Theorem, we can easily choose the $\lambda_i$ such that whenever $f_k(\mathbf{z}) - f_l(\mathbf{z})$ is a polynomial, then after the change of variables, the degree with respect to $z'_1$ is the total degree of $f_k - f_l$. Since for $\hat{F}(\mathbf{z}') = F(\mathbf{z})$,

$$M(r, F(\mathbf{z})) \leqq M(2r, \hat{F}(\mathbf{z}')) \leqq M(4r, F(\mathbf{z})),$$

the coefficients of a linear dependence relation have order $< \rho$ as functions of $\mathbf{z}$ exactly when they do as functions of $\mathbf{z}'$. Consequently by Theorem 1, for some $1 \leqq k < l \leqq m$, $\hat{f}_k(\mathbf{z}') - \hat{f}_l(\mathbf{z}') \in \mathbf{C}[\mathbf{z}']$ with degree with respect to $z'_1$ at most $\rho$. By changing the variables back to $z$, we see that $f_k(\mathbf{z}) - f_l(\mathbf{z}) \in \mathbf{C}[\mathbf{z}]$ has total degree $\leqq \rho$.

A. Ehrenfeucht has pointed out to me that one can also show by a Baire category argument that an entire function which is a polynomial of degree $\leqq \rho$ on each complex line through the origin is a polynomial of degree $\leqq \rho$. Thus one could give a proof more in the spirit of Narasimhan's approach of reducing the question in $n$ variables to the corresponding one in one variable.

To state the second theorem mentioned above, it is necessary to state a few conventions. Let $\lambda : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be monotonically increasing and, if non-constant, $\log r = O(\lambda(r))$. Here, as below, by $f = O(g)$ we mean that there is a constant $C > 0$ such that off a subset of $\mathbf{R}_{>0}$ of finite measure, $f(r) \leqq Cg(r)$. It follows from the standard inequalities of Nevanlinna theory (see Section I below) that the set $O_\lambda$ of meromorphic functions $f(z)$ with $T(r, f) = O(\lambda(r))$ is a differential field, where $T(r, f)$ denotes the Nevanlinna characteristic functions.

For $\rho_1, \ldots, \rho_n \in \mathbf{R}_{\geqq 0}$, let $O_{\lambda, \rho_1, \ldots, \rho_n}$ denote the field of meromorphic functions $f(\mathbf{z}_0)$ such that

a) $T(r, f) = O(\lambda(r))$ for every fixed $z_1, \ldots, z_n \in \mathbf{C}$

b) $\limsup_{r \to \infty} \dfrac{\log^+ T(r, f)}{\log r} \leqq \rho_i$ for every fixed $z_0, z_1, \ldots, \hat{z}_i, \ldots, z_n \in \mathbf{C}$

It is a fundamental result [**4**, Theorem 1.7] that for an entire function $f(z)$ of order $\rho$,

$$\rho = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

Thus condition $b$) says that as a function of $z_i$ alone, $f$ is of order $\leqq \rho_i$, whether $f$ is entire or not. Actually Theorem 1 holds over the *field* $O_{\rho_1, \ldots, \rho_n}$ of meromorphic functions of order $\leqq \rho_i$ with respect to $z_i$, $1 \leqq i \leqq n$, as one sees from the following result.

THEOREM 2. *Let $f_1(\mathbf{z}_0), \ldots, f_m(\mathbf{z}_0)$ be meromorphic functions. Then* $\exp(f_1), \ldots, \exp(f_m)$ *are linearly dependent over $O_{\lambda, \rho_1, \ldots, \rho_n}$ if and only if, for some $1 \leqq k < l \leqq m$,*

$$f_k(\mathbf{z}_0) - f_l(\mathbf{z}_0) = \Sigma_\nu\, b_\nu(z_0)\mathbf{z}^\nu$$

*where $0 \leqq \nu_i \leqq \rho_i$, $1 \leqq i \leqq n$, and each $b_\nu(z_0)$ is entire with* $\exp(b_\nu) \in O_\lambda$. *Here, as below, $\nu = (\nu_1, \ldots, \nu_n)$ and $\mathbf{z}^\nu = z_1^{\nu_1} \ldots z_n^{\nu_n}$.*

Theorems 1 and 2 are equivalent to theorems about the algebraic independence of certain exponential functions. For example, Theorem 1 is equivalent to the following result.

THEOREM 1a. *Let $f_1(\mathbf{z}), \ldots, f_m(\mathbf{z})$ be meromorphic functions. Then* $\exp(f_1), \ldots, \exp(f_m)$ *are algebraically dependent over $O_{\rho_1, \ldots, \rho_n}$ if and only if there are integers $r_1, \ldots, r_m$, not all zero, with*

$$r_1 f_1(\mathbf{z}) + \ldots + r_m f_m(\mathbf{z}) \in \mathbf{C}[\mathbf{z}]$$

*having degree at most $\rho_i$ with respect to $z_i$, $1 \leqq i \leqq n$.*

Clearly the conditions listed in the theorems are sufficient to guarantee dependence. The proofs below will only be concerned with necessity.

Moreover Theorem 2 for the $n$ variables $z_0, z_1, \ldots, z_{n-1}$ implies Theorem 1 for the $n$ variables $z_1, \ldots, z_n$: We simply rename $z_0$ to be $z_n$ and take the special case $\lambda(r) = r^{\rho_n + \epsilon}$, $0 < \epsilon < 1$. The linear dependence implies that each $b_\nu(z_n)$ is entire and by the remark at the end of Section I,

$$M(r, b_\nu) \leqq 6T(2r, e^{b_\nu}) + O(1) = O(r^{\rho_n + \epsilon}).$$

Consequently $b_\nu$ is a polynomial of degree at most $\rho_n$ by the Cauchy inequalities.

For the case $n = 0$, we indicate briefly below the changes necessary in Nevanlinna's classical proof of Borel's theorem to achieve the desired result. To obtain a correlation between the growth and degree in each of the $n + 1$ variables and to allow more general growth in a distinguished variable, we had to abandon Narasimhan's technique for reduction to one variable. Instead we induct on the number of variables, employing a variation on Hurwitz's proof, as it appears in [**1**], of the theorem enunciated by Weierstrass that an entire function which is a polynomial in each variable separately is indeed a polynomial.

This paper is based on work done while the author had a visiting position at the University of Colorado. He wishes to express his thanks to his colleagues there for their hospitality and to K. T. Hahn for pointing out the classical roots of the problem of linear dependence of functions.

**I. Estimates from Nevanlinna theory.** We collect here the estimates used to show that $O_\lambda$ is a different field. For meromorphic $f(z)$ and $r \geqq 0$, set

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta$$

$$N(r,f) = \int_0^r (n(t,f) - n(0,f))dt/t + n(0,f)\log r,$$

where $n(t,f)$ is the number of poles of $f(z)$, counting multiplicites, in $|z| \leqq t$. Then the *characteristic function of* $f$ is defined to be

$$T(r,f) = m(r,f) + N(r,f).$$

Evidently for entire functions, $T(r,f) = m(r,f)$.

When $f(z)$ is meromorphic and $r \geqq 0$, we have

$$T\left(r, \sum_{i=1}^{p} f_i\right) \leqq \sum_{i=1}^{p} T(r,f_i) + \log p$$

$$T\left(r, \prod_{i=1}^{p} f_i\right) \leqq \sum_{i=1}^{p} T(r,f_i)$$

and the corresponding inequalities on replacing $T$ by $m$ [**4**, p. 5] from the corresponding inequalities on $\log^+$ applied to sums and products.

If $f(z)$ is a meromorphic function with a zero (or pole) of order $\lambda \geqq 0$ (or $-\lambda \geqq 0$) at $z = 0$, then Jensen's formula applied to $r^\lambda f(z)/z^\lambda$ on the circle $|z| = r$ shows that

$$T(r, 1/f) = T(r,f) - \log|C_\lambda|$$

where $C_\lambda$ is the first non-zero coefficient of the Laurent series expansion for $f(z)$ about $z = 0$ [**4**, pp. 3, 4].

If $f(z)$ is meromorphic and not constant, then outside a set $E(f)$ of finite length in $\mathbf{R}_{\geqq 0}$, we have [**4**, p. 40]

$$m(r, f'/f) \leqq 10 \log^+ T(r, f) + 10 \log r.$$

Of course if $f(z)$ is a non-zero constant, then $m(r, f'/f) \equiv 0$. In particular we conclude that for $f$ entire,

$$m(r, f') \leqq 10 \log^+ m(r, e^f) + 10 \log r$$

outside an exceptional set.

Moreover, when $f(z)$ is meromorphic with only finitely many zeros and poles and $T(r, f) = O(\log r)$, then $f(z)$ is a rational function [**4**, p. 21].

Finally, when $f(z)$ is entire with real part $u(z)$, we consider Schwarz's Formula in polar coordinates for $|z| = r < R$:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Re^{i\theta} + z}{Re^{i\theta} - z} u(Re^{i\theta}) d\theta.$$

On taking $R = 2r$, we find that

$$\begin{aligned} |f(z)| &\leqq \frac{3}{2\pi} \int_0^{2\pi} |u(2re^{i\theta})| d\theta \\ &\leqq \frac{3}{2\pi} \int_0^{2\pi} \log^+ |\exp f(2re^{i\theta})| + \log^+ |\exp(-f(2re^{i\theta}))| d\theta \\ &\leqq 3T(2r, e^f) + 3T(2r, e^{-f}) \\ &\leqq 6T(2r, e^f) + O(1). \end{aligned}$$

**II. Proof of Theorem 2 when n = 0.** The desired result could be deduced as a corollary of the following result of R. Nevanlinna [**6**, p. 116] mentioned above:

THEOREM. *Let* $\varphi_1, \ldots, \varphi_m$ *be meromorphic functions, linearly independent over* $\mathbf{C}$, *which satisfy the relation*

(1) $\varphi_1 + \ldots + \varphi_m = 1.$

*Then for* $1 \leqq i \leqq m$,

$$T(r, \varphi_i) < \sum_{k=1}^{m} N(r, 1/\varphi_k) + N(r, \varphi_i) + N(r, D) - \sum_{k=1}^{m} N(r, \varphi_k) - N(r, 1/D) + O(\max_j \log(r, T(r, \varphi_j))),$$

*where* $D$ *is the Wronskian of* $\varphi_1, \ldots, \varphi_m$.

To apply Nevanlinna's result to our situation, we assume that we have a minimal non-trivial linear dependence

$$g_1 \exp f_1 + \ldots + g_{m+1} \exp f_{m+1} = 0$$

with $g_i \in O_\lambda$. Then define, for $1 \leqq i \leqq m$,

(2) $\varphi_i = -g_i/g_{m+1} \exp(f_i - f_{m+1})$

to obtain a relation of the form (1). By the minimality of our relation, the $\varphi_i$ are **C**-linearly independent (even $O_\lambda$-linearly independent). We can apply Nevanlinna's theorem as soon as we know the $\varphi_i$ are meromorphic. However, after differentiating our relation (1) $m$ times, we obtain a system of linear equations in the $\exp(f_i - f_{m+1})$. The determinant of coefficients is equal to the (non-zero) Wronskian $D$ multiplied by $\Pi(\exp(f_{m+1} - f_i)$ and is thus a rational function in the $g_i$, $f'_i - f'_{m+1}$ and their derivatives. Since the determinant of coefficients is non-zero, we can solve for each $\exp(f_i - f_{m+1})$ by Cramer's rule to show that it is a rational function in the $g_i, f'_1 - f'_{m+1}$ and derivatives, and hence is a meromorphic function. Consequently each $f_i - f_{m+1}$ is entire, which is what we needed to apply Nevanlinna's result. However, the argument we have just given is essentially the first part of Nevanlinna's proof. (So the reader may consult [**6**, Chapter V] for details if need be.) It thus comes as no surprise that we can now just as easily finish the argument directly as to appeal to Nevanlinna's result.

For since each $\exp(f_i - f_{m+1})$ is a rational function in the $g_i, f'_i - f'_{m+1}$ and derivatives, the inequalities of Section I show that each

$$\begin{aligned} T(r, \exp(f_i - f_{m+1})) &= O(\lambda(r) + \max_j T(r, f'_j - f'_{m+1})) \\ &= O(\lambda(r) + \log r \\ &\qquad + \max_j \log T(r, \exp(f_j - f_{m+1}))). \end{aligned}$$

Thus if some $f_i - f_{m+1}$ is not a constant (the only way $\exp(f_i - f_{m+1})$ could be a rational function),

$$\max_i T(r, \exp(f_i - f_{m+1})) \leqq O(\lambda(r))$$

as claimed.

**III. Proof of Theorem 2.** The proof proceeds by induction on $n$. We may thus assume Theorem 2 for $n = 0$ and Theorem 1 for arbitrary $n$ in order to prove Theorem 2 for $n$. In the following discussion $\nu = (\nu_1, \ldots, \nu_n)$ will run through all possibilities with $0 \leqq \nu_i \leqq \rho_i$, $1 \leqq i \leqq n$.

Let every proper subset of $\{\exp(f_1), \ldots, \exp(f_m)\}$ be linearly independent over $O_{\lambda,\rho_1,\ldots,\rho_n}$, but

(3) $g_1(\mathbf{z}_0)\exp(f_1(\mathbf{z}_0)) + \ldots + g_m(\mathbf{z}_0)\exp(f_m(\mathbf{z}_0)) \equiv 0$

be a non-trivial dependence relation over $O_{\lambda,\rho_1,\ldots,\rho_n}$. Since by the Weierstrass Preparation Theorem, the zeros of non-constant holomorphic functions locally

https://doi.org/10.4153/CJM-1976-094-9 Published online by Cambridge University Press

form finitely many hyper-surfaces, there is an open set $\emptyset \neq D_0 \times D_n \subseteq \mathbf{C}^{n+1}$ with $D_0$ open in $\mathbf{C}$ and $D_n$ open in $\mathbf{C}^n$ such that none of the $g_1(\mathbf{z}_0), \ldots, g_m(\mathbf{z}_0)$ vanish anywhere on $D_0 \times D_n$ and such that $f_1(\mathbf{z}_0), \ldots, f_m(\mathbf{z}_0)$ remain bounded there.

A. Let $U$ be a non-empty open subset of $D_n$. For fixed $u \in U$, define an equivalence relation on $f_1, \ldots, f_m$ by $f_i \sim f_j$ if and only if $f_i(z_0, u) - f_j(z_0, u)$ is an entire function of $z_0$ and, as a function of $z_0$, $\exp(f_i(z_0, u) - f_j(z_0, u)) \in O_\lambda$. If there are $r$ equivalence classes, then we select representatives $f_{i_k}$ for the classes and deduce from (3) that for $\mathbf{z}_0 = (z_0, u)$, $z_0 \in \mathbf{C}$ arbitrary,

$$(4) \quad \sum_{k=1}^{r} G_k(\mathbf{z}_0) \exp f_{i_k}(\mathbf{z}_0) \equiv 0,$$

where

$$G_k = \sum g_j \exp(f_j - f_{i_k})$$

and the sum for $G_k$ is over all $j$ having $f_j \sim f_{i_k}$ with respect to this particular $u$.

Then, as a function of $z_0$ alone, each $G_k \in O_\lambda$. By Theorem 2 in the case $n = 0$, we conclude that each $G_k(z_0, u)$ vanishes identically as a function of $z_0$, else some $f_{i_k} \sim f_{i_l}$, $k \neq l$, contrary to our selection of representatives. Multiplying by $\exp(f_{i_k}(z_0, u)$ gives, for our $u$, a subrelation

$$(5) \quad \sum g_j(z_0, u) \exp f_j(z_0, u) = 0$$

for all $z_0$, where the sum is over all $j$ with $f_j \sim f_{i_k}$ at $u$.

By continuity in the last $n$ variables, we see that the set of $u \in U$ for which each subrelation (5) holds is closed in $U$. But for given $u \in U$ and given $k$, $1 \leqq k \leqq m$, $f_k$ is involved in at least one subrelation (5) derived in the above manner. Thus for fixed $k$, $U$ can be written as a finite union of relatively closed subsets indexed by the relations (5) which hold on them. (That is, to each sub-relation (5) which comes from the relations at *any* point $u$ of $U$ and which involves $f_k$, there corresponds the subset of $U$ on which (5) holds.)

Consequently we know that for given $f_k$, there is a relation (5) holding on some subset $U_k$ of $U$ with a non-empty interior $I$. By construction we know that there is a $u_0 \in U_k$, but, as far as we know thus far, not necessarily in $I$ such that for all $f_j$ involved in the relation (5) corresponding to $U_k$, $f_j \sim f_k$ at $u_0$.

Since (5) holds on $\mathbf{C} \times I$, it holds on any open connected set containing $\mathbf{C} \times I$ where the $f_j$ involved remain finite by the Monodromy Theorem. We conclude that (5) holds on all of $\mathbf{C}^{n+1}$. But by the minimality of (3), equations (3) and (5) must be the same. Since $U$ was an arbitrary open subset of $D_n$ and since $k$ was arbitrary, we have just shown that there is a dense set $S$ of points $u \in D_n$ such that for any $1 \leqq k \leqq m$ and any $u \in S$,

$$\exp(f_k(z_0, u) - f_m(z_0, u)) \in O_\lambda.$$

Let $N = \prod_{i=1}^{n} (1 + [\rho_i])$. Since $S \times \ldots \times S$ is dense in $D_n \times \ldots \times D_n$, the

Weierstrass Preparation Theorem shows that we can choose $\mathbf{z}_1, \ldots, \mathbf{z}_N \in S$ such that

$$(6)\quad \det \begin{bmatrix} \mathbf{z}_1^{\nu(1)} & \ldots & \mathbf{z}_1^{\nu(N)} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \mathbf{z}_N^{\nu(1)} & & \mathbf{z}_N^{\nu(N)} \end{bmatrix} \neq 0,$$

where the $\nu_{(j)}$ run through the distinct $\nu$ with $0 \leqq \nu_i \leqq \rho_i$, $1 \leqq i \leqq n$.

B. Now for any fixed $w \in D_0$, we know from Theorem 1 for $n$ that there is a $k$, $1 \leqq k \leqq m - 1$, and there are complex numbers $a_{\nu,k}(w)$ such that

$$(7)\quad f_k(w, \mathbf{z}) - f_m(w, \mathbf{z}) = \sum_\nu a_{\nu\,k}(w)\mathbf{z}^\nu$$

for all $\mathbf{z} \in \mathbf{C}^n$. We can write $a_{\nu,k}(w) = b_{\nu,k}(w)/c_k(w)$ where

$$(8)\quad |c_k(w)|^2 + \sum_\nu |b_{\nu,k}(w)|^2 = 1.$$

Take a convergent sequence $\{w_i\}_{i=1}^\infty$ from $D_0$ for which (7) involves the same $k$. Then from (8), there is a subsequence $\{w_{i_j}\}_{j=1}^\infty$ with

$$c_k(w_{i_j}) \rightarrow c \neq 0, \quad b_{\nu,k}(w_{i_j}) \rightarrow b_{\nu,k}.$$

Then by the continuity of $f_k(\mathbf{z}_0) - f_m(\mathbf{z}_0)$ in $z_0$, (7) still holds in the limit. So the set $S_k$ of $w \in D_0$ for which (7) involves the same $k$ is closed in $D_0$. But $D_0 = \bigcup_{k=1}^{m-1} S_k$. So for at least one $k$, $S_k$ contains a non-empty open set $I_0$, and (7) holds on the whole open set $I_0 \times \mathbf{C}^n$, where $a_{\nu,k}(w)$ varies with $w \in I_0$.

C. Now for any $\mathbf{z}_0 = (z_0, \mathbf{z}) \in I_0 \times D_n$, (7) shows that the determinant is zero:

$$(9)\quad \det \begin{bmatrix} f_k(z_0, \mathbf{z}) - f_m(z_0, \mathbf{z}) & \mathbf{z}^{\nu(1)} & \ldots & \mathbf{z}^{\nu(N)} \\ f_k(z_0, \mathbf{z}_1) - f_m(z_0, \mathbf{z}_1) & \mathbf{z}_1^{\nu(1)} & & \mathbf{z}_1^{\nu(N)} \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ f_k(z_0, \mathbf{z}_N) - f_m(z_0, \mathbf{z}_N) & \mathbf{z}_N^{\nu(1)} & & \mathbf{z}_N^{\nu(N)} \end{bmatrix} \equiv 0.$$

Thus (9) holds on an open set in $\mathbf{C}^{n+1}$ and, by the Monodromy Theorem, on all of $\mathbf{C}^{n+1}$. In the usual expansion of this determinant, the coefficient of $f_k(\mathbf{z}_0) - f_m(\mathbf{z}_0)$ is the non-zero constant of (6). But this gives the desired expression for $f_k(\mathbf{z}_0) - f_m(\mathbf{z}_0)$, for we know that $\mathbf{z}_1, \ldots, \mathbf{z}_n \in S$ and thus, as a function of $z_0$, each

$$\exp(f_k(z_0, \mathbf{z}_i) - f_m(z_0, \mathbf{z}_i)) \in O_\lambda.$$

It is an open problem to determine to what extent more rapid growth can be allowed with respect to $z_1, \ldots, z_n$ in the coefficients of a dependence relation.

https://doi.org/10.4153/CJM-1976-094-9 Published online by Cambridge University Press

## References

1. S. Bochner and W. T. Martin, *Several complex variables* (Princeton University Press, Princeton, 1948).
2. E. Borel, *Sur les zéros des fonctions entières*, Acta Math. *20* (1897), 357–396.
3. P. Bundschuh, *Ein funktiontheoretisches Analogon zum Satz von Lindemann*, Archiv. d. Mathematik *25* (1974), 45–51.
4. W. K. Hayman, *Meromorphic functions* (Oxford University Press, Oxford, 1964).
5. R. Narasimhan, *Un analogue holomorphe du théorème de Lindemann*, Ann. Inst. Fourier (Grenoble) *21* (1971), 271–278.
6. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes* (Gauthier-Villars, Paris, 1929).


*Pennsylvania State University,*
*University Park, Pennsylvania*

https://doi.org/10.4153/CJM-1976-094-9 Published online by Cambridge University Press