Proceedings of the Edinburgh Mathematical Society (1976-77), **20**, 317-327 ©

A VARIANT OF SEPARABILITY IN DUAL SYSTEMS

by J. O. POPOOLA† and I. TWEDDLE

(Received 19th November 1975)

1. Introduction

In (**12**) we introduced the concept of *essential separability* and used it to define two classes of locally convex spaces, *δ-barrelled spaces* and *infra-δ-spaces*, which serve as domain and range spaces respectively in certain closed graph theorems (**12**, Theorems 3 and 7). In this note we continue the study of these ideas. The relevant definitions are reproduced below.

Section 2 is concerned with characterisations of essential separability and its connection with weak compactness properties. In Section 3 we discuss some relationships between δ-barrelled, barrelled and countably barrelled spaces. Finally, in Section 4 we consider the associated δ-barrelled topology of an infra-δ-space in connection with a completeness result of V. Eberhardt and N. Adasch for infra-s-spaces.

Generally we follow the topological vector space notation of (**13**). Except where alternative symbols are introduced in the text, E^* will denote the algebraic dual of a vector space E, and when E is a separated locally convex space, E' will represent its (continuous) dual. When we refer to the dimension of E (dim E) we shall always mean its vector space dimension. $\xi|_A$ denotes the induced topology on a subset A of a topological space (X, ξ), $|B|$ is the cardinality of a set B and c is the cardinal number of the real field.

We are grateful to the referee for improving our original version of Theorem 2, which now appears as a corollary.

2. Essentially separable sets

We begin by reformulating the definition of essential separability which was given in (**12**). Let (E, F) be a dual pair. We regard E as a subspace of F^* and say that a subset A of E is *essentially separable for the dual pair* (E, F) if it is contained in a $\sigma(F^*, F)$-separable set. When the dual pair is clearly indicated, we simply say that A is essentially separable. In particular, if E is a separated locally convex space and A and B are subsets of E and E' respectively, we will usually write "A (resp. B) is essentially

† The first author was supported by a Commonwealth Academic Staff Scholarship which he gratefully acknowledges.

separable" for "A (resp. B) is essentially separable for the dual pair (E, E') (resp. (E', E))".

Theorem 1. *If A is essentially separable for the dual pair (E, F) then $\sigma(E, F)|_A$ has a base consisting of at most c sets.*

Proof. This is trivial if $A = \emptyset$. Otherwise let H be the linear span of A and let G be the $\sigma(F^*, F)$-closed linear span of A. Since $(F/H^o)^*$ is isomorphic to G and G is $\sigma(F^*, F)$-separable (**12**, Corollary to Theorem 1), it follows that $(F/H^o)^*$ is isomorphic to a product of at most c copies of the scalar field (**4**, Chapter VIII, Theorem 7.2). Thus the dimension and consequently the cardinality of F/H^o are at most c. Let Φ be the set of all non-empty finite subsets of F/H^o, so that $|\Phi| \leq c$, and let $\{x_n: n \in \mathbb{N}\}$ be an at most countable $\sigma(F^*, F)$-dense subset of G. Note that $\sigma(F^*, F)$, $\sigma(G, F/H^o)$ and $\sigma(E, F)$ all coincide on A.

Let

$$\mathcal{T} = \{\{x \in A: |\langle x - x_n, x'\rangle| < 1, x' \in \phi\}: \phi \in \Phi, n \in \mathbb{N}\}.$$

Certainly $|\mathcal{T}| \leq c$ and each element of $\mathcal{T}$ is $\sigma(E, F)|_A$-open. Let $y \in A$ and let U be any $\sigma(E, F)|_A$-neighbourhood of y. There exists $\phi_0 \in \Phi$ such that

$$V = \{x \in A: |\langle x - y, x'\rangle| < 1, x' \in \phi_0\} \subseteq U.$$

Also there exists $n_0 \in \mathbb{N}$ such that

$$x_{n_0} \in \{x \in G: |\langle x - y, x'\rangle| < 1, x' \in 2\phi_0\}.$$

Then $W = \{x \in A: |\langle x - x_{n_0}, x'\rangle| < 1, \quad x' \in 2\phi_0\} \in \mathcal{T}, \quad y \in W$ and $W \subseteq V \subseteq U$. Thus $\mathcal{T}$ is a base for $\sigma(E, F)|_A$.

Since any topological space has a dense subset of cardinality at most that of a given base for its topology, we have immediately:

Corollary. *If A is essentially separable for the dual pair (E, F), then A has a $\sigma(E, F)$-dense subset of cardinality at most c.*

The next result and its corollary are analogues of (**9**, Proposition 1.3).

Theorem 2. *Let E be a topological vector space with topology ξ, let A be an absolutely convex subset of E and let $\mathcal{A}$ be a base of neighbourhoods of 0 for $\xi|_A$. For each $W \in \mathcal{A}$, let W' be an open balanced ξ-neighbourhood of 0 such that $W' \cap A \subseteq W$. Then if D is a dense subset of A, the sets $(d + W') \cap A$ $(d \in D, W \in \mathcal{A})$ form a base for $\xi|_A$.*

Proof. Let $y \in A$ and let Y be any $\xi|_A$-neighbourhood of y. There exist ξ-neighbourhoods U and V of 0 and $W \in \mathcal{A}$ such that

$$(y + U) \cap A \subseteq Y, \quad V + V + V \subseteq U \quad \text{and} \quad W \subseteq V(*).$$

Choose $d \in (y + (V \cap W')) \cap D$ and let $x \in X = (d + W') \cap A$. Since A is absolutely convex and W' is balanced, $\frac{1}{2}(x-d) \in A \cap W'$ and so by $(*)$

$$x = y + 2(\tfrac{1}{2}(x-d)) + (d-y) \in (y + 2W + V) \cap A \subseteq Y.$$

The result now follows since X is an $\xi|_A$-open set which contains y.

Corollary. *Let (E, F) be a dual pair and let A be an absolutely convex subset of E. Then $\sigma(E, F)|_A$ has a base consisting of at most c sets if and only if*

(i) 0 *has a base of neighbourhoods for $\sigma(E, F)|_A$ consisting of at most c sets,*

(ii) *A has a $\sigma(E, F)$-dense subset of cardinality at most c.*

Proof. The conditions are clearly necessary. An application of Theorem 2 establishes their sufficiency.

Let (E, F) be a dual pair and let A be a non-empty $\sigma(E, F)$-bounded set. The $\sigma(F^*, F)$-closed absolutely convex envelope B of A is $\sigma(F^*, F)$-compact. Let H be the linear span of A and let L be the linear span of B. $(F/H^o, L)$ is a dual pair and F/H^o is a normed space under $\tau(F/H^o, L)$ with B as the closed unit ball of the dual space L. We denote this normed space by $\mathcal{N}(F, A)$ and its completion by $\mathcal{B}(F, A)$. We now characterise essential separability for A in terms of these spaces.

Lemma 1. *If a normed space E has a total subset D with $|D| \leqslant c$, then* $\dim E \leqslant c$.

Proof. The linear span X of D has cardinality at most c and since each element of E is the limit of a sequence in X, it follows that $|E| \leqslant c^{\aleph_0} = c$. Thus $\dim E \leqslant c$.

Theorem 3. *Let (E, F) be a dual pair and let A be a non-empty $\sigma(E, F)$-bounded set. The following are equivalent:*

(i) *A is essentially separable;*

(ii) $\dim \mathcal{N}(F, A) \leqslant c$;

(iii) $\dim \mathcal{B}(F, A) \leqslant c$.

Proof. The argument used in the first part of the proof of Theorem 1 shows that (i)$\Rightarrow$(ii), for $\mathcal{N}(F, A)^*$ is isomorphic to the $\sigma(F^*, F)$-closed linear span of A. ((ii)$\Rightarrow$(iii)) follows from Lemma 1, while ((iii)$\Rightarrow$(ii)) is trivial.

Suppose that (ii) holds. The $\sigma(F^*, F)$-closed linear span of A is $\sigma(F^*, F)$-separable, being isomorphic to a product of at most c copies of the scalar field (4, Chapter VIII, Theorem 7.2). Thus A is essentially separable.

As a corollary we have a partial converse of Theorem 1.

Corollary. *Let A be a non-empty absolutely convex $\sigma(E, F)$-bounded* *set. Then A is essentially separable if and only if 0 has a base of neighbourhoods for $\sigma(E,F)|_A$ consisting of at most c sets.*

Proof. The necessity of the condition is immediate by Theorem 1. If the condition is satisfied, there is a set $\{\phi_\lambda: \lambda \in \Lambda\}$ of non-empty finite subsets of F such that $|\Lambda| \leq c$ and $\{\{x \in A: |\langle x, x'\rangle| \leq 1, x' \in \phi_\lambda\}: \lambda \in \Lambda\}$ is a base of neighbourhoods of 0 for $\sigma(E,F)|_A$. The $\sigma(F^*,F)$-closure B of A is $\sigma(F^*,F)$-compact and absolutely convex, and by the bipolar theorem $\{\{x \in B: |\langle x, x'\rangle| \leq 1, x' \in \phi_\lambda\}: \lambda \in \Lambda\}$ is a base of neighbourhoods of 0 under $\sigma(F^*,F)|_B$.

Let $z \in B\backslash\{0\}$. Then there exists $\lambda_0 \in \Lambda$ such that $|\langle z, x'\rangle| > 1$ for some $x' \in \phi_{\lambda_0}$. It follows that $D = \cup\{\phi_\lambda: \lambda \in \Lambda\}$ separates the elements of B and so the set of equivalence classes in $\mathcal{N}(F, A)$ of the elements of D is total. Since $|D| \leq c$, the result now follows from Lemma 1 and Theorem 3.

Remark. The Corollary to Theorem 1 does not have a similar converse. If $E = l''_\infty$ and $F = l'_\infty$, the closed unit ball A of l_∞ is a $\sigma(l''_\infty, l'_\infty)$-dense subset of the closed unit ball B of l''_∞ and $|A| = c$. Clearly $\mathcal{N}(l'_\infty, A) = \mathcal{B}(l'_\infty, A) \simeq l'_\infty$. Now it follows from (**17**, Theorem 2.3 and Note 1.8(a)) that $\dim l'_\infty \geq 2^c$. In fact $\dim l'_\infty = 2^c$ for $\dim l_\infty = c$ and $\dim l'_\infty \leq \dim l^*_\infty = c^c = 2^c$. Thus neither A nor B is essentially separable for the dual pair (l''_∞, l'_∞). Note however that A is essentially separable for the dual pair (l_∞, l_1).

We now identify some particular essentially separable sets.

Lemma 2. *Let E be a normed space and let B be the closed unit ball of E'. If $|B| = c$ then* $\dim E \leq c$.

Proof. There is a set Λ with cardinality at most c and a bijection $\lambda \mapsto x'_\lambda$ of Λ onto $A = \{x' \in B: \|x'\| = 1\}$. For each $\lambda \in \Lambda$, choose $x_\lambda \in E$ such that $\langle x_\lambda, x'_\lambda\rangle \neq 0$. Let M be the closed vector subspace of E generated by $\{x_\lambda: \lambda \in \Lambda\}$. Then $M = E$, for otherwise we would be able to find $\lambda_0 \in \Lambda$ such that $\langle x, x'_{\lambda_0}\rangle = 0$ for all $x \in M$, contradicting $\langle x_{\lambda_0}, x'_{\lambda_0}\rangle \neq 0$.

The result now follows from Lemma 1.

Corollary. *Let (E,F) be a dual pair and let A be a $\sigma(E,F)$-compact convex set. If $|A| \leq c$ then A is essentially separable.*

Proof. Let C be the balanced hull of A and let B be the closed absolutely convex envelope of A. Then C is $\sigma(E,F)$-compact, $|C| \leq c$ and since $B \subseteq C + C + C + C$, it follows that B is a $\sigma(E,F)$-compact set with cardinality at most c. In fact $|B| = 0$, 1 or c. In the case $|B| = c$ the corollary now follows from Lemma 2 and Theorem 3. The other cases are trivial.

Theorem 4. *Let E be a separated locally convex space whose topology is defined by at most c seminorms and let F be the completion of E. Then each* *subset of E which is $\sigma(F, E')$-relatively compact and whose cardinality is at most c is essentially separable.*

Proof. Since the topology of F is also defined by at most c seminorms and since the dual of F is (isomorphic to) E', it is enough to establish the result when E is complete. We may regard E as a subspace of a product $\Pi\{E_\lambda: \lambda \in \Lambda\}$ of Banach spaces E_λ $(\lambda \in \Lambda)$ where $|\Lambda| \leq c$ (**13**, Chapter V, Proposition 16 and Corollary to Proposition 19). For each $\lambda \in \Lambda$ let p_λ be the canonical projection of the product onto E_λ.

Let A be a non-empty $\sigma(E, E')$-relatively compact set with $|A| \leq c$. For each $\lambda \in \Lambda$, $p_\lambda(A)$ is $\sigma(E_\lambda, E'_\lambda)$-relatively compact and so by Krein's theorem (**10**, Section 24, 5(4)) the $\sigma(E_\lambda, E'_\lambda)$-closed absolutely convex envelope B_λ of $p_\lambda(A)$ is $\sigma(E_\lambda, E'_\lambda)$-compact. Now $|p_\lambda(A)| \leq c$ and so the absolutely convex envelope D_λ of $p_\lambda(A)$ has cardinality at most c. Since each element of B_λ is the $\sigma(E_\lambda, E'_\lambda)$-limit of a sequence in D_λ (**10**, Section 24, 1(7)), it is easily shown that $|B_\lambda| \leq c$. By the Corollary to Lemma 2, B_λ and therefore $p_\lambda(A)$ are essentially separable for the dual pair (E_λ, E'_λ). Let C_λ be a $\sigma(E'^*_\lambda, E'_\lambda)$-separable set which contains $p_\lambda(A)$. We now have

$$A \subseteq \prod \{p_\lambda(A): \lambda \in \Lambda\} \subseteq C = \prod \{C_\lambda: \lambda \in \Lambda\}$$

and C is $\sigma(\Pi_{\lambda\in\Lambda} E'^*_\lambda, \Sigma_{\lambda\in\Lambda} E'_\lambda)$-separable (**4**, Chapter VIII, Theorem 7.2). Thus A is essentially separable for the dual pair $(\Pi_{\lambda\in\Lambda} E_\lambda, \Sigma_{\lambda\in\Lambda} E'_\lambda)$.

Let H be the linear span of A and let $H^\bullet, E^\bullet$ be the polars of H and E respectively in $\Sigma_{\lambda\in\Lambda} E'_\lambda$ and let $H^\bullet$ be the polar of H in E'. Then

$$\left(\sum_{\lambda\in\Lambda} E'_\lambda\right)\Big/H^\bullet \simeq \left(\left(\sum_{\lambda\in\Lambda} E'_\lambda\right)\Big/E^\bullet\right)\Big/(H^\bullet/E^\bullet) \simeq E'/H^\bullet.$$

The result now follows from Theorem 3.

Corollary. *Let E, F be as in the theorem and let B be a subset of E which is $\sigma(F, E')$-relatively compact. If x is an element of the $\sigma(F, E')$-closure of B, there is an essentially separable subset A of B such that x is in the $\sigma(F, E')$-closure of A.*

Proof. By (**16**, (b)) there is a subset A of B with cardinality at most c such that x is in the $\sigma(F, E')$-closure of A. The result now follows from the theorem since A is also $\sigma(F, E')$-relatively compact.

As an application of this corollary we obtain in Theorem 5 criteria for weak compactness and weak relative compactness in a separated locally convex space whose topology is defined by at most c seminorms. These would appear to be the natural analogues of the well-known sequential criteria in a metrizable locally convex space (**10**, Section 24, 3(8), (9)).

Theorem 5. *Let E be a separated locally convex space whose topology* *is defined by at most c seminorms. A subset B of E is $\sigma(E, E')$-relatively compact (resp. $\sigma(E, E')$-compact) if and only if each essentially separable subset of B is $\sigma(E, E')$-relatively compact (resp. $\sigma(E, E')$-relatively compact and has its $\sigma(E, E')$-closure contained in B).*

Proof. The conditions are clearly necessary.

Under either condition, B is $\sigma(E, E')$-relatively countably compact, since a countable set is trivially essentially separable. Then if F is the completion of E, B is $\sigma(F, E')$-relatively compact by Eberlein's theorem (**10**, Section 24, 2(1)). It follows from the Corollary to Theorem 4 that each $\sigma(F, E')$-point of closure x of B is already in E so that B is $\sigma(E, E')$-relatively compact under either condition. Under the bracketed condition, $x \in B$ so that B is $\sigma(E, E')$-compact.

The concept of *Schauder dimension* for Banach spaces was introduced in (**7**). We end the present section by combining this idea with a property of essentially separable sets. The terminology is that of (**7**).

Theorem 6. *Let E be a Banach space and suppose that every subset of the closed unit ball of E' has a $\sigma(E', E)$-dense subset of cardinality at most c. If E has a Schauder dimension, then* $\dim E \leq c$.

Proof. Let $\{x_\lambda : \lambda \in \Lambda\}$ be a maximal strongly linearly independent subset of E and let $\{x'_\lambda : \lambda \in \Lambda\}$ be a subset of E' such that $\langle x_\mu, x'_\lambda \rangle = \delta_{\lambda\mu}$ for all $\lambda, \mu \in \Lambda$. Now $\{\|x'_\lambda\|^{-1} x'_\lambda : \lambda \in \Lambda\}$ is a subset of the closed unit ball of E' with no proper $\sigma(E', E)$-dense subset, for if $\lambda \neq \mu$, $\langle x_\lambda, \|x'_\lambda\|^{-1} x'_\lambda - \|x'_\mu\|^{-1} x'_\mu \rangle = \|x'_\lambda\|^{-1}$. Thus $|\Lambda| \leq c$. Since $\{x_\lambda : \lambda \in \Lambda\}$ is total in E (**7**, Proposition 1), the result now follows from Lemma 1.

3. δ-barrelled spaces

We gave the following definition in (**12**).

A separated locally convex space E is *δ-barrelled* if each essentially separable $\sigma(E', E)$-bounded set is equicontinuous.

We showed by example that a δ-barrelled space need not be barrelled even in its associated Mackey topology. On the other hand, since δ-barrelled spaces are necessarily σ-barrelled, a separable δ-barrelled space is barrelled (**3**, Corollary 4a). Further a δ-barrelled space which has a strongly dense subset of cardinality at most c is always barrelled (**12**, Corollary 2 of Theorem 3). We now give two generalisations of this last result.

Theorem 7. *Let E be a δ-barrelled space with completion F. Suppose that there is a family $(X_\lambda)_{\lambda \in \Lambda}$ of subsets of E such that*

(i) $|\Lambda| \leq c$,

(ii) $\cup\{X_\lambda: \lambda \in \Lambda\}$ *is total in* E *under* $\beta(E, E')$,

(iii) *for each* $\lambda \in \Lambda$, X_λ *is* $\sigma(F, E')$*-relatively compact.*

Then $(E, \tau(E, E'))$ *is barrelled.*

Proof. Let Y_λ be the $\sigma(F, E')$-closed absolutely convex envelope of X_λ $(\lambda \in \Lambda)$. By Krein's theorem (**10**, Section 24, 5(4)) and (iii), each Y_λ is $\sigma(F, E')$-compact. Denote by G the subspace of F spanned by $\cup\{Y_\lambda: \lambda \in \Lambda\}$ and by H the subspace of E spanned by $\cup\{X_\lambda: \lambda \in \Lambda\}$.

Let B be a $\sigma(E', E)$-closed bounded set and let A be a subset of B which is essentially separable for the dual pair (E', G). Certainly A is essentially separable for the dual pair (E', H) and since H is $\beta(E, E')$-dense in E, it follows easily from Theorem 3 that A is essentially separable for the dual pair (E', E). Since E is δ-barrelled, A is equicontinuous. If C is the $\sigma(E', E)$-closure of A, then $C \subseteq B$ and C is also $\sigma(E', F)$-compact (**13**, Chapter VI, Corollary 3 of Theorem 2) and therefore $\sigma(E', G)$-compact. Since each Y_λ is $\sigma(G, E')$-compact and absolutely convex, E' has a topology of the dual pair (E', G) defined by at most c seminorms and so by Theorem 5, B is $\sigma(E', G)$-compact. This implies that B is $\sigma(E', H)$-compact, and since E is contained in the completion of H for the topology induced by $\beta(E, E')$, we deduce that B is $\sigma(E', E)$-compact.

Remark. In Theorem 7, the initial δ-barrelled topology of E need not be $\tau(E, E')$. To see this, we refer to (**12**, Theorem 2 and Remark (i) following Theorem 8). If $|M| > c$, $l_2(M)$ is δ-barrelled but not barrelled under the topology of uniform convergence on the $\sigma(l_2(M), l_2(M))$-bounded essentially separable sets. However the conditions of Theorem 7 are satisfied by taking the closed unit ball of $l_2(M)$ as the single X_λ.

We require the following lemma for our other result in this direction. It is probably well-known but we include a proof for completeness.

Lemma 3. *Let* E *be a* σ*-barrelled space. If* $\Sigma_{\lambda\in\Lambda} x_\lambda$ *converges unconditionally in* E, *it also converges unconditionally under* $\beta(E, E')$ *to the same sum.*

Proof. It is enough to show that $\Sigma_{\lambda\in\Lambda} x_\lambda$ is unconditionally Cauchy under $\beta(E, E')$, for then the result will follow from (**10**, Section 18, 4(4)). Suppose that this is false and denote by Φ the set of all non-empty finite subsets of Λ. Then there is a $\sigma(E', E)$-bounded set B such that for each $\phi \in \Phi$, there exist $\phi' \in \Phi$ with $\phi' \cap \phi = \emptyset$ and $x' \in B$ such that $|\langle \Sigma_{\lambda\in\phi'} x_\lambda, x'\rangle| > 1$. We can thus determine sequences (ϕ_n) in Φ and (x'_n) in B such that $\phi_{n+1} \cap \cup_{r=1}^n \phi_r = \emptyset$ and $|\langle \Sigma_{\lambda\in\phi_n} x_\lambda, x'_n\rangle| > 1$ $(n \in \mathbb{N})$.

But $\{x'_n: n \in \mathbb{N}\}$ is equicontinuous and so there exists $\phi_0 \in \Phi$ such that $|\langle \Sigma_{\lambda\in\phi} x_\lambda, x'_n\rangle| \leq 1$ for all $n \in \mathbb{N}$ and for all $\phi \in \Phi$ with $\phi \cap \phi_0 = \emptyset$. Since $\phi_n \cap \phi_0 = \emptyset$ for all sufficiently large n we obtain a contradiction.

Theorem 8. *Let E be a δ-barrelled space and suppose that there is a family $(x_\mu)_{\mu\in M}$ of elements of E such that*

(a) *for each $x \in E$ there exist scalars α_μ $(\mu \in M)$ such that $\Sigma\, \alpha_\mu x_\mu$ is unconditionally convergent to x,*

(b) *there is a family $(z_\lambda)_{\lambda\in\Lambda}$ of elements of E such that $z_\lambda = \Sigma\, \alpha_\mu^{(\lambda)} x_\mu$ $(\lambda \in \Lambda)$, $|\Lambda| \leq c$ and for each $\mu \in M$, at least one $\alpha_\mu^{(\lambda)} \neq 0$.*

Then each $\sigma(E', E)$-bounded set is essentially separable and consequently E is barrelled.

Proof. Let A be a non-empty $\sigma(E', E)$-bounded set. For each $x \in E$ let $\bar{x}$ denote its equivalence class in $\mathcal{N}(E, A)$. If $x = \Sigma\, \alpha_\mu x_\mu$ as above, it follows from Lemma 3 that $\Sigma\, \alpha_\mu \bar{x}_\mu$ converges unconditionally to $\bar{x}$ in $\mathcal{N}(E, A)$. Since $\mathcal{N}(E, A)$ is a normed space, $\{\mu \in M: \alpha_\mu \bar{x}_\mu \neq 0\}$ is at most countable and so $\cup_{\lambda\in\Lambda} \{\mu \in M: \alpha_\mu^{(\lambda)} \bar{x}_\mu \neq 0\}$ has cardinality at most c. But by (b) this set is just $\{\mu \in M: \bar{x}_\mu \neq 0\}$. Since $\{\bar{x}_\mu: \mu \in M\}$ is total in $\mathcal{N}(E, A)$ the result now follows from Lemma 1 and Theorem 3.

Remark. It should be noted that the space E of Theorem 8 need not have a dense subset of cardinality at most c. Using the argument in part (3) of the proof of (**4**, Chapter VIII, Theorem 7.2), we see that $\mathbb{R}^M$ has no such subset if $|M| > 2^c$. However we may apply Theorem 8 to $\mathbb{R}^M$ with $x_\mu = (\delta_{\mu\gamma})_{\gamma\in M}$ and a single z_λ, viz $\Sigma\, x_\mu$. Theorem 8 is an analogue of (**15**, Theorem 1).

In (**12**, Theorem 2) we showed that a δ-barrelled space E is both δ-barrelled and countably barrelled (**8**) under the topology $\delta(E, E')$ of uniform convergence on the $\sigma(E', E)$-bounded essentially separable sets. We end this section by giving an example of a δ-barrelled space which is not countably barrelled. In (**14**, Proposition 4.4), J. Schmets describes a general method of constructing σ-barrelled spaces which are not countably barrelled. We adapt this technique to our present purpose, although our approach is rather different.

Let $E = \mathbb{R}^{(M)}$ and let $E' = \{(\xi_\mu) \in \mathbb{R}^M: |\{\mu: \xi_\mu \neq 0\}| \leq c\}$. For any subset A of E' let $\operatorname{supp} A = \{\nu \in M: \exists (\xi_\mu) \in A \text{ with } \xi_\nu \neq 0\}$. It follows from the Corollary to Theorem 1 that if A is essentially separable for the dual pair (E', E), $|\operatorname{supp} A| \leq c$ $(*)$. Thus if A is a $\sigma(E', E)$-bounded essentially separable set, it is $\sigma(E', E)$-relatively compact. Since the closed absolutely convex envelope of an essentially separable set is essentially separable, the Mackey-Arens theorem shows that $\delta(E, E')$ is a topology of the dual pair (E, E') under which E is δ-barrelled (cf. Example 1 of (**12**)).

For each non-empty subset B of M

$$S(B) = \{(\xi_\mu) \in \mathbb{R}^M: \xi_\mu = 0 \quad \text{if} \quad \mu \notin B, \Sigma\, |\xi_\mu| \leq 1\}$$

is easily seen to be a closed bounded absolutely convex subset of $\mathbb{R}^M$.

Therefore $S(B)$ is compact in $\mathbb{R}^M$ and since it is contained in E', it is $\sigma(E',E)$-compact.

We now take $M=\mathcal{P}(\mathbb{R})$, the power set of $\mathbb{R}$. In this case $E'\neq\mathbb{R}^M$ and $(E,\tau(E,E'))$ is not barrelled. Let $\mathcal{B}$ be the collection of all $\sigma(E',E)$-bounded essentially separable sets together with the sets $S(\mathcal{P}(C))$ where C is a compact subset of $\mathbb{R}$. The topology ξ on E of uniform convergence on the sets in $\mathcal{B}$ is then a δ-barrelled topology of the dual pair (E,E') and a base of neighbourhoods of the origin for ξ is given by all sets of the form $D^\circ\cap\epsilon S(\mathcal{P}(C))^\circ$ $(**)$, where D is a non-empty $\sigma(E',E)$-bounded essentially separable set, $\epsilon>0$ and C is a compact subset of $\mathbb{R}$.

Now $A=\bigcup_{n=1}^{\infty}S(\mathcal{P}([-n,n]))$ is a subset of $S(\mathcal{P}(\mathbb{R}))$ so that A is a $\sigma(E',E)$-bounded set which is the union of a sequence of ξ-equicontinuous sets. Given any set V of the form $(**)$, by $(*)$ and the fact that $|\mathcal{P}([-n,n])\backslash\mathcal{P}(C)|=2^c$ for all sufficiently large n, we may choose $\nu\in(\bigcup_{n=1}^{\infty}\mathcal{P}([-n,n]))\backslash\{(\text{supp } D)\cup\mathcal{P}(C)\}$. Then $(2\delta_{\mu\nu})_{\mu\in M}\in V$ so that $(\delta_{\mu\nu})_{\mu\in M}\notin V^\circ$. Since $(\delta_{\mu\nu})_{\mu\in M}\in A$, this shows that A is not ξ-equicontinuous and consequently (E,ξ) is δ-barrelled but not countably barrelled.

4. Infra-δ-spaces

Let E be a separated locally convex space and for each vector subspace H of E' let H^δ be the intersection of all vector subspaces G of E^* such that

(i) $H\subseteq G$,

(ii) the $\sigma(E^*,E)$-closure of each $\sigma(E^*,E)$-bounded subset of G which is essentially separable for the dual pair (E^*,E) is contained in G.

As in (**12**) we say that E is an infra-δ-space if for each $\sigma(E',E)$-dense vector subspace H, we have $E'\cap H^\delta=E'$.

For any separated locally convex space E, the upper bound topology η of the initial topology ξ of E and $\delta(E,(E')^\delta)$ is clearly the coarsest δ-barrelled topology on E which is finer than ξ. We call η the *associated δ-barrelled topology of E*. This definition is analogous to Adasch's definition of the associated barrelled topology (**1**), which is clearly finer than the associated δ-barrelled topology.

It is shown in (**5**, Theorem 1.5) and in (**2**, Section 4) that an infra-s-space (**1**) is complete in its associated barrelled topology. As pointed out in (**12**), the infra-δ-spaces form a proper subclass of the infra-s-spaces, so that this completeness result applies to infra-δ-spaces. However essentially the same proof as that given in (**5**) shows that an infra-δ-space is actually complete in its associated δ-barrelled topology. To show that this is a genuine improvement, we adapt ideas from (**6**) to give an example of an infra-δ-space for which the associated barrelled topology and the as-

https://doi.org/10.1017/S0013091500026559 Published online by Cambridge University Press

sociated δ-barrelled topology are not even topologies of the same dual pair.

Let $E=\mathbb{R}^M$ where $|M|=2^c$ and let $E'=\{(\xi_\mu)\in\mathbb{R}^M\colon |\{\mu\colon \xi_\mu\neq 0\}|\leq\aleph_0\}$. We show first of all that E is an infra-δ-space for any topology of the dual pair (E,E'). Let H be a $\sigma(E',E)$-dense vector subspace and let (x'_n) be a sequence in $H^\delta\cap E'$ which converges to $x'\in E'$ under $\sigma(E',E)$. Since $\{x'_n\colon n\in\mathbb{N}\}$ is (essentially) separable, its $\sigma(E^*,E)$-closure must be contained in each G considered in constructing H^δ. Thus $x'\in H^\delta$ and so $H^\delta\cap E'$ is $\sigma(E',E)$-sequentially closed. But as pointed out by V. Eberhardt in (**6**), Theorem 2.1 of (**11**) now shows that $H^\delta\cap E'$ is $\sigma(E',E)$-closed. Since $H\subseteq H^\delta\cap E'$, we must then have $H^\delta\cap E'=E'$.

It is clear that if B is any subset of $\mathbb{R}^M$ which is a product of intervals, $B\cap E'$ is $\sigma(\mathbb{R}^M,\mathbb{R}^{(M)})$-dense in B. It follows from this observation that the associated barrelled topology for any topology of the dual pair (E,E') is $\tau(\mathbb{R}^{(M)},\mathbb{R}^M)$. However if $F'=\{(\xi_\mu)\in\mathbb{R}^M\colon |\{\mu\colon\xi_\mu\neq 0\}|\leq c\}$, we know from the previous section that $\delta(E,F')$ is a δ-barrelled topology of the dual pair (E,F'). If we start with the topology $\sigma(E,E')$ on E, the associated δ-barrelled topology η must therefore be coarser than $\delta(E,F')$. (In fact it is not difficult to show that $\eta=\delta(E,F')$). Since $F'\neq\mathbb{R}^M$, we may take $(E,\sigma(E,E'))$ for the promised example.

REFERENCES

(**1**) N. ADASCH, Tonnelierte Räume und zwei Sätze von Banach, *Math. Ann.* **186** (1970), 209–214.

(**2**) N. ADASCH, Vollständigkeit und der Graphensatz, *J. Reine Angew. Math.* **249** (1971), 217–220.

(**3**) M. DE WILDE and C. HOUET, On increasing sequences of absolutely convex sets in locally convex spaces, *Math. Ann.* **192** (1971), 257–261.

(**4**) J. DUGUNDJI, *Topology* (Allyn and Bacon, Boston, 1966).

(**5**) V. EBERHARDT, Der Graphensatz von A.P. und W. Robertson für s-Räume, *Manuscripta Math.* **4** (1971), 255–262.

(**6**) V. EBERHARDT, Einige Vererbbarkeitseigenschaften von B-und B_r-vollständigen Räumen, *Math. Ann.* **215** (1975), 1-11.

(**7**) J. W. EVANS and R. A. TAPIA, Hamel versus Schauder dimension, *Amer. Math. Monthly* **77** (1970), 385–388.

(**8**) T. HUSAIN, Two new classes of locally convex spaces, *Math. Ann.* **166** (1966), 289–299.

(**9**) N. J. KALTON, Some forms of the closed graph theorem, *Proc. Cambridge Philos. Soc.* **70** (1971), 401–408.

(**10**) G. KÖTHE, *Topological Vector Spaces* I (Springer-Verlag, Berlin, 1969).

(11) N. NOBLE, The continuity of functions on cartesian products, *Trans. Amer. Math. Soc.* **149** (1970), 187–198.

(12) J. O. POPOOLA and I. TWEDDLE, On the closed graph theorem, *Glasgow Math. J.* **17**(1976), 89–97.

(13) A. P. ROBERTSON and W. J. ROBERTSON, *Topological Vector Spaces*, 2nd edition (Cambridge University Press, Cambridge, 1973).

(14) J. SCHMETS, *Espaces associés à un espace linéaire à semi-normes-applications aux espaces de fonctions continues* (Université de Liège, 1972–3).

(15) I. TWEDDLE, Unconditional convergence and bases, *Proc. Edinburgh Math. Soc.* **18** (Series II) (1973), 321–324.

(16) M. VALDIVIA, Some criteria for weak compactness, *J. Reine Angew. Math.* **255** (1972), 165–169.

(17) K. YOSIDA and E. HEWITT, Finitely additive measures, *Trans. Amer. Math. Soc.* **72** (1952), 46–66.

UNIVERSITY OF STIRLING
AND
UNIVERSITY OF LAGOS