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On a construction of unitary cocycles and the representation theory of amenable groups

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Abstract. If K is a countable amenable group acting freely and ergodically on a probability space (Γ, μ), and G is an arbitrary countable amenable group, we construct an injection of the space of unitary representations of G into the space of unitary 1-cocycles for K on (Γ, μ); this injection preserves intertwining operators. We apply this to show that for many of the standard non-type-I amenable groups H, the representation theory of H contains that of every countable amenable group.

1. *Introduction and conventions*

Throughout, G denotes a locally compact second countable group, and $\mathscr{H}_n$, $n = 1, 2, \ldots, \infty$, a separable Hilbert space of dimension n. The space $\text{Rep}_n\,(G)$ of continuous unitary representations of G on $\mathscr{H}_n$ is endowed with the standard Borel structure generated by the functions

$$\pi \in \text{Rep}_n\,(G) \to \langle \pi(s)\xi, \eta \rangle \qquad \text{for } s \in G \quad \text{and} \quad \xi, \eta \in \mathscr{H}_n.$$

$\text{Rep}\,(G) = \bigcup_n \text{Rep}_n\,(G)$ is given the usual Borel structure as a disjoint union of Borel spaces. $\underline{\text{Rep}}\,(G)$ denotes the category with objects $\text{Rep}\,(G)$ and

$$\text{Hom}\,(\sigma, \sigma') = \{T\colon T \textit{ is a bounded operator from } \mathscr{H}_\sigma \textit{ to } \mathscr{H}_{\sigma'} \textit{ with}$$
$$T\sigma(s) = \sigma'(s)T \quad \textit{for all } s \in G\}.$$

Similarly, one may consider categories $\underline{\text{Fac}}\,(G)$ and $\underline{\text{Irr}}\,(G)$ consisting of factor or irreducible representations.

The main purpose of the paper is to show that for a number of standard non-type-I countable amenable groups H, and for every countable amenable group G, $\underline{\text{Rep}}\,(G)$ is equivalent to a full subcategory of $\underline{\text{Rep}}\,(H)$, and that this is characteristic of amenable groups—see theorem 4 and subsequent remarks. The proof depends on a description of the representations of certain semidirect products $H = N \times_\alpha K$ in terms of certain measures on $\hat{N}$ (we assume N is abelian) and cocycles for the dual action of K on $\hat{N}$; this description is certainly well known, but as I am unaware of an exposition adapted to the present needs, the relevant material is given in § 3.

In addition we need to consider cohomology spaces of actions of groups on measure spaces as categories; this may be accomplished as follows. Let (Γ, μ) be a standard measure space, and let the discrete group K act on Γ (on the right) to leave μ quasi-invariant. If $\mathrm{U}(\mathscr{H}_n)$ denotes the unitary group of $\mathscr{H}_n$, with the standard Borel structure generated by the weak topology,

$$Z^1_\mu(\Gamma\times K, \mathrm{U}(\mathscr{H}_n))$$

denotes the space of measurable maps $\gamma\in\Gamma\to u(\gamma)\in \mathrm{U}(\mathscr{H}_n)$ satisfying

$$u(\gamma, k)u(\gamma\cdot k, k') = u(\gamma, kk') \qquad \mu\text{-a.e. in } \gamma \text{ for all } k, k'\in K,$$

where we identify two such maps if they agree μ-a.e. in γ for each k. We let

$$Z^1_\mu(\Gamma\times K, U) = \bigcup_n Z^1_\mu(\Gamma\times K, \mathrm{U}(\mathscr{H}_n)),$$

and for $u, u'\in Z^1(\Gamma\times K, U)$, with $u:\Gamma\times K\to\mathscr{H}$ and $u':\Gamma\times K\to\mathscr{H}'$, set

$$\begin{aligned}\mathrm{Hom}_\mu\,(u, u') = \{&\text{bounded measurable functions } \gamma\to T(\gamma), \text{ with} \\ &T(\gamma):\mathscr{H}\to\mathscr{H}' \text{ and } T(\gamma)u(\gamma, k) = u'(\gamma, k)T(\gamma\cdot k) \quad \text{a.e.}\}\end{aligned}$$

where we again identify maps which agree μ-a.e. If $T\in\mathrm{Hom}_\mu\,(u, u')$ and $S\in\mathrm{Hom}_\mu\,(u', u'')$ the pointwise product ST is in $\mathrm{Hom}_\mu\,(u, u'')$, so that $Z^1_\mu(\Gamma\times K, U)$ may be viewed as a category which is denoted $\underline{Z}^1_\mu(\Gamma\times K, U)$.

Note that $\underline{Z}^1_\mu(\Gamma\times K, U)$ depends only on the class of measure μ; furthermore, by the results of [**3**], $\underline{Z}^1_\mu(\Gamma\times K, U)$ depends only on the isomorphism class of the measured groupoid $(\Gamma\times K, \mu)$. In particular if K (respectively K') acts freely on (Γ, μ) (respectively (Γ', μ')) and K on (Γ, μ) is orbit equivalent, (see [**4**]), with K' on (Γ', μ'), we may identify $\underline{Z}^1_\mu(\Gamma\times K, U)$ with $\underline{Z}^1_{\mu'}(\Gamma'\times K', U)$. It is precisely this freedom, coupled with the existence of many different presentations of the same measured groupoid, which allows us to replicate $\underline{\mathrm{Rep}}\,(G)$ within $\underline{Z}^1_\mu(\Gamma\times K, U)$ whenever G and K are countable amenable and (Γ, μ) is an ergodic K-space with K acting freely and non-transitively—see § 2.

2. *A construction of cocyles*

Fix an amenable, countable group K and a free, ergodic action of K on a standard measure space (Γ, μ) with $\mu(\Gamma) = 1$; we suppose this action to be non-transitive and hence not of type I, [**4**]. For any other countable amenable group G, let (Z, m) be an ergodic free G space of type II_1, (for example, the Bernoulli shift based on G), and let

$$(Y, \nu) = \prod_{-\infty}^{\infty} (Z, m).$$

We let G act on Y via the diagonal action, and let $\mathbb{Z}$ act via the shift, obtaining an action of $\mathbb{Z}\times G$ with $\mathbb{Z}$ acting ergodically. Thus $K\times\mathbb{Z}\times G$ acts ergodically on $(\Gamma\times Y, \mu\times\nu)$, with $K\times\mathbb{Z}$ acting ergodically.

Define, for each unitary representation $\{\sigma, \mathscr{H}\}$ of G, a map

$$v_\sigma:(\Gamma\times Y)\times(K\times\mathbb{Z}\times G)\to\mathrm{U}(\mathscr{H}) \qquad \text{by } v_\sigma((\gamma, y), (k, n, g)) = \sigma(g).$$

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PROPOSITION 1. With the notation above

(i) $v_\sigma \in Z^1_{\mu\times\nu}((\Gamma\times Y)\times(K\times\mathbb{Z}\times G), \mathrm{U}(\mathscr{H}))$.

(ii) There is a natural linear isomorphism between $\mathrm{Hom}_G(\sigma, \sigma')$ and $\mathrm{Hom}_{\mu\times\nu}(v_\sigma, v_{\sigma'})$.

Proof. (i) Is trivial.

(ii) If $T:\mathscr{H}_\sigma\to\mathscr{H}_{\sigma'}$ is bounded and $T\sigma(g)=\sigma'(g)T$ for all $g\in G$, the constant map $(\gamma, y)\to T$ clearly lies in $\mathrm{Hom}_{\mu\times\nu}(v_\sigma, v_{\sigma'})$.

Conversely, if $(\gamma, y)\to T(\gamma, y)$ is a bounded measurable operator field satisfying

$$T(\gamma, y)v_\sigma((\gamma, y), (k, n, g)) = v_{\sigma'}((\gamma, y), (k, n, g))T((\gamma, y)\cdot(k, n, g))$$

$\mu\times\nu$-a.e. for all $(k, n, g)\in K\times\mathbb{Z}\times G$, then, taking $g=e$ and using the ergodicity of $K\times\mathbb{Z}$, we see $T(\gamma, y)$ is a constant T $\mu\times v$-a.e. Evidently $T\in\mathrm{Hom}_G(\sigma, \sigma')$ as required. □

COROLLARY 2. There is a functor w from $\underline{\mathrm{Rep}}\, G$ onto a full subcategory of $\underline{Z}^1_\mu(\Gamma\times K, U)$ which is a linear isomorphism on $\overline{\mathrm{Hom}}_G(\sigma, \sigma')$ for all $\sigma, \sigma'\in\mathrm{Rep}\, G$.

Proof. By [**4**], the actions of K on (Γ, μ) and of $K\times\mathbb{Z}\times G$ on $(\Gamma\times Y, \mu\times\nu)$ are orbit equivalent, so the resulting measured groupoids are isomorphic [**3**]. Since the cohomology categories involved depend only on the underlying groupoid, these are equivalent via a functor f which is a linear isomorphism on the relevant sets of morphisms; the desired functor is now given by $w=f\circ v$, where v is as constructed in proposition 1. □

Note also that for $\sigma\in\mathrm{Rep}\, G$, w provides a (normal) algebra isomorphism between the von Neumann algebras $\mathrm{Hom}_G(\sigma, \sigma)=\sigma(G)'$ and $\mathrm{Hom}_\mu((w_\sigma, w_\sigma)$.

3. *Representations of some semi-direct products*

Let N and K be countable groups with N abelian and K amenable, and let $\alpha: K\to\mathrm{Aut}\, N$ be a homomorphism such that the dual action

$$\hat{\alpha}: k\in K\to\mathrm{Aut}\,\hat{N}, \qquad \hat{\alpha}_k(\gamma)=\gamma\circ\alpha_k^{-1}$$

for $\gamma\in\Gamma=\hat{N}$ and $k\in K$, is free, i.e. $\hat{\alpha}_k(\gamma)=\gamma$ for some $\gamma\neq 1$ implies $k=e$—one can take $N=\prod_{-\infty}^{\infty}\mathbb{Z}_2$ and $K=\mathbb{Z}$ acting by translation. Let $H=N\times_\alpha K$, the semi-direct product, with product $(n, k)(n', k')=(n\alpha_k(n'), kk')$. We write $\gamma\cdot k=\hat{\alpha}_k^{-1}(\gamma)$ for convenience.

Let $\{\pi, \mathscr{H}\}$ be a unitary representation of H on a separable space $\mathscr{H}$, and let

$$\{\mathscr{H}, \pi|_N\} = \int_\Gamma^\oplus \{\mathscr{H}_\gamma, \gamma 1_{\mathscr{H}_\gamma}\}\, d\mu_\pi(\gamma)$$

be the central decomposition of the restriction of π to N. It is well known that μ_π is quasi-invariant for the dual action of K on Γ, and ergodic whenever π is a factor representation. In addition, if $u_\pi(k)=\pi(0, k)$, there are, see [**5**], Borel fields of unitaries $u_\pi(\gamma, k):\mathscr{H}_{\gamma\cdot k}\to\mathscr{H}_\gamma$ with

(1) $u_\pi(k)\xi\sim(\gamma\to\rho(\gamma, k)u_\pi(\gamma, k)\xi(\gamma\cdot k))$ where $\xi\in\mathscr{H}$, $\xi\sim\xi(\gamma)$ in the

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decomposition and

$$\rho(\gamma, k) = \left(\frac{d\mu_\pi(\cdot k)}{d\mu_\pi}(\gamma)\right)^{\frac{1}{2}};$$

(2) $u_\pi(\gamma, k)u_\pi(\gamma \cdot k, k') = u_\pi(\gamma, kk')$ μ_π-a.e. in γ for each $k, k' \in K$;

(3) $\operatorname{Ad} u_\pi(\gamma, k) \circ \pi_{\gamma\cdot k} = \pi_\gamma$ μ_π-a.e. in γ for each $k \in K$, where $\pi_\gamma = \gamma 1_{\mathcal{H}_\gamma}$

Thus $(\gamma, k) \to u_\pi(\gamma, k)$ defines an element $u_\pi \in Z^1_{\mu_\pi}(\Gamma \times K, U)$. Conversely each pair (μ, u), where μ is a quasi-invariant measure for K on Γ and $u \in Z^1_\mu(\Gamma \times K, U)$, defines a representation of H.

Proposition 3. Let $\pi, \pi' \in \operatorname{Fac}(H)$. Then if $\operatorname{Hom}_H(\pi, \pi') \neq \{0\}$, μ_π is equivalent to $\mu_{\pi'}$ and there is a natural linear bijection between $\operatorname{Hom}_H(\pi, \pi')$ and $\operatorname{Hom}_{\mu_\pi}(u_\pi, u_{\pi'})$. Conversely if μ is quasi-invariant for K on Γ and $u, u' \in Z^1_\mu(\Gamma \times K, U)$ there is a linear bijection between $\operatorname{Hom}_\mu(u, u')$ and $\operatorname{Hom}_H(\pi, \pi')$, where π, π' are the representations determined by (μ, u) and (μ, u') respectively.

Proof. Let $\pi, \pi' \in \operatorname{Fac}(H)$ with $\operatorname{Hom}_H(\pi, \pi') \neq \{0\}$. From [**1**, 5.2] we may suppose π is a subrepresentation of π' and hence that μ_π is dominated (in the sense of absolute continuity) by $\mu_{\pi'}$. Since both these measures are ergodic for K we conclude their equivalence.

If $T \in \operatorname{Hom}_H(\pi, \pi')$ and $T = \int_\Gamma^\oplus T_\gamma \, d\mu_\pi(\gamma)$ with $T_\gamma : \mathcal{H}_\gamma \to \mathcal{H}'_\gamma$ in the obvious notation, the map $\gamma \to T_\gamma$ provides the desired element of $\operatorname{Hom}_{\mu_\pi}(u_\pi, u_{\pi'})$. The final assertion is routine. □

The restriction to factor representations in the first part of the proposition may be removed if one permits an 'intertwining' relation between cocycles $u_\pi \in Z^1_{\mu_\pi}(\Gamma \times K, U)$ and $u_{\pi'} \in Z^1_{\mu_{\pi'}}(\Gamma \times K, U)$ for measures μ_π and $\mu_{\pi'}$ which are not necessarily equivalent but which dominate a common measure μ; we leave the details to the interested reader.

4. *Comparison of representation theories*

Theorem 4. Let G be a countable amenable group, and let $H = N \times_\alpha K$ be as in § 3, with the dual action of K on $\Gamma = \hat{N}$ being not smooth. Then there is a functor $F : \underline{\operatorname{Rep}}(G) \to \underline{\operatorname{Rep}}(H)$ which is onto a full subcategory and which gives a linear isomorphism of $\operatorname{Hom}_G(\sigma, \sigma')$ with $\operatorname{Hom}_H(F(\sigma), F(\sigma'))$ for all $\sigma, \sigma' \in \operatorname{Rep} G$.

Proof. Choose a measure μ on Γ which is quasi-invariant and ergodic under K, and not of type I, [**2**]. For $\sigma \in \operatorname{Rep}(G)$, let $w_\sigma \in Z^1_\mu(\Gamma \times K, U)$ be as provided by corollary 2, and let $F(\sigma)$ be the representation of H determined by the pair (μ, w_σ). By corollary 2 and proposition 3, F determines a functor with the desired properties. □

Corollary 5. Let G be a solvable locally compact second countable group, and let H be as in theorem 4. Then the conclusions of theorem 4 hold for G and H.

Proof. G has a dense amenable countable subgroup G_0; if $F_0: \underline{\mathrm{Rep}}\,(G_0) \to \underline{\mathrm{Rep}}\,(H)$ is as provided by theorem 4,

$$F(\sigma) = F_0(\sigma|_{G_0}) \qquad \text{for } \sigma \in \mathrm{Rep}\, G$$

has the desired properties. □

Remarks. (1) Theorem 4 characterizes countable amenable groups among discrete groups in the sense that if G is a countable group and $F: \underline{\mathrm{Rep}}\,(G) \to \underline{\mathrm{Rep}}\,(H)$ is a functor as in theorem 4, the commutant of the left regular representation λ^G of G is ismorphic to $F(\lambda^G)'$ and hence hyperfinite; thus G is amenable.

(2) The restriction that K acts on Γ in such a way that $\gamma \cdot k = \gamma$ for some $\gamma \neq 1$ implies $k = e$ is stronger than necessary and has been assumed for ease of exposition only – all that is necessary is that Γ carries a measure μ quasi-invariant and ergodic for K and such that the action of K on (Γ, μ) is free in the measure-theoretic sense.

(3) The functor F constructed in theorem 4 has many pleasant properties – indeed if $\underline{\mathrm{Rep}}\,(G)$, $\underline{\mathrm{Rep}}\,(H)$ are given their usual standard Borel structures, the map

$$\sigma \in \underline{\mathrm{Rep}}\,(G) \to F(\sigma) \in \underline{\mathrm{Rep}}\,(H)$$

is Borel. The crux of the matter is an examination of the proof of Krieger's Theorem [**4**] and verification that all choices made be made in a Borel manner. We omit the gory details. In addition one may verify by routine arguments that F preserves direct integrals of representations.

REFERENCES

[**1**] J. Dixmier. *Les C^*-algèbres et leurs représentations.* (2^e ed.). Gauthier-Villars: Paris, 1969.

[**2**] E. Effros. Transformation groups and C^*-algebras. *Ann. Math.* **81** (1965), 38–55.

[**3**] J. Feldman & C. C. Moore. Ergodic equivalence relations, von Neumann algebras, and cohomology, I. *Trans. Amer. Math. Soc.* **234** (1977), 289–324.

[**4**] W. Krieger. On ergodic flows and the isomorphism of factors. *Math. Ann.* **223** (1976), 19–70.

[**5**] C. Sutherland. Cohomology and extensions of von Neumann algebras, II. *Publ. R.I.M.S., Kyoto,* **16,** No 1, (1980), 135–174.