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REAL HYPERSURFACES WITH CYCLIC-PARALLEL STRUCTURE JACOBI OPERATORS IN A NONFLAT COMPLEX SPACE FORM

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Abstract

It is known that there are no real hypersurfaces with parallel structure Jacobi operators in a nonflat complex space form. In this paper, we classify real hypersurfaces in a nonflat complex space form whose structure Jacobi operator is cyclic-parallel.

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1. Introduction

A complex n-dimensional Kähler manifold with Kähler structure J of constant holomorphic sectional curvature $4c$ is called a complex space form and denoted by $M_n(c)$. As is well known, a connected complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}$ or a complex hyperbolic space $H_n\mathbb{C}$ if $c > 0$, $c = 0$ or $c < 0$, respectively.

The study of real hypersurfaces in complex projective space $P_n\mathbb{C}$ was initiated by Takagi [14], who proved that all homogeneous real hypersurfaces in $P_n\mathbb{C}$ could be divided into six types which are said to be of type A_1, A_2, B, C, D and E. He showed also in [15] and [16] that if a real hypersurface M in $P_n\mathbb{C}$ has two or three distinct constant principal curvatures, then M is locally congruent to one of the homogeneous ones of type A_1, A_2 or B. In particular, real hypersurfaces of type A_1, A_2 and B in $P_n\mathbb{C}$ have been studied by several authors (see Cecil and Ryan [3], Maeda [8] and Okumura [11]).

In the case of complex hyperbolic space $H_n\mathbb{C}$, Montiel and Romero started the study of real hypersurfaces in [9] and constructed some homogeneous real hypersurfaces in $H_n\mathbb{C}$ which are said to be of type A_0, A_1 and A_2. Those hypersurfaces have a lot of nice geometric properties (see Berndt [1] and Montiel and Romero [9]). In 2007 Berndt and Tamaru [2] classified all homogeneous real hypersurfaces in $H_n\mathbb{C}$.

Real hypersurfaces of each type in $M_n(c)$ have been described in detail by Niebergall and Ryan [10].

On the other hand, the Jacobi operator field with respect to X in a Riemannian manifold M is defined by $R_X = R(\cdot, X)X$, where R denotes the Riemannian curvature tensor of M. Let M be a real hypersurface in $M_n(c)$, $c \neq 0$, and ν a unit normal vector field on M. Then a tangent vector field $\xi := -J\nu$ to M is called the *structure vector field* on M. We will call the Jacobi operator on M with respect to ξ the *structure Jacobi operator* on M. The structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ has a fundamental role in contact geometry. Cho and the first author started the study of real hypersurfaces in a complex space form by using the operator R_ξ (see [4]). Recently Ortega *et al.* [12] proved the nonexistence of real hypersurfaces in nonflat complex space forms with parallel structure Jacobi operator. More generally, such a result has been extended by [13] due to them.

Now let M be a real hypersurfaces in a complex space form $M_n(c)$, $c \neq 0$. The structure Jacobi operator R_ξ of M is said to be *cyclic-parallel* if it satisfies

$$\mathfrak{S} R'_\xi(X, Y, Z) = \mathfrak{S} g(\nabla_X R_\xi(Y), Z) = 0$$

for any vector fields X, Y and Z, where $\mathfrak{S}$ and ∇ denote the cyclic sum and the Riemannian connection, respectively. In Section 5, the structure Jacobi operator of real hypersurfaces in $P_n\mathbb{C}$ of type A_1, A_2 and a special case of type B, and $H_n\mathbb{C}$ of type A_0, A_1 and A_2 are cyclic-parallel. The purpose of this paper is to investigate this converse problem.

THEOREM 1.1. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If the structure Jacobi operator is cyclic-parallel, then M is locally congruent to one of the following.*

(I) *In the case where $M_n(c) = P_n\mathbb{C}$:*

- (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/\sqrt{4c}$;*
- (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$, where $0 < r < \pi/\sqrt{4c}$;*
- (B) *a tube of radius r over complex quadric Q_{n-1}, where $\cot r = (\sqrt{2c+4} + \sqrt{2c})/2$.*

(II) *In the case where $M_n(c) = H_n\mathbb{C}$:*

- (A_0) *a horosphere;*
- (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$;*
- (A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, \ldots, n-2\}$.*

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

2. Preliminaries

We denote by $M_n(c)$, $c \neq 0$, a nonflat complex space form with the Fubini–Study metric $\tilde{g}$ of constant holomorphic sectional curvature $4c$ and Levi-Civita connection $\tilde{\nabla}$.

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For an immersed $(2n-1)$-dimensional Riemannian manifold $\tau : M \to M_n(c)$, the Levi-Civita connection ∇ of induced metric and the shape operator H of the immersion are characterized by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(HX, Y)\nu, \quad \tilde{\nabla}_X \nu = -HX,$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde{g}$ and ν a unit normal vector on M. In the sequel the indices $i, j, k, l, \ldots$ run over the range $\{1, 2, \ldots, 2n-1\}$ unless otherwise stated. For a local orthonormal frame field $\{e_i\}$ of M, we denote the dual 1-forms by $\{\theta_i\}$. Then the connection forms θ_{ij} are defined by

$$d\theta_i + \sum_j \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0.$$

Then

$$\nabla_{e_i} e_j = \sum_k \theta_{kj}(e_i) e_k = \sum_k \Gamma_{kij} e_k,$$

where we put $\theta_{ij} = \sum_k \Gamma_{ijk} \theta_k$. The almost contact metric structure $(\phi = (\phi_{ij}), \xi = \sum_i \xi_i e_i)$ is induced on M by the equation

$$J(e_i) = \sum_j \phi_{ji} e_j + \xi_i \nu.$$

Then the structure tensor ϕ and the structure vector ξ satisfy

$$\begin{gathered}\sum_k \phi_{ik}\phi_{kj} = \xi_i \xi_j - \delta_{ij}, \quad \sum_j \xi_j \phi_{ij} = 0, \quad \sum_i \xi_i^2 = 1, \quad \phi_{ij} + \phi_{ji} = 0, \\ d\phi_{ij} = \sum_k (\phi_{ik}\theta_{kj} - \phi_{jk}\theta_{ki} - \xi_i h_{jk}\theta_k + \xi_j h_{ik}\theta_k), \\ d\xi_i = \sum_j \xi_j \theta_{ji} - \sum_{j,k} \phi_{ji} h_{jk} \theta_k.\end{gathered} \tag{2.1}$$

We denote the components of the shape operator or the second fundamental tensor H of M by h_{ij}. The components $h_{ij;k}$ of the covariant derivative of H are given by $\sum_k h_{ij;k}\theta_k = dh_{ij} - \sum_k h_{ik}\theta_{kj} - \sum_k h_{jk}\theta_{ki}$. Then we have the equations of Gauss and Codazzi,

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + 2\phi_{ij}\phi_{kl}) + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.2}$$

$$h_{ij;k} - h_{ik;j} = c(\xi_k \phi_{ij} - \xi_j \phi_{ik} + 2\xi_i \phi_{kj}), \tag{2.3}$$

respectively.

From (2.2) the structure Jacobi operator $R_\xi = (\Xi_{ij})$ is given by

$$\Xi_{ij} = \sum_{k,l} h_{ik}h_{jl}\xi_k\xi_l - \sum_{k,l} h_{ij}h_{kl}\xi_k\xi_l + c\xi_i\xi_j - c\delta_{ij}. \tag{2.4}$$

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First, we give the following lemma.

LEMMA 2.1. *Let U be an open set in M and F a smooth function on U. We put $dF = \sum_i F_i \theta_i$. Then*

$$F_{ij} - F_{ji} = \sum_k F_k \Gamma_{kij} - \sum_k F_k \Gamma_{kji}.$$

PROOF. Taking the exterior derivative of $dF = \sum_i F_i \theta_i$ immediately gives the formula . □

REMARK 2.2. We have already proved [5, Lemma 2.1].

Now we take a local orthonormal frame field e_i in such a way that:

(1) $e_1 = \xi$;
(2) e_2 is in the direction of $\sum_{i=2}^{2n-1} h_{1i} e_i$; and
(3) $e_3 = \phi e_2$.

Then

$$\xi_1 = 1, \quad \xi_i = 0 \ (i \geq 2), \quad h_{1j} = 0 \ (j \geq 3) \quad \text{and} \quad \phi_{32} = 1. \tag{2.5}$$

We put $\alpha := h_{11}$, $\beta := h_{12}$, $\gamma := h_{22}$, $\varepsilon := h_{23}$ and $\delta := h_{33}$.

Hereafter the indices $p, q, r, s, \ldots$ run over the range $\{4, 5, \ldots, 2n-1\}$ unless otherwise stated.

Since $d\xi_i = 0$,

$$\begin{aligned} \theta_{12} &= \varepsilon\theta_2 + \delta\theta_3 + \sum_p h_{3p}\theta_p, \\ \theta_{13} &= -\beta\theta_1 - \gamma\theta_2 - \varepsilon\theta_3 - \sum_p h_{2p}\theta_p, \\ \theta_{1p} &= \sum_q \phi_{qp} h_{q2}\theta_2 + \sum_q \phi_{qp} h_{q3}\theta_3 + \sum_{q,r} \phi_{qp} h_{qr}\theta_r. \end{aligned} \tag{2.6}$$

We put

$$\theta_{23} = \sum_i X_i\theta_i, \quad \theta_{2p} = \sum_i Y_{pi}\theta_i, \quad \theta_{3p} = \sum_i Z_{pi}\theta_i. \tag{2.7}$$

Then it follows from $d\phi_{2i} = 0$ that $Y_{pi} = -\sum_q \phi_{pq} Z_{qi}$ or $Z_{pi} = \sum_q \phi_{pq} Y_{qi}$. Equation (2.4) is rewritten as

$$\Xi_{ij} = -\alpha h_{ij} + h_{1i}h_{1j} + c\delta_{i1}\delta_{j1} - c\delta_{ij}. \tag{2.8}$$

Some fundamental properties about the structure vector and the principal curvature are stated for later use.

PROPOSITION 2.3 (Meada [8], Ki and Suh [6]). *Let M be a real hypersurface in $M_n(c)$, $c \neq 0$. If the structure vector ξ is principal, then the corresponding principal curvature α is locally constant.*

PROPOSITION 2.4 (Niebergall and Ryan [10]). *Let M be a real hypersurface in $M_n(c)$, $c \neq 0$. Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $HX = rX$ for $X \perp \xi$, then $(2r - \alpha)H\phi X = (\alpha r + 2c)\phi X$.*

3. Real hypersurfaces with cyclic-parallel structure Jacobi operator

First we assume that $\mathfrak{S}R'_\xi(X, Y, Z) = 0$ for any vector fields X, Y and Z. The components $\Xi_{ij;k}$ of the covariant derivative of $R_\xi = (\Xi_{ij})$ are given by

$$\sum_k \Xi_{ij;k}\theta_k = d\Xi_{ij} - \sum_k \Xi_{kj}\theta_{ki} - \sum_k \Xi_{ik}\theta_{kj}.$$

Substituting (2.8) into the above equation,

$$\begin{aligned}\sum_k \Xi_{ij;k}\theta_k = &-(d\alpha)h_{ij} - \alpha dh_{ij} + (dh_{1i})h_{1j} + h_{1i}(dh_{1j})\\ &+ \alpha\sum_k h_{kj}\theta_{ki} - \alpha h_{1j}\theta_{1i} - \beta h_{1j}\theta_{2i} - c\delta_{j1}\theta_{1i}\\ &+ \alpha\sum_k h_{ik}\theta_{kj} - \alpha h_{1i}\theta_{1j} - \beta h_{1i}\theta_{2j} - c\delta_{i1}\theta_{1j}.\end{aligned} \tag{3.1}$$

Our assumption $\mathfrak{S}R'_\xi(X, Y, Z) = 0$ for any vector fields X, Y and Z is equivalent to $\Xi_{ij;k} + \Xi_{jk;i} + \Xi_{ki;j} = 0$. This equation is rewritten as

$$\begin{aligned}&\alpha_k h_{ij} + \alpha_i h_{jk} + \alpha_j h_{ki} + \alpha(h_{ijk} + h_{jki} + h_{kij})\\ &\quad - h_{1j}h_{1ik} - h_{1k}h_{1ji} - h_{1i}h_{1kj} - h_{1j}h_{1ki} - h_{1k}h_{1ij} - h_{1i}h_{1jk}\\ &\quad + \alpha h_{1j}(\Gamma_{1ik} + \Gamma_{1ki}) + \alpha h_{1k}(\Gamma_{1ji} + \Gamma_{1ij}) + \alpha h_{1i}(\Gamma_{1kj} + \Gamma_{1jk})\\ &\quad + \beta h_{1j}(\Gamma_{2ik} + \Gamma_{2ki}) + \beta h_{1k}(\Gamma_{2ji} + \Gamma_{2ij}) + \beta h_{1i}(\Gamma_{2kj} + \Gamma_{2jk})\\ &\quad + c\delta_{1j}(\Gamma_{1ik} + \Gamma_{1ki}) + c\delta_{1k}(\Gamma_{1ji} + \Gamma_{1ij}) + c\delta_{1i}(\Gamma_{1kj} + \Gamma_{1jk})\\ &\quad - \alpha\sum_l h_{lj}(\Gamma_{lik} + \Gamma_{lkj}) - \alpha\sum_l h_{lk}(\Gamma_{lji} + \Gamma_{lij})\\ &\quad - \alpha\sum_l h_{li}(\Gamma_{lkj} + \Gamma_{ljk}) = 0,\end{aligned} \tag{3.2}$$

because of (3.1).

In the following we assume that $\beta \neq 0$. Let $i, j, k \in \{1, 2, 3, p, q\}$. Then equation (3.2) can be stated as follows:

$$\varepsilon = 0, \quad h_{3p} = 0, \tag{3.3}$$

$$\alpha\delta + c = 0, \tag{3.4}$$

$$(\beta^2 - \alpha\gamma)_1 - 2\alpha\sum_p h_{2p}Y_{p1} = 0, \tag{3.5}$$

$$(\alpha\gamma)_3 + 2(\beta^2 - \alpha\gamma - c)X_2 + 2\alpha\sum_p h_{2p}(Y_{p3} + Z_{p2}) = 0, \tag{3.6}$$

$$(\beta^2 - \alpha\gamma - c)(X_1 - \delta) + \alpha\sum_p h_{2p}Z_{p1} = 0, \tag{3.7}$$

$$\begin{aligned}&(\beta^2 - \alpha\gamma)_p - 2(\alpha h_{2p})_2 - 2(\beta^2 - \alpha\gamma)Y_{p2}\\ &\quad + 2\alpha\sum_q h_{2q}(\Gamma_{qp2} - Y_{qp}) = 0,\end{aligned} \tag{3.8}$$

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$$(\alpha h_{2p})_1 + \alpha \sum_q h_{pq} Y_{q1} + (\beta^2 - \alpha\gamma) Y_{p1} - \alpha \sum_q h_{2q} \Gamma_{qp1} + \alpha \sum_{q,r} h_{2q} \phi_{rq} h_{rp} = 0, \tag{3.9}$$

$$\alpha h_{2p}(X_1 - \delta) - \sum_q (\alpha h_{qp} + c\delta_{pq}) Z_{q1} = 0, \tag{3.10}$$

$$\delta Z_{p3} + h_{2p} X_3 = 0, \tag{3.11}$$

$$(\beta^2 - \alpha\gamma - c) X_3 + \alpha \sum_p h_{2p} Z_{p3} = 0, \tag{3.12}$$

$$(\alpha h_{2p})_3 - (\beta^2 - \alpha\gamma)(X_p + Y_{p3}) + \alpha\delta(Z_{p2} - X_p) + \alpha h_{2p} X_2 + \alpha \sum_r h_{2r}(\Gamma_{rp3} - Z_{rp}) = 0, \tag{3.13}$$

$$(\alpha h_{pq})_1 - \alpha h_{2q} Y_{p1} - \alpha \sum_r h_{rq} \Gamma_{rp1} - \alpha h_{2p} Y_{q1} - \alpha \sum_r h_{pr} \Gamma_{rq1} + c(\Gamma_{1qp} - \Gamma_{1pq}) = 0. \tag{3.14}$$

Henceforth we shall use (3.3) without further mention.

Properly speaking, we should denote equation (2.3) by $(23)_{ijk}$, for example. In this paper we denote it simply by (ijk). Then we have the following equations (112)–$(q3p)$.

(112) $\alpha_2 - \beta_1 = 0,$

(212) $\beta_2 - \gamma_1 - 2 \sum_p h_{2p} Y_{p1} = 0,$

(312) $(\alpha - \delta)\gamma - \beta X_2 + (\gamma - \delta) X_1 - \beta^2 - \sum_p h_{2p} Z_{p1} = -c,$

(113) $\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0,$

(213) $\beta_3 - \alpha\delta + \gamma\delta + (\gamma - \delta) X_1 - \beta^2 - \sum_p h_{2p} Z_{p1} = c,$

(313) $\beta X_3 + \delta_1 = 0,$

(223) $\gamma_3 - 2\beta\delta + 2 \sum_p h_{2p} Y_{p3} + (\gamma - \delta) X_2 - \beta\gamma - \sum_p h_{2p} Z_{p2} = 0,$

(323) $\sum_p h_{2p} Z_{p3} - \delta_2 - (\gamma - \delta) X_3 = 0,$

$(1p1)$ $\alpha_p + \beta Y_{p1} = 0,$

$(12p)$ $\beta_p + 2 \sum_{q,r} h_{2q} \phi_{rq} h_{rp} + \beta Y_{p2} + \alpha \sum_q \phi_{qp} h_{2q} = 0,$

$(13p)$ $-2\delta h_{2p} + \beta Y_{p3} + \alpha h_{2p} - \beta X_p = 0,$

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$$(22p)\quad \gamma_p + 2\sum_q h_{2q}Y_{qp} - h_{2p2} - \sum_q h_{qp}Y_{q2} + \beta\sum_q \phi_{qp}h_{2q} + \gamma Y_{p2} + \sum_q h_{2q}\Gamma_{qp2} = 0,$$

$$(23p)\quad \delta X_p + \beta h_{2p} - \gamma X_p + \sum_q h_{2q}Z_{qp} - h_{2p3} - \sum_q h_{qp}Y_{q3} + \gamma Y_{p3} + \sum_q h_{2q}\Gamma_{qp3} = 0,$$

$$(33p)\quad \delta_p + h_{2p}X_3 - \sum_q h_{qp}Z_{q3} + \delta Z_{p3} = 0,$$

$$(21p)\quad \beta_p + \sum_{q,r} h_{2q}\phi_{rq}h_{rp} - h_{2p1} - \sum_q h_{qp}Y_{q1} + \gamma Y_{p1} + \sum_q h_{2q}\Gamma_{qp1} = 0,$$

$$(31p)\quad -\delta h_{2p} + \alpha h_{2p} - \beta X_p + h_{2p}X_1 - \sum_q h_{qp}Z_{q1} + \delta Z_{p1} = 0,$$

$$(32p)\quad \delta X_p + \beta h_{2p} - \gamma X_p + \sum_q h_{2q}Z_{qp} + h_{2p}X_2 - \sum_q h_{pq}Z_{q2} + \delta Z_{p2} = 0,$$

$$(2pq)\quad h_{2pq} + \sum_r h_{rp}Y_{rq} - \beta\sum_r \phi_{rp}h_{rq} - \gamma Y_{pq} - \sum_r h_{2r}\Gamma_{rpq} - h_{2qp} - \sum_r h_{rq}Y_{rp} + \beta\sum_r \phi_{rq}h_{rp} + \gamma Y_{qp} + \sum_r h_{2r}\Gamma_{rqp} = 0,$$

$$(q1p)\quad \sum_{r,s} h_{rq}\phi_{sr}h_{sp} - \alpha\sum_r \phi_{rq}h_{rq} - \beta Y_{qp} - h_{pq1} + h_{2q}Y_{p1} + \sum_r h_{rq}\Gamma_{rp1} + h_{2p}Y_{q1} + \sum_r h_{rp}\Gamma_{rq1} = c\phi_{pq},$$

$$(q3p)\quad -\delta Z_{qp} - h_{2q}X_p + \sum_r h_{qr}Z_{rp} - h_{qp3} + h_{q2}Y_{p3} + \sum_r h_{qr}\Gamma_{rp3} + h_{p2}Y_{q3} + \sum_r h_{pr}\Gamma_{rq3} = 0.$$

REMARK 3.1. We have omitted equations $(1pq)$, $(3pq)$, $(p2q)$ and (pqr) since we do not need them.

4. Key lemma

Suppose that $\beta \neq 0$. From (3.10) and $(31p)$,

$$\alpha h_{2p} = \beta X_p. \tag{4.1}$$

This and $(13p)$ imply that

$$2\delta h_{2p} = \beta Y_{p3}, \tag{4.2}$$

and so

$$\sum_p h_{2p}Z_{p3} = 0. \tag{4.3}$$

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LEMMA 4.1. *$H(e_2) \in \mathrm{span}\{e_1, e_2\}$.*

PROOF. It follows from (3.4), (3.11) and (4.3) that

$$\sum_p (h_{2p})^2 X_3 = 0.$$

If $X_3 \neq 0$, then obviously $h_{2p} = 0$. Then $X_3 = 0$, which together with (3.4), (3.11), (4.2) and $Y_{pi} = -\sum_q \phi_{pq} Z_{qi}$ implies that $h_{2p} = 0$. □

LEMMA 4.2. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If the structure Jacobi operator is cyclic-parallel, then the structure vector is principal.*

PROOF. From Lemma 4.1 the matrix (h_{pq}) is diagonalizable; that is, for a suitable choice of a orthonormal frame field $\{e_p\}$ we can set

$$h_{pq} = \lambda_p \delta_{pq}.$$

Here we shall set out some equations obtained from Lemma 4.1.

From (4.1), (4.2) and (3.13),

$$X_p = Y_{p3} = Z_{p3} = Y_{p2} = Z_{p2} = 0. \tag{4.4}$$

Equations (3.5), (3.6), (3.7) and (3.12) imply that

$$(\beta^2 - \alpha\gamma)_1 = 0, \tag{4.5}$$
$$(\alpha\gamma)_3 + (\beta^2 - \alpha\gamma - c)X_2 = 0, \tag{4.6}$$
$$(\beta^2 - \alpha\gamma - c)(X_1 - \delta) = 0, \tag{4.7}$$
$$(\beta^2 - \alpha\gamma - c)X_3 = 0. \tag{4.8}$$

Put $p = q$ in (3.14). Then

$$(\alpha\lambda_p)_1 = 0. \tag{4.9}$$

Moreover, from (112)–(32p),

$$\alpha_2 - \beta_1 = 0, \tag{4.10}$$
$$\beta_2 - \gamma_1 = 0, \tag{4.11}$$
$$(\alpha - \delta)\gamma - \beta X_2 + (\gamma - \delta)X_1 - \beta^2 + c = 0, \tag{4.12}$$
$$\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0, \tag{4.13}$$
$$\beta_3 + \gamma\delta + (\gamma - \delta)X_1 - \beta^2 = 0, \tag{4.14}$$
$$\delta_1 + \beta X_3 = 0, \tag{4.15}$$
$$\gamma_3 - 2\beta\delta + (\gamma - \delta)X_2 - \beta\gamma = 0, \tag{4.16}$$
$$\delta_2 + (\gamma - \delta)X_3 = 0, \tag{4.17}$$
$$\beta_p = 0, \tag{4.18}$$
$$\gamma_p = 0, \tag{4.19}$$

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$$\alpha_p = \delta_p = 0, \tag{4.20}$$
$$Y_{p1} = Z_{p1} = 0. \tag{4.21}$$

It follows from $(q1p)$ and (3.14) that

$$\alpha\beta Y_{qp} = (\alpha\lambda_p\lambda_q - \alpha^2\lambda_p + c\lambda_p - c\lambda_q - c\alpha)\phi_{pq} + \alpha_1\lambda_p\delta_{pq}. \tag{4.22}$$

From this, $(2pq)$ and $(q3p)$,

$$\begin{aligned}[\alpha\beta^2(\lambda_p + \lambda_q) - (\lambda_p - \gamma)\{\alpha\lambda_p\lambda_q - \alpha^2\lambda_q - c\alpha + c(\lambda_p - \lambda_q)\}\\ - (\lambda_q - \gamma)\{\alpha\lambda_p\lambda_q - \alpha^2\lambda_p - c\alpha + c(\lambda_q - \lambda_p)\}]\phi_{pq} = 0,\end{aligned} \tag{4.23}$$
$$\begin{aligned}(\lambda_q - \delta)[\alpha\{(\lambda_q)^2 - \alpha\lambda_q - c\}\delta_{pq} + \alpha_1\lambda_q\phi_{pq}]\\ - \alpha\beta\{h_{qp3} + (\lambda_p - \lambda_q)\Gamma_{qp3}\} = 0.\end{aligned} \tag{4.24}$$

If $p = q$ in the above equation, then

$$(\lambda_p - \delta)\{(\lambda_p)^2 - \alpha\lambda_p - c\} - \beta(\lambda_p)_3 = 0. \tag{4.25}$$

In the following we shall abbreviate the expression 'take account of the coefficient of θ_i in the exterior derivative of . . .' to 'see θ_i of d of . . .'.

Case I. Suppose that $\beta^2 - \alpha\gamma - c \neq 0$. From (4.7) and (4.8),

$$X_3 = 0, \quad X_1 = \delta. \tag{4.26}$$

It follows from (4.10), (4.15), (4.17), (4.5) and (4.11) that

$$\alpha_1 = \delta_1 = \alpha_2 = \delta_2 = \beta_1 = \beta_2 = \gamma_1 = 0. \tag{4.27}$$

From (4.12), (4.13), (4.14) and (4.26),

$$\beta X_2 + (\beta^2 - \alpha\gamma - c) + \delta^2 = 0, \tag{4.28}$$
$$\alpha_3 + 4\beta\delta - \alpha\beta = 0, \tag{4.29}$$
$$\beta_3 - \beta^2 + 2\gamma\delta - \delta^2 = 0. \tag{4.30}$$

Seeing $\theta_1 \wedge \theta_3$ of d of θ_{23},

$$\delta_3 = -\beta\delta - 2X_2\delta, \tag{4.31}$$

which, together with (4.28) and (4.29), implies that

$$-2\beta^2\delta + \alpha\delta^2 + \alpha(\beta^2 - \alpha\gamma - c) = 0.$$

Seeing θ_2 of d of the above equation, we have that $\gamma_2 = 0$.

Now put $F = \alpha, \beta, \gamma$, and $i = 1,\ j = 2$ in Lemma 2.1. Then,

$$\alpha_3(\gamma + \delta) = \beta_3(\gamma + \delta) = \gamma_3(\gamma + \delta) = 0.$$

If $\gamma + \delta \neq 0$, then from (4.6) and (4.31) we have a contradiction. Thus $\gamma + \delta = 0$, which also contradicts (4.6) and (4.31).

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Case II-1. Suppose that

$$\alpha_1 = 0, \tag{4.32}$$
$$\beta^2 - \alpha\gamma - c = 0. \tag{4.33}$$

Seeing θ_2 of d of (4.33),

$$(\beta^2 - \alpha\gamma)_3 = 2\beta\beta_3 - \gamma\alpha_3 - \alpha\gamma_3 = 0. \tag{4.34}$$

From (4.12), (4.13, (4.14), (4.16) and (4.33),

$$-\delta\gamma - \beta X_2 + (\gamma - \delta)X_1 = 0, \tag{4.35}$$
$$\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0, \tag{4.36}$$
$$\beta_3 + (\gamma - \delta)X_1 + \gamma\delta - \alpha\gamma - c = 0, \tag{4.37}$$
$$\gamma_3 - 2\beta\delta + (\gamma - \delta)X_2 + \beta\gamma = 0. \tag{4.38}$$

On substituting (4.36), (4.37) and (4.38) into (4.34),

$$(\delta - \gamma)(X_1 - 4\alpha) = 0,$$

by virtue of (4.35). If $\delta = \gamma$, then by (4.33) we have a contradiction and hence

$$X_1 = 4\alpha. \tag{4.39}$$

Substituting this equation into (4.35), (4.36) and (4.37),

$$\beta X_2 = 4\alpha(\gamma - \delta) - \delta\gamma, \tag{4.40}$$
$$\alpha_3 + 3\beta\delta + 3\alpha\beta = 0, \tag{4.41}$$
$$\beta_3 + 3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0. \tag{4.42}$$

It follows from (4.16), (4.33) and (4.40) that

$$\alpha\gamma_3 + \beta(3\alpha\gamma - 6\alpha\delta - \gamma\delta) = 0. \tag{4.43}$$

From (4.15) and (4.32) we have $X_3 = 0$ and therefore $\beta_1 = \alpha_2 = \delta_2 = 0$ because of (4.10) and (4.17). Hence, by (4.5), we have $\gamma_1 = 0$, and so $\beta_2 = 0$. From (3.6) we have $(\alpha\gamma)_3 = 0$. This, together with (4.33), implies that $\beta_3 = 0$. Therefore it follows from (4.18) that β is constant.

Now put $F = \alpha$ and β in Lemma 2.1. Then

$$\alpha_3(\gamma + X_2) = 0, \quad \beta_3(\gamma + X_2) = 0.$$

If $\gamma + X_2 \neq 0$, then $\alpha_3 = \beta_3 = 0$. It follows from (4.20) that α and δ are constant. Furthermore, by (4.33) we see that γ is constant. Thus from (4.41), (4.42) and (4.43),

$$\alpha + \delta = 0,$$
$$3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0,$$
$$3\alpha\gamma - 6\alpha\delta - \gamma\delta = 0.$$

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Hence, by (3.4) and (4.33), $\alpha^2 - c = 0$ and $2\beta^2 + c = 0$, which is a contradiction. Therefore $X_1 = -\gamma$, which, together with (4.39), implies that $\gamma = -X_1 = -4\alpha$. Thus it follows from (4.41) that $\gamma_3 = -4\alpha_3 = 12\beta(\delta + \alpha)$. Hence, from (4.43) we have a contradiction $\alpha\delta = 0$.

Case II-2. Suppose that

$$\beta^2 - \alpha\gamma - c = 0 \quad \text{and} \quad \alpha_1 \neq 0. \tag{4.44}$$

Here we assert that if $\phi_{pq} \neq 0$, then $\lambda_p = \lambda_q$. To prove this, we assume that there exist indices p and q such that

$$\phi_{pq} \neq 0, \quad \lambda_p - \lambda_q \neq 0.$$

Then from (4.23) and (4.44),

$$(\lambda_p\lambda_q - 2c)(\lambda_p + \lambda_q) - 2(\alpha + \gamma - \delta)\lambda_p\lambda_q - 2c\gamma - \delta\{(\lambda_p)^2 + (\lambda_q)^2\} = 0. \tag{4.45}$$

Multiply the above equation by α^3 and see θ_1 of d of this equation. Then from (4.9),

$$(\alpha\gamma)_1(\lambda_p\lambda_q - c) = 2\alpha_1(c\gamma - c\lambda_p - c\lambda_q - \alpha\lambda_p\lambda_q). \tag{4.46}$$

On the other hand, by (4.5), (4.10), (4.15) and (4.17),

$$(\alpha\gamma)_1 = 2(\gamma - \delta)\alpha_1, \tag{4.47}$$

which, together with (4.46) and (4.47), implies that

$$\lambda_p\lambda_q(\alpha + \gamma - \delta) = c(2\gamma - \delta - \lambda_p - \lambda_q).$$

Eliminate $(\alpha + \gamma - \delta)\lambda_p\lambda_q$ from this and (4.45):

$$(\alpha\lambda_p)(\alpha\lambda_q)(\alpha\lambda_p + \alpha\lambda_q) - 2c\alpha^2(\alpha\gamma - \alpha\delta) - (\alpha\delta)\{(\alpha\lambda_p)^2 + (\alpha\lambda_q)^2\} = 0.$$

Multiply this equation by α^3 and see θ_1 of d of this equation. Then by (4.9), $(\alpha\gamma)_1 = -2(\gamma - \delta)\alpha_1$ and so $\gamma = \delta$ by virtue of (4.44) and (4.47). Thus from (3.4) and (4.43) we have contradiction. Therefore, for all p, q such that $\phi_{pq} \neq 0$, we have $\lambda_p = \lambda_q$.

We now take p, q such that $\phi_{pq} \neq 0$. Then, from (4.24) and $\lambda_p = \lambda_q$,

$$\beta^2\lambda_p - (\lambda_p - \gamma)\{(\lambda_p)^2 - \alpha\lambda_p - c\} = 0. \tag{4.48}$$

Furthermore, from $(q3p)$, (4.9) and (4.22),

$$\lambda_p(\lambda_p - \delta) = 0.$$

Note that (4.25) implies that $\lambda_p \neq 0$. Hence we have $\lambda_p = \delta$. From (4.48) we have $(\alpha + \delta)(\delta - \gamma) = 0$. If $\alpha + \delta = 0$, then α and δ are constant, which contradicts (4.44). Hence $\delta - \gamma = 0$. However, from (4.44) we have $\beta = 0$, which is a contradiction.

Consequently we have proved $\beta = 0$, which completes the proof. □

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5. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. Suppose that the structure Jacobi operator is cyclic-parallel. From $d\xi_i = 0$, (2.1) and (2.5),

$$\Gamma_{1ij} = \lambda_j \phi_{ji}. \tag{5.1}$$

If $\alpha = 0$, then (3.2) implies that

$$(\lambda_i - \lambda_j)\phi_{ij} = 0 \quad \text{for } i,\ j \geq 2.$$

Thus H and ϕ are commutative. By the classification theorems of real hypersurfaces in $M_n(c)$, $c \neq 0$, due to Okumura [11] and Montiel and Romero [9], M is locally congruent to one of the real hypersurfaces of type A_1 or A_2 in $P_n\mathbb{C}$ or of type A_0, A_1 or A_2 in $H_n\mathbb{C}$.

Suppose that $\alpha \neq 0$. We first prove that all principal curvatures are constant. From Lemma 4.1 and Proposition 2.3 we see that α is constant in M. We denote equation (3.2) by $(32)_{ijk}$. By (213) and (313),

$$\gamma_1 = \delta_1 = 0. \tag{5.2}$$

It follows from $(32)_{iii}$ and $(32)_{1pp}$ that

$$(\lambda_p)_1 = (\lambda_i)_i = 0 \quad \text{for } i \geq 2. \tag{5.3}$$

We take the indices i and j such that $i \neq 1$, $j \neq 1$ and $i \neq j$. Then eliminating $(\lambda_i - \lambda_j)\Gamma_{iji}$ from (iij) and $(32)_{iij}$, we have

$$(\lambda_i)_j = 0. \tag{5.4}$$

Hence, from (5.2), (5.3) and (5.4) all principal curvatures are constant. By the classification theorems of real hypersurfaces in $M_n(c)$, $c \neq 0$, due to Kimura [7] and Berndt [1], M is locally congruent to one of the homogeneous real hypersurfaces of type $A_1 \sim E$ in $P_n\mathbb{C}$ or of type $A_0 \sim B$ in $H_n\mathbb{C}$. So, we shall check equation (3.2) one by one for the above model spaces.

Here since all principal curvatures are constant, we shall rewrite the condition (3.2). For a suitable choice of a orthonormal frame field $\{e_i\}$ of each model space we set $h_{ij} = \lambda_i \delta_{ij}$. The Codazzi equation (2.3) asserts that

$$(\lambda_i - \lambda_j)\Gamma_{ijk} = (\lambda_k - \lambda_i)\Gamma_{kij} - c(\xi_k \phi_{ij} + 2\xi_i \phi_{kj} - \xi_j \phi_{ik}). \tag{5.5}$$

Therefore the cyclic-parallel structure Jacobi operator condition (3.2) is rewritten as

$$\begin{aligned} 3\alpha(\lambda_i - \lambda_j)\Gamma_{ijk} + 3\alpha c(\xi_j \phi_{ki} - \xi_i \phi_{jk}) - (\alpha^2 + c)\{\delta_{i1}(\lambda_j - \lambda_k)\phi_{jk} \\ + \delta_{j1}(\lambda_k - \lambda_i)\phi_{ki} + \delta_{k1}(\lambda_i - \lambda_j)\phi_{ij}\} = 0. \end{aligned} \tag{5.6}$$

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Case I. $M_n(c) = H_n\mathbb{C}$. Let M be of type A_0. Then M has two distinct constant principal curvatures $\alpha = 2$ and 1. Then it is easy to see that M satisfies (5.6).

Let M be of type A_1 or A_2. Then M has three distinct constant principal curvatures $\sqrt{-c}/t$, $\sqrt{-c}t$ and $\alpha = \sqrt{-c}(t + 1/t)$, where $t = \tanh r$. For $i, j, k \geq 2$ we have (5.6) because the same principal curvatures exist in $\{\lambda_i, \lambda_j, \lambda_k\}$. Therefore, by (5.1) we obtain (5.6). For $j = 1$ the left-hand side of (5.6) can be expressed as

$$(\lambda_i - \lambda_k)(\alpha^2/2 - c)\phi_{ik}. \tag{5.7}$$

Since $2\sqrt{-c}t - \alpha \neq 0$ and $2\sqrt{-c}/t - \alpha \neq 0$, it follows from Proposition 2.4 that $\phi V_{\sqrt{-c}/t} = V_{\sqrt{-c}/t}$ and $\phi V_{\sqrt{-c}t} = V_{\sqrt{-c}t}$, where V_λ denotes the eigenspace of H with eigenvalue λ. Hence we have $(\lambda_i - \lambda_j)\phi_{ij} = 0$ for any i, j. This, together with (5.7), implies (5.6). Thus the manifold M satisfies (5.6).

Let M be of type B. Put $j = 1$ in (5.6). Thus by an argument similar to that above,

$$(\lambda_i - \lambda_k)(\alpha^2 - 2c)\phi_{ik} = 0. \tag{5.8}$$

M has three distinct constant principal curvatures $\sqrt{-c}/t$, $\sqrt{-c}t$ and $\alpha = 4\sqrt{-c}t/(t^2 + 1)$, where $t = \tanh r$. Then, from Proposition 2.4, $\phi V_{1/t} = V_t$ and therefore there exist indices i and j such that $(\lambda_i - \lambda_j)\phi_{ij} \neq 0$. This contradicts (5.8).

Case II. $M_n(c) = P_n\mathbb{C}$. Let M be of type A_1, A_2 or B. By an argument similar to that in Case I, real hypersurfaces of type A_1 or A_2 satisfy the condition (5.6). Moreover if $\alpha^2 = 2c$, then real hypersurfaces of type B satisfy (5.6). The equation $\alpha^2 = 2c$ tells us that $\cot r = (\sqrt{2c + 4} + \sqrt{2c})/2$.

Let M be of type C, D or E. Then M has five distinct constant principal curvatures. Put $j = 1$ in (5.5) to get

$$(\lambda_k - \lambda_i)\Gamma_{ki1} = (\alpha\lambda_k - \lambda_i\lambda_k + c)\phi_{ki},$$

by virtue of (5.1), which implies that

$$2(\lambda_k - \lambda_i)\Gamma_{ki1} = \alpha(\lambda_k - \lambda_i)\phi_{ki}. \tag{5.9}$$

Suppose that $i, j, k \geq 2$ and $\lambda_i \neq \lambda_j$. Then it follows from (5.6) that $\Gamma_{ijk} = 0$ and therefore $\theta_{ij} = \Gamma_{ij1}\theta_1$. Seeing $\theta_i \wedge \theta_j$ of d of θ_{ij},

$$\lambda_i\lambda_j + c + (\phi_{ij})^2\{\lambda_i\lambda_j + (\alpha/2)(\lambda_i + \lambda_j) + 3c\} = 0,$$

because of (5.1) and (5.9). Put $i = 1$ in (5.5). Then

$$\{2\lambda_j\lambda_k - \alpha(\lambda_j + \lambda_k) - 2c\}\phi_{jk} = 0 \quad \text{for } j, k \geq 2.$$

From the above two equations it is easy to see that

$$\lambda_i\lambda_j + c = 0 \quad \text{for } i, j \geq 2 \text{ and } \lambda_i \neq \lambda_j,$$

which implies that M has at most three constant principal curvatures. This is a contradiction.

This completes the proof of Theorem 1.1.

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