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ON THE CARDINALITY OF URYSOHN SPACES

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ABSTRACT. In this paper some cardinal inequalities for Urysohn spaces are established. In particular the following two theorems are proved:

(i) *if* $A \subset X$ *then* $|[A]_\theta| \leqq |A|^{\chi(X)}$, *where* $[A]_\theta$ *denotes the* θ*-closed hull of* A, *i.e., the smallest* θ*-closed subset of* X *containing* A;

(ii) $|X| \leqq 2^{\chi(X)aL(X,X)}$, *where* $aL(X, X)$ *is the smallest cardinal number* m *such that for every open cover* $\mathcal{U}$ *of* X *there is a subfamily* $\mathcal{U}_0 \subset \mathcal{U}$ *for which* $X = \cup_{U \in \mathcal{U}_0} \bar{U}$ *and* $|\mathcal{U}_0| \leqq m$.

Our aim in this paper is to study some cardinality properties of Urysohn spaces. We begin with a theorem that gives an upper bound for the cardinality of a θ-closed set and then we establish some inequalities that improve for non-regular spaces the well known Arkhangel'skii's formula $|X| \leqq 2^{\chi(X)L(X)}$ (see [1] and [5]).

For notations and definitions not explicitly mentioned here we refer to [3] and [4]. m, k will denote cardinal numbers and α, β ordinal numbers. m^+ is the successor cardinal of m and $\alpha + 1$ the successor ordinal of α. All cardinal numbers are assumed to be initial ordinals. For any set S we denote by $\exp_m(S)$ the collection of all subsets of S whose cardinality is at most m and by $|S|$ the cardinality of S. All topological spaces considered here are assumed to be infinite. For any space X and any family Γ of subsets of X we denote by $\bar{\Gamma}$ the family $\{\bar{A} : A \in \Gamma\}$ and we briefly write $\cap\Gamma$ (resp. $\cup\Gamma$) for $\cap_{A \in \Gamma} A$ (resp. $\cup_{A \in \Gamma} A$) $\cdot$ $\chi(X)$, $\psi_c(X)$, $\pi\chi(X)$, $t(X)$ and $L(X)$ denote respectively the character, the closed-pseudocharacter, the π-character, the tightness and the Lindelöf number of a space X.

We recall the following:

DEFINITION 1. (see [6]) *Let* X *be a topological space and* A *a subset of* X. *The* θ*-closure of* A, *denoted by* $\mathrm{cl}_\theta(A)$, *is the set of all points* $x \in X$ *such that every closed neighbourhood of* x *intersects* A. *The* θ*-interior of* A, *denoted by* $\mathrm{int}_\theta(A)$, *is*

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the set $X - \mathrm{cl}_\theta(X - A)$, i.e., $x \in \mathrm{int}_\theta(A)$ *if and only if there is some closed neighbourhood of* x *contained in* A. A *is said to be* θ*-closed if* $A = \mathrm{cl}_\theta(A)$.

Note that the θ-closure operator is not in general idempotent. This suggests that we introduce the following:

DEFINITION 2. *Let* X *be a topological space and* A *a subset of* X. *The* θ*-closed hull of* A, *denoted by* $[A]_\theta$, *is the smallest* θ*-closed subset of* X *containing* A.

It is clear that $[A]_\theta = \cap\{C:A \subset C$ and $\mathrm{cl}_\theta(C) = C\}$. If the space is assumed to be regular we have: $\bar{A} = \mathrm{cl}_\theta(A) = [A]_\theta$, but for more general spaces the gap between the closure and the θ-closure of certain subsets can be very large as the next example shows:

EXAMPLE 1. Let N be the set of natural numbers. We recall (see [3], Theorem 3.6.18) that there exists a family $\{N_t\}_{t \in T}$ of infinite subsets of N such that $|T| = 2^{\aleph_0}$ and $N_t \cap N_{t'}$ is finite for any $t \neq t'$. Let R be the real line and put $\mathcal{I} = \{(2n, 2n + 1):n \in N\}$. Since $|\mathcal{I}| = \aleph_0$ there is a family $\{\mathcal{I}_t\}_{t \in T}$ of subsets of $\mathcal{I}$ such that $|T| = 2^{\aleph_0}$, $|\mathcal{I}_t| = \aleph_0$ and $|\mathcal{I}_t \cap \mathcal{I}_{t'}| < \aleph_0$ for any $t \neq t'$. For any $t \in T$ we write $\mathcal{I}_t = \{i_1^t, i_2^t, \ldots\}$. Let $X = R \cup T$ topologized as follows

(i) for any $x \in R$ a fundamental system of neighbourhoods of x is the family $\{(x - 1/n, x + 1/n):n \in N\}$;

(ii) for any $t \in T$ a fundamental system of neighbourhoods is the family $\{U_t^n\}_{n \in N}$, where $U_t^n = \{t\} \cup \bigcup_{m \geqq n} i_m^t$.

It is easy to see that X is a first countable Urysohn space. The set N is closed in X, but $\mathrm{cl}_\theta(N) = N \cup T$ and hence $|\mathrm{cl}_\theta(N)| = 2^{\aleph_0} > \aleph_0 = |\bar{N}|$.

An upper bound for the θ-closed hull is given in the following.

THEOREM 1. *Let* X *be a Urysohn space. If* A *is a subset of* X *then* $|[A]_\theta| \leqq |A|^{\chi(X)}$.

PROOF. Let $m = \chi(X)$ and $\kappa = |A|$. Let us denote by $\mathcal{U}_x$ a fundamental system of neighbourhoods at a point $x \in X$ such that $|\mathcal{U}_x| \leqq m$. If $x \in \mathrm{cl}_\theta(A)$, for every $U \in \mathcal{U}_x$, let us choose a point in the set $\bar{U} \cap A$ and let A_x be the set of all these points. It is clear that $x \in \mathrm{cl}_\theta(A_x)$ and $A_x \in \exp_m(A)$. Let Γ_x be the family $\{\bar{U} \cap A_x:U \in \mathcal{U}_x\}$. Since $x \in \mathrm{cl}_\theta(\bar{U} \cap A_x)$ and moreover

$$\bigcap_{U \in \mathcal{U}_x} \mathrm{cl}_\theta(\bar{U} \cap A_x) \subset \bigcap_{U \in \mathcal{U}_x} \mathrm{cl}_\theta(\bar{U}) \subset \{x\}$$

by the fact that X is a Urysohn space, we have $\cap_{U \in \mathcal{U}_x} \mathrm{cl}_\theta(\bar{U} \cap A_x) = \{x\}$. This implies that the correspondence $x \to \Gamma_x$ defines a one to one map from $\mathrm{cl}_\theta(A)$ into $\exp_m(\exp_m(A))$. As

$$|\exp_m(\exp_m(A))| \leqq (\kappa^m)^m = \kappa^m$$

we have $|\mathrm{cl}_\theta(A)| \leqq \kappa^m = |A|^{\chi(X)}$.

Now let $A_0 = A$ and, by transfinite induction, let us define, for any $\alpha \in m^+$ sets A_α, in such a way that $A_\alpha = \mathrm{cl}_\theta(\cup_{\beta \in \alpha} A_\beta)$. It is easy to see that

$$\bigcup_{\alpha \in m^+} A_\alpha \subset [A]_\theta.$$

For any $x \in \mathrm{cl}_\theta(\cup_{\alpha \in m^+} A_\alpha)$ we can choose a set $B \in \exp_m(\cup_{\alpha \in m^+} A_\alpha)$ so that $x \in \mathrm{cl}_\theta(B)$. Since $\mathrm{cf}(m^+) = m^+$ there is an ordinal $\alpha \in m^+$ for which $B \subset A_\alpha$ and consequently,

$$x \in \mathrm{cl}_\theta(A_\alpha) \subset A_{\alpha+1} \subset \bigcup_{\alpha \in m^+} A_\alpha,$$

i.e., $\cup_{\alpha \in m^+} A_\alpha$ is θ-closed. By definition, the last assertion implies that $[A]_\theta \subset \cup_{\alpha \in m^+} A_\alpha$ and hence $[A]_\theta = \cup_{\alpha \in m^+} A_\alpha$. To complete the proof it suffices to show that $|\cup_{\alpha \in m^+} A_\alpha| \leqq \kappa^m$ or, equivalently, that $|A_\alpha| \leqq \kappa^m$ for every $\alpha \in m^+$. Let us assume the contrary. So, let $\tilde{\alpha}$ be the smallest ordinal number α such that $|A_\alpha| > \kappa^m$. We have $|A_\beta| \leqq \kappa^m$ for any $\beta \in \tilde{\alpha}$ and therefore $|\cup_{\beta \in \tilde{\alpha}} A_\beta| \leqq \kappa^m$. Since $A_{\tilde{\alpha}} = \mathrm{cl}_\theta(\cup_{\beta \in \tilde{\alpha}} A_\beta)$ from what stated above it follows that

$$|A_{\tilde{\alpha}}| \leqq \left|\bigcup_{\beta \in \tilde{\alpha}} A_\beta\right|^{\chi(X)} \leqq (\kappa^m)^m = \kappa^m.$$

This is a contradiction and the proof is complete.

REMARK 1. Note that the proof of the above theorem also shows that if X is a Urysohn space and $\chi(X) \leqq m$ then the θ-closed hull of any subset of X can be obtained iterating the θ-closed operator at most m^+ times. In [8] Willard and Dissanayake introduced the following:

DEFINITION 3. *Let X be a topological space. The almost Lindelöf degree of X, denoted by $aL(X)$, is defined as $aL(X) = \sup\{aL(F, X)$: F is a closed subset of $X\}$, where $aL(F, X)$ is the smallest cardinal number m with the property that for any family $\mathcal{U}$ of open sets of X such that $F \subset \cup\mathcal{U}$ there is a subfamily $\mathcal{U}_0 \in \exp_m(\mathcal{U})$ for which $F \subset \cup\bar{\mathcal{U}}_0$.*

For any space X we have $aL(X, X) \leqq aL(X)$ and, for non regular spaces this inequality can be proper as the next example shows:

EXAMPLE 2. For a given cardinal number m let us consider the set $X = \{a, b_\alpha, c_{\alpha,n}: \alpha \in m^+, n \in N\}$. We topologize X as follows:

(i) for any $\alpha \in m^+$ and $n \in N$ the point $c_{\alpha,n}$ is isolated;

(ii) for any $\alpha \in m^+$ the family $\{\{b_\alpha\} \cup \{c_{\alpha,k}: k \geqq n\}\}_{n \in N}$ is a fundamental system of neighbourhoods of the point b_α;

(iii) the family $\{\{a\} \cup \{c_{\beta,n}: \beta \in m^+ - \alpha, n \in N\}\}_{\alpha \in m^+}$ is a fundamental system of neighbourhoods of the point a.

It is easy to see that X is a Urysohn space and moreover $aL(X, X) = m$ while $aL(X) = m^+$.

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REMARK 2. Perhaps it would be more convenient to call $aL(X, X)$ almost Lindelöf degree and $aL(X)$ c-Lindelöf degree, in analogy with the well known notions of almost compactness and c-compactness (see [7]).

In [8] Willard and Dissanayake proved that $|X| \leqq 2^{\chi(X)aL(X)}$ for a Hausdorff space X. As a consequence of Theorem 1 above we can sharpen this result when the space is assumed to be Urysohn.

THEOREM 2. *If X is a Urysohn space then* $|X| \leqq 2^{\chi(X)aL(X,X)}$.

PROOF. Let $m = \chi(X)aL(X, X)$ and let $\mathcal{U}_x$ be a fundamental system of open neighbourhoods at a point $x \in X$, such that $|\mathcal{U}_x| \leqq m$. By transfinite induction we construct a family $\{F_\alpha\}_{\alpha \in m^+}$ of subsets of X satisfying the following properties:

(i) for any $\alpha \in m^+$ F_α is θ-closed;

(ii) for any $\alpha \in m^+$ $|F_\alpha| \leqq 2^m$;

(iii) if $\alpha \in \beta \in m^+$ then $F_\alpha \subset F_\beta$;

(iv) for any $\alpha \in m^+$, if $X - \cup\bar{\mathcal{U}} \neq \emptyset$, where

$$\mathcal{U} \in \exp_m\left(\cup\left\{\mathcal{U}_x : x \in \bigcup_{\beta\in\alpha} F_\beta\right\}\right),$$

then $F_\alpha - \cup\bar{\mathcal{U}} \neq \emptyset$.

Let $x_0 \in X$ and $F_0 = \{x_0\}$. Let us suppose we have already defined the sets F_β for every $\beta \in \alpha$ satisfying properties (i)–(iv). Let $\mathcal{U}_\alpha = \cup\{\mathcal{U}_x : x \in \cup_{\beta\in\alpha} F_\beta\}$. It is clear that $|\mathcal{U}_\alpha| \leqq 2^m$. For any $\mathcal{U} \in \exp_m(\mathcal{U}_\alpha)$ for which $X - \cup\bar{\mathcal{U}} \neq \emptyset$ we choose a point in $X - \cup\bar{\mathcal{U}}$ and let E be the set so obtained. To complete the induction, let us put $F_\alpha = [E \cup (\cup_{\beta\in\alpha} F_\beta)]_\theta$. Clearly F_α satisfies properties (i), (iii), (iv) and, thanks to Theorem 1, also property (ii). Let $F = \cup_{\alpha\in m^+} F_\alpha$. We have $|F| \leqq 2^m$ and moreover F is θ-closed because if $x \in \mathrm{cl}_\theta(F)$ then there is a set $F_x \in \exp_m(F)$ such that $x \in \mathrm{cl}_\theta(F_x)$, but for some $\alpha \in m^+$ we have $F_x \subset F_\alpha$ and consequently $x \in \mathrm{cl}_\theta(F_\alpha) = F_\alpha$.

To complete the proof it suffices to show that $F = X$. Let us assume the contrary. So, let p be a point in $X - F$. For any $x \in F$ we can choose an open neighbourhood $U_x \in \mathcal{U}_x$ such that $p \notin \bar{U}_x$ and, for any $x \in X - F$, an open neighbourhood $U_x \in \mathcal{U}_x$ such that $\bar{U}_x \cap F = \emptyset$. The family $\{U_x\}_{x\in X}$ is an open cover of X. Since $aL(X, X) \leqq m$ there exists a set $C \in \exp_m(X)$ such that $X = \cup_{x\in C} \bar{U}_x$. We clearly have $F \subset \cup_{x\in C\cap F} \bar{U}_x$ and $p \notin \cup_{x\in C\cap F} \bar{U}_x$. As $\mathrm{cf}(m^+) = m^+$ and $|C \cap F| \leqq m$ there is an ordinal $\alpha \in m^+$ for which $C \cap F \subset F_\alpha$, but this leads to a contradiction because $F_{\alpha+1}$ must satisfy property (iv) and so the proof is complete.

As another consequence of Theorem 1 we will prove a theorem for a special class of Urysohn spaces, recently introduced by Dikranjan and Giuli.

DEFINITION 4. (see [2]) *Let X be a topological space. A family $\mathcal{S}$ of subsets of X is said to be a U-cover if $X = \cup_{S \in \mathcal{S}} \mathrm{int}_\theta(S)$. A space X is Ury-closed if it is a Urysohn space and every open U-cover has a finite subcover. The class of Ury-closed spaces properly lies between the class of H-closed spaces and the class of Urysohn closed spaces in the classical sense.*

DEFINITION 5. *Let X be a T_1 space. We denote by $\tilde{\psi}(X)$ the smallest cardinal number m such that every closed subset of X can be expressed as the intersection of at most m open sets.*

THEOREM 3. *If X is a Ury-closed space then $|X| \leqq 2^{\chi(X)\tilde{\psi}(X)}$.*

PROOF. Let $m = \chi(X)\tilde{\psi}(X)$ and let $\mathcal{U}_x$ be a fundamental system of neighbourhoods at a point $x \in X$ such that $|\mathcal{U}_x| \leqq m$. For any $U \in \mathcal{U}_x$ let $\mathcal{G}_U$ be a family of open sets such that $|\mathcal{G}_U| \leqq m$ and $\bar{U} = \cap \mathcal{G}_U$. Letting $\mathcal{G}_x = \cup_{U \in \mathcal{U}_x} \mathcal{G}_U$ we have $|\mathcal{G}_x| \leqq m$ and $\cap \mathcal{G}_x = \{x\}$. As in Theorem 2 we construct a family $\{F_\alpha\}_{\alpha \in m^+}$ of subsets of X satisfying the following properties:

(i) for any $\alpha \in m^+$ F_α is θ-closed;

(ii) for any $\alpha \in m^+$ $|F_\alpha| \leqq 2^m$;

(iii) if $\alpha \in \beta \in m^+$ then $F_\alpha \subset F_\beta$;

(iv) for any $\alpha \in m^+$, if $X - \cup \mathcal{G} \neq \emptyset$, where $\mathcal{G}$ is a finite subset of

$$\cup \left\{ \mathcal{G}_x : x \in \bigcup_{\beta \in \alpha} F_\beta \right\},$$

then $F_\alpha - \cup \mathcal{G} \neq \emptyset$.

The set $F = \cup_{\alpha \in m^+} F_\alpha$ is θ-closed and moreover $|F| \leqq 2^m$. We claim that $F = X$. Let us assume the contrary. So, let $p \in X - F$. For any $x \in F$ let us choose an open set $G_x \in \mathcal{G}_x$ for which $p \notin G_x$. The family $\{X - F\} \cup \{G_x : x \in F\}$ is an open U-cover of X. Since the space is Ury-closed there exists a finite set of points $\{x_1, \ldots, x_n\} \subseteq F$ such that $X = G_{x_1} \cup \ldots \cup G_{x_n} \cup (X - F)$ and so $F \subset G_{x_1} \cup \ldots \cup G_{x_n}$. For a suitable $\alpha \in m^+$ we have $\{x_1, \ldots, x_n\} \subset F_\alpha$ and this leads to a contradiction because $F_{\alpha+1}$ must satisfy property (iv).

To conclude, we make some observations on a result given by Willard and Dissanayake in [8]. Precisely we want to show that in the inequality proved in [8] Theorem 2.4, i.e., $|X| \leqq 2^{\psi_c(X)t(X)\pi\chi(X)aL(X)}$ for a Hausdorff space X, the π-character can be omitted.To this end let us consider the following:

LEMMA. *If X is a Hausdorff space and A is a subset of X, then $|\bar{A}| \leqq |A|^{\psi_c(X)t(X)}$.*

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PROOF. The proof is essentially the same of Proposition 2.5 in [4]. Replacing Lemma 2.3 in [8] with the above lemma it is easy to obtain the following:

THEOREM 4. *If X is a Hausdorff space, then* $|X| \leqq 2^{\psi_c(X)t(X)aL(X)}$.

QUESTION. Does Theorem 1 or Theorem 2 remain true if the space X is assumed to be Hausdorff?

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REFERENCES

1. A. Arkhangel'skii, *The power of bicompacta with the first axiom of countability*, Dokl. Akad. Nauk S.S.S.R. **187** (1969), pp. 967-970; (Soviet Math. Dokl. **10** (1969), pp. 951-955).
2. D. Dikranjan and E. Giuli, *S(n)-θ-Closed spaces*, Topology Appl., to appear.
3. R. Engelking, *General topology*, (Monografie Matematyczne, Warszawa 1977).
4. I. Juhasz, *Cardinal functions in topology*, Math. Centre Tracts **34**, Amsterdam (1971).
5. R. Pol, *Short proofs of two theorems on cardinality of topological spaces*, Bull. Acad. Polon. Sci. Ser. Math. Astr. Phys. **22** (1974), pp. 1245-1249.
6. N. Veličko, *H-closed topological spaces*, Mat. Sb. (N.S.) **70** (112) (1966), pp. 98-112; Amer. Math. Soc. Transl. **78** (Ser. 2) (1969), pp. 103-118.
7. G. Viglino, *C-compact spaces*, Duke Math. J. **36** (1969), pp. 761-764.
8. S. Willard and B. Dissanayake, *The almost-Lindelöf degree*, Canad. Math. Bull. **27** (4) (1984), pp. 1-4.

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