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THE INDIVIDUAL ERGODIC THEOREM FOR CONTRACTIONS WITH FIXED POINTS

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Let $(X, \mathfrak{X}, \mu)$ be a σ-finite measure space and let T take L_p to L_p, p fixed, $1<p<\infty$, $\|T\|_p \le 1$. We shall say that *the individual ergodic theorem holds for T* if for any uniform sequence $k_1, k_2, \ldots$ (for the definition, see [2]) and for any $f \in L_p(X)$, the limit

$$f^*(X) = \lim_n \frac{1}{n} \sum_{i=1}^{n} T^{k_i} f(X)$$

exists and is finite almost everywhere. Using Akcoglu's ergodic theorem ([1]) to modify slightly the proof in [4] that the individual ergodic theorem holds for T a positive contraction of L_1 and L_p, some $p>1$, we see that the individual ergodic theorem holds for any positive $T: L_p \to L_p$, $1<p<\infty$, $\|T\|_p \le 1$. The purpose of this note is to point out that if there exists $h \ne 0$ with $Th = h$, and in addition $T: L_\infty \to L_\infty$, $\|T\|_\infty \le 1$, then the individual ergodic theorem holds for T, without the restriction of positivity.

In [3], De La Torre proves that if there exists $h \ne 0$, $Th = h$, and $\|T\|_\infty \le 1$, then the Dominated Ergodic Theorem holds for T: i.e.,

$$\left\| \sup_n \frac{1}{n} \left| \sum_{k=1}^{n-1} T^k f(x) \right| \right\|_p \le \frac{p}{p-1} \|f\|_p$$

for all $f \in L_p$.

In proving his result, De La Torre proves three lemmas. In the first he proves that if there exists an h such that $Th = h$, $|h| = 1$, then the Dominated Ergodic Theorem holds for T. The other two lemmas show that if there exists $g \ne 0$, $Tg = g$, then there exists such an h.

In the proof of his first lemma, De La Torre defines the operator $Sf = \bar{h}T(hf)$, where $|h| = 1$, $Th = h$, which is a contraction of L_p and L_∞, and shows that S is positive. Hence the individual ergodic theorem holds for S. But $T^i f = hS^i \bar{h} f$, so

$$\lim_n \frac{1}{n} \sum_{i=1}^{n-1} T^{k_i} f = h \cdot \lim_n \left(\frac{1}{n} \sum_{i=1}^{n-1} S^{k_i} \bar{h} f \right)$$

which must now exist and be finite almost everswhere for every $f \in L_p$.

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We summarize this discussion by stating the result as a theorem.

THEOREM. *Let T be a contraction of L_p and L_∞, p fixed, $1 < p < \infty$, and let there exist an $h \neq 0$ such that $Th = h$. Then the individual ergodic theorem holds for T.*

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