A CYCLIC INEQUALITY AND AN EXTENSION OF IT. I.

by P. H. DIANANDA
(Received 23rd November, 1961)

1. Introduction

For positive integral n and positive real $x_1, \ldots, x_n$ let

$$S_n(x_1, \ldots, x_n) = \sum_{r=1}^{n} \frac{x_r}{x_{r+1}+x_{r+2}}, \quad \ldots\ldots(1)$$

where

$$x_{n+r} = x_r \quad (\text{all } r), \quad \ldots\ldots(2)$$

and let

$$\lambda(n) = \frac{1}{n} \inf_{x_1, \ldots, x_n} S_n(x_1, \ldots, x_n). \quad \ldots\ldots(3)$$

In a recent paper (**3**) Rankin has proved that $\lambda(n) \geqq 0 \cdot 3307\ldots$, thus improving the inequality $\lambda(n) \geqq \frac{1}{3}(\sqrt{2}-\frac{1}{2}) = 0 \cdot 3047\ldots$, which he had obtained in 1957 (see (**3**)).

It is also known (**1**), (**2**) that

$$\lambda(n) = \tfrac{1}{2} \quad (n \leqq 6),$$
$$\geqq \frac{3}{n} \quad (n \geqq 7).\dagger$$

In this paper we shall prove in Theorem 1 that $\lambda(n) \geqq \frac{1}{2}(\sqrt{2}-\frac{1}{2}) = 0 \cdot 4571\ldots$. We shall give two proofs of this result. The first is based on more elementary ideas than the second and is also simpler and shorter. The second, which was obtained before the first, will only be given in outline. It is based essentially on Rankin's method of proof in which properties of convex functions were used. Our first proof also uses certain ideas introduced by Rankin.

We shall prove also Theorem 2, which is a slight improvement of Theorem 1 for odd n, and Theorem 3, which is an extension of Theorems 1 and 2.

2. First Proof of Theorems 1 and 2

Following Rankin, we write

$$\phi_L(x_0, \ldots, x_{L+1}) = \sum_{r=0}^{L-1} \frac{x_r}{x_{r+1}+x_{r+2}} \quad \ldots\ldots(4)$$

for positive integral L and positive real $x_0, \ldots, x_{L+1}$.

† See note at the end of this paper.

Lemma 1. *If* $x_1 \leqq x_2 \leqq \ldots \leqq x_L$ *and* $x_L \geqq x_{L+1}$, *then* $\phi_L(x_0, \ldots, x_{L+1}) \geqq \psi_L(x_0, \ldots, x_L)$, *where*

$$\psi_L(x_0, \ldots, x_L)$$

$$= \frac{x_0}{x_1+x_2} + \frac{x_1+x_2}{x_3+x_4} + \frac{x_3+x_4}{x_5+x_6} + \ldots + \frac{x_{L-4}+x_{L-3}}{x_{L-2}+x_{L-1}} + \frac{x_{L-2}+x_{L-1}}{2x_L} \quad (L \text{ odd}),$$

$$= \frac{x_0}{x_1+x_2} + \frac{x_1+x_2}{x_3+x_4} + \frac{x_3+x_4}{x_5+x_6} + \ldots + \frac{x_{L-3}+x_{L-2}}{x_{L-1}+x_L} + \frac{x_{L-1}+x_L}{2x_L} - \tfrac{1}{2} \quad (L \text{ even}).$$

Lemma 2. *The functions* $\frac{1}{x}(2^{\pm x}-1)$ *increase steadily for* $x>0$.

Lemma 3. *The function* $2^x - x$ *decreases steadily for* $0 \leqq x \leqq \frac{1}{2}$.

These three lemmas have obvious proofs.

To prove the theorem itself, let $a_1, \ldots, a_{s+1}$ be a set of integers for which

$$1 \leqq a_1 < \ldots < a_s \leqq n < a_{s+1} = a_1 + n, \quad \ldots\ldots(5)$$

$$x_{a_k} \geqq x_{a_k+1} \text{ and } x_{a_k+1} \leqq x_{a_k+2} \leqq \ldots \leqq x_{a_{k+1}} \quad (k = 1, \ldots, s).$$

Then, from (1), (2) and (4),

$$S_n(x_1, \ldots, x_n) = \sum_{k=1}^{s} \phi_{a_{k+1}-a_k}(x_{a_k}, \ldots, x_{a_{k+1}+1}) \quad \ldots\ldots(6)$$

$$\geqq \sum_{k=1}^{s} \psi_{a_{k+1}-a_k}(x_{a_k}, \ldots, x_{a_{k+1}}),$$

by Lemma 1. Using the expressions given in Lemma 1 for the terms in the last sum we see that (this sum)$+\frac{1}{2}c$ = a sum of $\frac{1}{2}(n+s+c)$ terms whose product is 2^{-s}, where

$$c = \text{the number of even } a_{k+1}-a_k \text{ for } 1 \leqq k \leqq s. \quad \ldots\ldots(7)$$

Thus

$$S_n(x_1, \ldots, x_n) \geqq \tfrac{1}{2}(n+s+c)2^{\frac{-2s}{n+s+c}} - \tfrac{1}{2}c$$

$$= \frac{n+s+c}{2s}\left(2^{\frac{-2s}{n+s+c}}-1\right)s + \frac{n+s}{2} \equiv F(s, c), \quad \ldots\ldots(8)$$

by the inequality between arithmetic and geometric means.

We shall prove later

Lemma 4. $$F(s, c) \geqq \tfrac{1}{2}n(\sqrt{2}-\tfrac{1}{2}) \quad (n \text{ even}), \quad \ldots\ldots(9)$$

$$\geqq \tfrac{1}{2}n\left(2^{\frac{n-1}{2n}} - \frac{n-1}{2n}\right) \quad (n \text{ odd}). \quad \ldots\ldots(10)$$

From (3) and (8) and Lemmas 3 and 4 we then obtain the following results.

Theorem 1. $$\lambda(n) \geqq \tfrac{1}{2}(\sqrt{2}-\tfrac{1}{2}) = 0{\cdot}4571\ldots.$$

Theorem 2.

$$\lambda(n) \geqq \tfrac{1}{2}\left(2^{\frac{n-1}{2n}} - \frac{n-1}{2n}\right) > \tfrac{1}{2}(\sqrt{2}-\tfrac{1}{2}) = 0{\cdot}4571\ldots \quad (n \text{ odd}).$$

Theorems 1 and 2 respectively contain the best known lower bounds for $\lambda(n)$ for even and odd $n \geqq 7$. (See note at the end of this paper.)

To prove Lemma 4 we consider separately two main cases, (i) $s \leqq [\frac{1}{2}(n+1)]$ and (ii) $s \geqq [\frac{1}{2}(n+1)]$. It is convenient to make (i) and (ii), which together exhaust all possibilities, overlap.

Case (i): $1 \leqq s \leqq [\frac{1}{2}(n+1)]$. Using (5) and (7), we find that $0 \leqq c \leqq s$ (n even) and $0 \leqq c \leqq s-1$ (n odd).

Subcase 1: n even. Clearly $0 \leqq c \leqq s \leqq \frac{1}{2}n$. Thus, from (8) and Lemma 2,

$$F(s,\,c) \geqq F(s,\,s) = \frac{n+2s}{n}\,(2^{\frac{n}{n+2s}}-1)\,\frac{n}{4} + \frac{n}{4} \equiv G(s).$$

Also, by Lemma 2, $G(s) \geqq G(\frac{1}{2}n) = \frac{1}{2}n(\sqrt{2}-\frac{1}{2})$, and (9) is proved.

Subcase 2: n odd. Clearly $0 \leqq c \leqq s-1 \leqq \frac{1}{2}(n-1)$. Thus, from (8) and Lemma 2,

$$F(s,\,c) \geqq F(s,\,s-1) = \frac{n-1+2s}{n-1}\,(2^{\frac{n-1}{n-1+2s}}-1)\,\frac{n-1}{4} + \frac{n+1}{4} \equiv H(s).$$

Also, by Lemma 2,

$$H(s) \geqq H\left(\frac{n+1}{2}\right) = \tfrac{1}{2}n\left(2^{\frac{n-1}{2n}} - \frac{n-1}{2n}\right),$$

and (10) follows.

Case (ii): $[\frac{1}{2}(n+1)] \leqq s \leqq n$. Using (5) and (7), we have that $0 \leqq c \leqq n-s$, and so from (8) and Lemma 2,

$$F(s,\,c) \geqq F(s,\,n-s) = \tfrac{1}{2}n\left(2^{\frac{n-s}{n}} - \frac{n-s}{n}\right).$$

Hence, by Lemma 3, (9) and (10) follow, since $s \geqq \frac{1}{2}n$ (n even) and $s \geqq \frac{1}{2}(n+1)$ (n odd).

This concludes the proof of Lemma 4, and hence of Theorems 1 and 2.

3. Second Proof of Theorems 1 and 2

For positive real t and non-negative real x we define functions $f_t(x)$, $g_t(x)$, $F_t(x)$ and $G_t(x)$ as follows:

$$\begin{aligned}
f_t(x) &= \tfrac{1}{2}tx^{\frac{2}{t}} && (0 \leqq x \leqq 2^{-\frac{1}{2}t}),\\
&= \tfrac{1}{2}(t+2)(\tfrac{1}{2}x)^{\frac{2}{t+2}} - \tfrac{1}{2} && (x \geqq 2^{-\frac{1}{2}t});\\
g_t(x) &= \tfrac{1}{2}(t+1)(\tfrac{1}{2}x)^{\frac{2}{t+1}};\\
F_t(x) &= \frac{2}{t}\,f_t(x^{\frac{1}{2}t}) = x && (0 \leqq x \leqq \tfrac{1}{2}),\\
&= \frac{t+2}{t}\,2^{\frac{-2}{t+2}}\,x^{\frac{t}{t+2}} - \frac{1}{t} && (x \geqq \tfrac{1}{2});\\
G_t(x) &= \frac{2}{t}\,g_t(x^{\frac{1}{2}t}) = \frac{t+1}{t}\,2^{\frac{-2}{t+1}}\,x^{\frac{t}{t+1}}.
\end{aligned}$$

It is seen that $f_2(x) = F_2(x) \equiv f(x)$ is the function $f(x)$ used by Rankin. The functions defined above are all convex functions of $\log x$ for $x>0$, but the only convexity property we shall use is that of $f(x)$. This and some other properties of $f(x)$ are given in Lemma 1 of (3), from which we have

Lemma 5. *$f(x)$ is a convex function of $\log x$ for $x>0$. Further,*

$$f(x) \geqq \sqrt{(2x)} - \tfrac{1}{2}$$

for $x \geqq 0$.

Lemma 6. *For $t \geqq t' > 0$ and $x \geqq 0$, $G_t(x) \geqq F_t(x) \geqq F_{t'}(x)$.*

For $x \leqq \frac{1}{2}$, Lemma 6 follows if we use the fact that, for $t>0$,

$$-\log\left(1-\frac{1}{t+1}\right) > \frac{1}{t+1} \quad \text{so that} \quad \left(1+\frac{1}{t}\right)^{t+1} > e.$$

For $x \geqq \frac{1}{2}$, Lemma 6 follows since $G_t(x)-F_t(x)$ and $F_t(x)-F_{t'}(x)$ have zero minima at $x = 2^{2/t}$ and $x = \frac{1}{2}$ respectively.

From Lemma 1 we obtain

Lemma 7. *If $x_1 \leqq x_2 \leqq \ldots \leqq x_L$ and $x_L \geqq x_{L+1}$, then*

$$\begin{aligned} \phi_L(x_0, \ldots, x_{L+1}) &\geqq g_L(x_0/x_L) \quad (L \text{ odd}), \\ &\geqq f_L(x_0/x_L) \quad (L \text{ even}). \end{aligned}$$

The particular cases $L \leqq 2$ are included in Lemmas 2 and 3 of (3). The proof of Lemma 7 is straightforward except when L is even and $0 \leqq x_0/x_L \leqq 2^{-\frac{1}{2}L}$, in which case we use the facts that

$$\begin{aligned} \psi_L(x_0, \ldots, x_L) &= \frac{x_0}{x_1+x_2} + \frac{x_1+x_2}{x_3+x_4} + \ldots + \frac{x_{L-3}+x_{L-2}}{x_{L-1}+x_L} + \frac{x_{L-1}}{2x_L} \\ &\geqq \frac{L}{2}\left(\frac{x_0}{x_{L-1}+x_L}\right)^{\frac{2}{L}} + \frac{x_{L-1}}{2x_L} \equiv h\left(\frac{x_0}{x_L}, \frac{x_{L-1}}{x_L}\right), \end{aligned}$$

and

$$\frac{\partial}{\partial u} h(x, u) \geqq 0 \quad (u \geqq 0,\ 0 \leqq x \leqq 2^{-\frac{1}{2}L}).$$

From Lemmas 5 to 7 we easily obtain

Lemma 8. *If $L \geqq 2$ and $x = x_0/x_L$, then $\phi_L(x_0, \ldots, x_{L+1}) \geqq \frac{1}{2}Lf(x^{\frac{2}{L}})$.*

We now use the equality (6). Suppose that, in the sum on the right-hand side of (6), the number of terms for which $a_{k+1}-a_k$ is unity is $n-p$ and that the product of the corresponding $x_{a_k}/x_{a_{k+1}}$ is x. Then, since $p \leqq n$, using (8) and Lemmas 3, 5 and 8, we have

$$\begin{aligned} 2S_n(x_1, \ldots, x_n) &\geqq pf\left(x^{-\frac{2}{p}}\right) + (n-p)x^{\frac{1}{n-p}} \\ &\geqq p\left(2^{\frac{1}{2}}x^{-\frac{1}{p}} - \tfrac{1}{2}\right) + (n-p)x^{\frac{1}{n-p}} \\ &\geqq n\left(2^{\frac{p}{2n}} - \frac{p}{2n}\right) \geqq n(\sqrt{2}-\tfrac{1}{2}). \end{aligned}$$

Theorem 1 then follows from (3).

This method can also be modified to give Theorem 2. If at least one of the $(a_{k+1}-a_k)$ is unity, then $p \leqq n-1$ and the modification is straightforward. If not, since n is odd, there is a k for which $a_{k+1}-a_k$ is odd and has the value $n-p$ (say) $\geqq 3$. Let $x_{a_k}/x_{a_{k+1}} = x$. Then, since $p \leqq n-1$, we have by (8) and Lemmas 2, 5 and 8, that

$$
\begin{aligned}
2S_n(x_1, \ldots, x_n) &\geqq pf\left(x^{-\frac{2}{p}}\right)+2g_{n-p}(x) \\
&\geqq \frac{n+p+1}{n-1}\left(2^{\frac{n-1}{n+p+1}}-1\right)\frac{n-1}{2}+\frac{n+1}{2} \\
&\geqq n\left(2^{\frac{n-1}{2n}}-\frac{n-1}{2n}\right).
\end{aligned}
$$

Using (3), we thus complete the proof of Theorem 2.

4. Extensions of Theorems 1 and 2

Both methods of proof enable us to extend Theorems 1 and 2. Their extensions are given in

Theorem 3. *Let n be a positive integer, let $x_{n+r} = x_r > 0$ for all r, and let $H_r(u, v)$ (for each positive $r \leqq n$) be a homogeneous function in u, v of degree d satisfying the inequalities*

$$
\begin{aligned}
&0 < H_r(u, v) \leqq \tfrac{1}{2}H_r(1, 1)(u^d+v^d) \qquad (v \geqq u > 0), \\
&0 < H_r(u, v) \leqq H_r(1, 1)u^d \qquad (u \geqq v > 0).
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{1}{n}\max_r H_r(1, 1)\sum_{r=1}^{n}\frac{x_r^d}{H_r(x_{r+1}, x_{r+2})} &\geqq \sqrt{2}-\tfrac{1}{2} && (n \text{ even}), \\
&\geqq 2^{\frac{n-1}{2n}}-\frac{n-1}{2n} > \sqrt{2}-\tfrac{1}{2} && (n \text{ odd}).
\end{aligned}
$$

The most general linear and quadratic forms $H_r(u, v)$ satisfying the conditions of the theorem are of the types

$$H_r(u, v) = au+bv \quad (a \geqq b \geqq 0,\ a+b > 0)$$

and

$$H_r(u, v) = au^2+buv+cv^2 \quad (a \geqq c \geqq 0,\ a+b \geqq c \geqq -b,\ a+b+c > 0).$$

Note that the sum appearing in the conclusion of the theorem is cyclic if the $H_r(u, v)$ are independent of r.

From Theorem 3 we deduce (for example) the

Corollary. *If n is a positive integer and $x_{n+r} = x_r > 0$ for all r, then*

$$
\left.
\begin{aligned}
&\text{(i)}\ \frac{2}{n}\sum_{r=1}^{n}\frac{x_r^2}{2x_{r+1}^2-x_{r+1}x_{r+2}+x_{r+2}^2} \\
&\textit{and}\ \ \text{(ii)}\ \frac{4}{n}\sum_{r=1}^{n}\frac{x_r}{3x_{r+1}+x_{r+2}+|x_{r+1}-x_{r+2}|}
\end{aligned}
\right\}
\geqq
\begin{cases}
\sqrt{2}-\tfrac{1}{2} & (n \text{ even}), \\
2^{\frac{n-1}{2n}}-\dfrac{n-1}{2n} \\
> \sqrt{2}-\tfrac{1}{2} & (n \text{ odd}).
\end{cases}
$$

Remark. (ii) *of the corollary implies Theorems* 1 *and* 2, *and* (i) *of the corollary.* This follows from the facts that, for positive u and v,

$$3u+v+|u-v|\geqq 2(u+v), \text{ i.e. } |u-v|\geqq v-u, \text{ and}$$

$$3u^2+v^2+|u^2-v^2|\geqq 2(2u^2-uv+v^2), \text{ i.e. } u+v\geqq|u-v|.$$

Note (added in proof). I am grateful to Professor Rankin for informing me that Djoković has proved that $\lambda(8)=\frac{1}{2}$. From this result I was able to deduce that $\lambda(7)=\frac{1}{2}$. Proofs of these results will appear shortly in *Proc. Glasgow Math. Assoc.*

I have proved that $\lambda(n)\geqq 0{\cdot}4612\ldots$ in a sequel to the present paper, to appear soon in these *Proceedings*.

REFERENCES

(1) P. H. Diananda, Extensions of an inequality of H. S. Shapiro, *Amer. Math. Monthly*, **66** (1959), 489-491.

(2) L. J. Mordell, On the inequality $\sum_{r=1}^{n} x_r/(x_{r+1}+x_{r+2})\geqq\frac{1}{2}n$ and some others, *Abh. Math. Sem. Univ. Hamburg*, **22** (1958), 229-240.

(3) R. A. Rankin, A cyclic inequality, *Proc. Edinb. Math. Soc.*, **12** (1961), 139-147.

Department of Mathematics
University of Malaya in Singapore †

† Now, University of Singapore.