BULL. AUSTRAL. MATH. SOC. 34C05, 58F21
VOL. 38 (1988) [1–10]

QUADRATIC SYSTEMS WITH A DEGENERATE CRITICAL POINT

W. A. COPPEL

It is shown that a quadratic system with a degenerate critical point has at most one limit cycle.

A *critical point* of a two-dimensional autonomous system

$$x' = P(x, y), \qquad y' = Q(x, y), \tag{1}$$

is a point $M = (x_0, y_0)$ for which

$$P(x_0, y_0) = Q(x_0, y_0) = 0.$$

The critical point is *degenerate* if both the Jacobian $P_x Q_y - P_y Q_x$ and the divergence $P_x + Q_y$ are zero at M.

The system (1) is said to be *quadratic* if

$$P(x, y) = \sum_{i+k=0}^{2} a_{ik} x^i y^k \quad Q(x, y) = \sum_{i+k=0}^{2} b_{ik} x^i y^k \tag{2}$$

are relatively prime real polynomials of degree at most two which are not both linear. We propose to prove

THEOREM 1. *If a quadratic system has a degenerate critical point then it either has a centre or has at most one periodic orbit. Moreover, if there is a unique periodic orbit it is a limit cycle with a non-zero characteristic exponent.*

By [2], Theorem 6 we may suppose that the quadratic system has a focus or centre at the origin, which is surrounded by a periodic orbit. Moreover, by a non-singular linear transformation we may suppose that it has the form

$$x' = \lambda x - y + lx^2 + mxy + ny^2$$
$$y' = x + \lambda y + ax^2 + bxy + cy^2.$$

Received 1 September 1987

Furthermore we may suppose that the degenerate critical point is situated at $(0, 1)$. Then $n = 1$, $c = -\lambda$ and $m = 0$, $b = -1 - \lambda^2$. Thus

$$x' = \lambda x - y + lx^2 + y^2$$
$$y' = x + \lambda y + ax^2 - (1 + \lambda^2)xy - \lambda y^2.$$

If we put $z = y + \lambda x$ this takes the form

$$x' = 2\lambda x - z + Lx^2 - 2\lambda xz + z^2$$
$$z' = (1 + \lambda^2)x + Ax^2 - (1 + \lambda^2)xy.$$

By scaling we can obtain finally a system of the form

$$\begin{aligned} x' &= dx - y + lx^2 - dxy + y^2 \\ y' &= x + ax^2 - xy \end{aligned} \tag{3}$$

where $-2 < d < 2$.

Since $y' = ax^2$ for $y = 1$ any periodic orbit γ of (3) which surrounds the origin must lie in the half-plane $y < 1$. Moreover, since the interior of γ is a convex region ([**2**, Theorem 1]), the intersections of γ with the y-axis are the only points on γ where $y' = 0$. Hence γ must also lie in the half-plane $1 + ax - y > 0$.

The system (3) has a centre at the origin if $d = a(2l - 1) = 0$, by the standard criteria for a weak focus to be a centre (see [**2**]). Thus this case may be excluded. If we set $B(y) = |1 - y|^\alpha$ then for the system (3)

$$(BP)_x + (BQ)_y = [d(1 - y)^2 + (2l - 1 - \alpha)x(1 - y) - \alpha a x^2]B/(1 - y).$$

If we take $\alpha = 2l - 1$ then it follows from Dulac's criterion that $ad(2l - 1) > 0$. If we take $\alpha = 1 - 2l$ then it follows similarly that $(2l - 1)^2 > ad(2l - 1)$. By changing the signs of x and t we may suppose that $d > 0$. Then we must have

$$0 < d < (2l - 1)/a. \tag{4}$$

It will now be shown that the quadratic system (3) has no focus or centre besides the origin, and hence that every periodic orbit must surround the origin. Besides the critical points $(0, 0)$ and $(0, 1)$ the system (3) also has the critical point (x_0, y_0), where

$$x_0 = -a/(a^2 - ad + l), \quad y_0 = 1 + ax_0 = (l - ad)/(a^2 - ad + l).$$

If we set $x = x_0 + \xi$, $y = y_0 + \eta$ then

$$\xi' = (2l - ad)x_0\xi + [1 + (2a - d)x_0]\eta + \dots$$
$$\eta' = ax_0\xi - x_0\eta + \dots .$$

https://doi.org/10.1017/S0004972700027167 Published online by Cambridge University Press

Thus the linear terms have trace $T = (2l - ad - 1)x_0$ and determinant $D = ax_0$. If the critical point (x_0, y_0) is a focus or centre then $T^2 < 4D$, that is

$$(2l - 1 - ad)^2 < -4(a^2 - ad + l)$$

or, equivalently,

$$(2l + 1 - ad)^2 < 4(l - a^2).$$

Thus $a^2 - ad + l < 0$ and $l > a^2$. Since $d > 0$, these inequalities imply $a > 0$. Then (4) implies $2l + 1 - ad > 2$ and hence $1 < l - a^2$. On the other hand $a^2 - ad + 1 > 0$, since $0 < d < 2$, and hence $l < 1$. Thus we have a contradiction.

In order to show that the quadratic system (3) has at most one periodic orbit surrounding the origin we now make some changes of variables. The linear fractional transformation

$$\xi = x/(1 - y),\ \eta = y/(1 - y),\ d\tau/dt = 1 - y$$

with inverse

$$x = \xi/(1 + \eta),\ y = \eta/(1 + \eta),\ dt/d\tau = 1 + \eta$$

replaces (3) by

$$\xi' = d\xi + (1 + l)\xi^2 + a\xi^3 - \eta$$

$$\eta' = (\xi + a\xi^2)(1 + \eta).$$

We wish to show that this system has at most one periodic orbit in the region $\eta > -1,\ 1 + a\xi > 0$.

If we now put $\xi = x,\ \eta = e^y - 1,\ \tau = t$ we obtain a system

$$\begin{aligned} x' &= F(x) - \varphi(y) \\ y' &= g(x) \end{aligned} \tag{5}$$

with

$$\begin{aligned} F(x) &= dx + (1 + l)x^2 + ax^3, \\ g(x) &= x + ax^2, \\ \varphi(y) &= e^y - 1. \end{aligned} \tag{6}$$

In order to show that (5)–(6) has at most one periodic orbit in the region $1 + ax > 0$ we make use of the following criterion:

THEOREM 2. *Let φ be a continuously differentiable function such that $\varphi(0) = 0, \varphi'(y) > 0$ for $-\infty < y < \infty$. Also let f, g be continuously differentiable functions on the open interval (a, b), where $a < 0 < b$, such that*

(i) $g(x) \gtrless 0$ *according as* $x \gtrless 0$,

(ii) $f(0) > 0$,

(iii) $f(x)/g(x)$ *is a decreasing function for $x > 0$ and for $x < 0$.*

Then the system (5), where $F(x) = \int_0^x f(\xi)d\xi$, has at most one periodic orbit and, if it exists, it is a stable limit cycle.

Theorem 2 differs only trivially from the theorem in Zhang Zhifen [5] and may be proved in exactly the same way. (This theorem was already announced, with a sketch of the proof, in [4].) It will now be shown that the functions (6) satisfy the conditions of Theorem 2. The assumptions concerning φ are obviously satisfied. The hypothesis (i) is satisfied on the open interval where $1 + ax > 0$ and the hypothesis (ii) is also satisfied, since

$$f(x) = d + 2(1 + l)x + 3ax^2.$$

Finally the hypothesis (iii) is satisfied, since

$$\frac{d}{dx}(f/g) = -[d + 2adx + a(2l - 1)x^2]/g^2(x)$$

and the quadratic in square brackets has negative discriminant.

Hence the quadratic system (3) has a unique periodic orbit, which is a stable limit cycle. To complete the proof of Theorem 1 it only remains to show that this periodic orbit has a negative characteristic exponent.

Let γ now denote the unique periodic orbit of the system (5)–(6). Let $x_2 > 0$ be the maximum value of x on γ and put

$$\overline{f}(x) = f(x) - \alpha g(x),$$

where $\alpha = f(x_2)/g(x_2)$. The hypotheses (i)–(iii) of Theorem 2 imply that $\alpha > 0$, $\overline{f}(x) < 0$ for $x > x_2$ and $\overline{f}(x) > 0$ for $0 \leqslant x < x_2$. The characteristic exponent of γ is

$$h = \int_\gamma \overline{f}(x)dt,$$

since

$$\int_\gamma g(x)dt = \int_\gamma dy = 0.$$

Since γ is stable we must have $h \leqslant 0$. We will assume $h = 0$ and derive a contradiction.

Evidently $\overline{f}(x_1) = 0$ for some $x_1 < 0$ and the minimum value of x on γ is less than x_1. The hypotheses (i)–(iii) of Theorem 2 imply that $\overline{f}(x) < 0$ for $x < x_1$ and $\overline{f}(x) > 0$ for $x_1 < x < 0$. Put

$$\Psi(x) = \begin{cases} \exp \quad [-(x_1 - x)^{-2}] \text{ for } x < x_1, \\ 0 \qquad \text{for } x \geqslant x_1. \end{cases}$$

and set

$$f_\mu(x) = f(x) + \mu\Psi(x)g(x),$$

where $\mu > 0$. Then the corresponding system

$$(5_\mu) \qquad \begin{aligned} x' &= F_\mu(x) - \varphi(y) \\ y' &= g(x) \end{aligned}$$

also satisfies the hypotheses of Theorem 2. Moreover (5_μ) has a periodic orbit γ_μ in the interior of γ, since the origin is an unstable critical point and orbits of (5_μ) which intersect γ to the left of the line $x = x_1$ cross it from the outside to the inside. Thus γ_μ is the unique periodic orbit of (5_μ) and is a stable limit cycle.

We now make use of some results from Andronov et al. [**1**, Chapter 13]. Although the system (5_μ) is C^∞, rather than analytic, and although it is not obtained from the analytic system (5) by a simple rotation of the vector field, nevertheless the quantity u_{01} defined there by (36) is still non-zero. Consequently Theorems 71 and 72 continue to hold. Theorem 71 shows that the limit cycle γ of (5) must be of odd multiplicity. Theorem 72 then shows that for all sufficiently small $\mu > 0$ the limit cycle γ_μ of (5_μ) has characteristic exponent $h_\mu < 0$.

Finally we compare h_μ with h, in the same manner as in the proof of Theorem 2. We refer to Figure 1. Since γ is a limit cycle of odd multiplicity greater than 1, the argument in [**1**] shows that the distance σ of B_1 from A_0 is given asymptotically by $\sigma \sim c_1\mu^\alpha$ for $\mu \to +0$, where $c_1 > 0$ and $0 < \alpha \leqslant 1/3$. Evidently

$$J_1 = \int_{A_1 B_1} \overline{f}(x)dt < 0, \quad J_2 = \int_{C_1 D_1} \overline{f}(x)dt < 0.$$

Since $x' < 0$ along $F_1 A_1$ and $C_0 A_0$ these arcs can be described by equations $y = y(x)$ and $y = y_\mu(x)$ respectively. Then

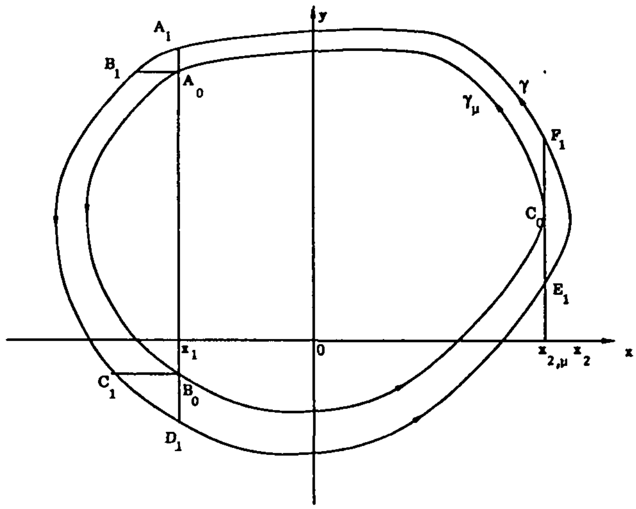

Figure 1

$$
\begin{aligned}
J_3 &= \int_{F_1 A_1} \overline{f}(x)dt - \int_{C_0 A_0} \overline{f}_\mu(x)dt \\
&= -\int_{x_1}^{x_{2,\mu}} \frac{\overline{f}(x)dx}{F(x)-\varphi(y(x))} + \int_{x_1}^{x_{2,\mu}} \frac{\overline{f}(x)dx}{F(x)-\varphi(y_\mu(x))} \\
&= \int_{x_1}^{x_{2,\mu}} \frac{\overline{f}(x)[\varphi(y_\mu(x))-\varphi(y(x))]dx}{[F(x)-\varphi(y(x))][F(x)-\varphi(y_\mu(x))]} \\
&< 0,
\end{aligned}
$$

since $y_\mu(x) < y(x)$ for $x_1 < x < x_{2,\mu}$. Similarly

$$J_4 = \int_{D_1 E_1} \overline{f}(x)dt - \int_{B_0 C_0} \overline{f}_\mu(x)dt < 0.$$

Since $y' < 0$ along $B_1 C_1$ and $A_0 B_0$ these arcs can be described by equations $x = x(y)$ and $x = x_\mu(y)$ respectively. Let $y_1, y_2 (y_1 < 0 < y_2)$ be the ordinates of B_0 and A_0. Then since $(f/g)' < 0$ and $\alpha < 1$ we have, for small $\mu > 0$,

$$
\begin{aligned}
J_5 &= \int_{B_1 C_1} \overline{f}(x)dt - \int_{A_0 B_0} \overline{f}_\mu(x)dt \\
&= \int_{y_1}^{y_2} [\frac{f(x_\mu(y))}{g(x_\mu(y))} - \frac{f(x(y))}{g(x(y))}]dy + \mu \int_{y_1}^{y_2} \Psi(x_\mu(y))dy \\
&< -c_2\mu^\alpha
\end{aligned}
$$

where $c_2 > 0$. On the arc E_1F_1 we have $x = \tilde{x}(y)$. Since $\bar{f}(x_2) = 0$ it follows that

$$J_6 = \int_{E_1F_1} \bar{f}(x)dt = \int \frac{\bar{f}\left(\tilde{x}(y)\right)}{g\left(\tilde{x}(y)\right)}dy = o(\mu^\alpha).$$

Adding these estimates we obtain, for small $\mu > 0$,

$$h - h_\mu = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 < 0.$$

Since $h = 0$ and $h_\mu < 0$, this is the required contradiction. This completes the proof of Theorem 1.

As Poincaré already pointed out, it is useful to extend the domain of definition of a quadratic system to the real projective plane. Then, in addition to the finite critical points, there are also critical points at infinity. In [**3**] I have shown that if a quadratic system has a critical point at infinity which is degenerate then it either has a centre or has at most one periodic orbit. Moreover, if there is a unique periodic orbit it is a limit cycle with a non-zero characteristic exponent. Thus the statement of Theorem 1 may be understood to refer to either a finite or an infinite critical point.

The proof given in [**3**] for a degenerate critical point at infinity depended on comparing the quadratic system with certain reflected systems and on the fact that the vector field defined by a quadratic system has at most six points of contact with a conic section, unless the conic section is invariant. In the same paper it was shown by a quite different method that a quadratic system with an invariant line has at most one limit cycle. That proof depended on the following uniqueness theorem for the Liénard equation

$$x'' - f(x)x' + g(x) = 0, \tag{7}$$

or rather the equivalent system

$$\begin{aligned} x' &= F(x) - y \\ y' &= g(x), \end{aligned} \tag{8}$$

where

$$F(x) = \int_0^x f(\xi)d\xi.$$

THEOREM 3. *Let f, g be continuously differentiable functions on the open interval (a, b), where $a < 0 < b$, such that*

(i) $g(x) \gtrless 0$ *according as* $x \gtrless 0$,

(ii) *$f(x) \gtrless 0$ according as $x \gtrless x_0$, where $x_0 < 0$,*

(iii) *the simultaneous equations*

$$F(x_1) = F(x_2),\ f(x_1)/g(x_1) = f(x_2)/g(x_2) \tag{9}$$

have at most one solution x_1, x_2 with $a < x_1 < x_0$ and $0 < x_2 < b$.

(iv) *if $F(\xi_0) = 0$ for some $\xi_0 < x_0$ then $f(x)F(x)/g(x)$ is a decreasing function for $a < x < \xi_0$.*

Then the system (8) has at most one periodic orbit and, if it exists, it has a negative characteristic exponent.

If in the proof of Theorem 3 given in [**3**] we replace the path $\tilde{\gamma}$ through the point C by the path $\overline{\gamma}$ through the point D then in a completely analogous way we can prove

THEOREM 3$'$. *The conclusions of Theorem 3 still hold if the hypotheses (i)–(iii) are retained but the hypothesis (iv) is replaced by*

(iv)$'$ *$f(x)F(x)/g(x)$ is an increasing function for $0 < x < b$ and*

$$\lim_{x \to a+} F(x) \leqslant \lim_{x \to b-} F(x).$$

In conclusion we show that Theorem 1, for the case of a degenerate critical point at infinity, can actually be deduced from Theorem 3$'$. Proceeding as in [**3**], we see that it is sufficient to show that the quadratic system

$$\begin{aligned} x' &= y + cx^2 \\ y' &= -x + dy + nx^2 + mxy, \end{aligned} \tag{10}$$

where $d > 0$, $n < (3c-1)d$ and $m + 2c = 1$, has at most one periodic orbit surrounding the origin. Eliminating y we obtain a Liénard equation (7) with

$$f(x) = d + x,\ g(x) = xq(x),$$

where

$$q(x) = 1 + (cd - n)x + c(1 - 2c)x^2.$$

We will show that the hypotheses of Theorem 3$'$ are satisfied by this Liénard equation.

Let γ be a periodic orbit of (8) which surrounds the origin. If $q(\xi) = 0$ then on the line $x = \xi$ x' has opposite signs on opposite sides of the critical point $(\xi, -c\xi^2)$. Since the interior of γ is a convex region containing no critical point besides the origin it follows that γ cannot intersect the line $x = \xi$. We now construct an open interval

https://doi.org/10.1017/S0004972700027167 Published online by Cambridge University Press

(a, b) with $a < 0 < b$, such that $q(x) > 0$ for $a < x < b$ and any periodic orbit of (8) which surrounds the origin must lie in the strip $a < x < b$. Then the hypothesis (i) will be satisfied.

If $q(x)$ has no real zeros we take $a = -\infty, b = +\infty$. If $q(x)$ has two real zeros $\xi_1 < \xi_2 < 0$, or if $q(x)$ has only one zero ξ_2 and $\xi_2 < 0$, we take $a = \xi_2, b = +\infty$. If $0 \leqslant c \leqslant 1/2$ then one of the preceding possibilities must occur, since $n - cd < 0$ and $c(1-2c) \geqslant 0$. If $c < 0$ or if $c > 1/2$ then $q(x)$ has two real zeros $\xi_1 < 0 < \xi_2$ and we take $a = \xi_1, b = \xi_2$.

If $-d \leqslant a$ the system (8) has no periodic orbits, since the divergence $f(x)$ is of constant sign for $a < x < b$. Hence we may suppose $a < -d$ and then the hypothesis (ii) is obviously satisfied with $x_0 = -d$.

Since $F(x) = dx + x^2/2$ it is readily shown that the simultaneous equations (9) with $x_1 < 0 < x_2$ are equivalent to

$$x_1 + x_2 = -2d, \quad dq(-2d) = Hx_1x_2, \tag{11}$$

where

$$H = n - cd + 3cd(1-2c).$$

Since $a < x_1 < -2d$ we must have $q(-2d) > 0$ and since $x_1x_2 < 0$ we must have $H < 0$. Since the first equation (11) defines x_2 as a decreasing function of x_1 and the second equation (11) defines x_2 as an increasing function of x_1, it follows that the hypothesis (iii) is also satisfied.

If we put $\omega(x) = f(x)F(x)/g(x)$ then

$$\begin{aligned}\omega'(x)/\omega(x) &= \frac{1}{d+x} + \frac{1}{2d+x} - \frac{q'(x)}{q(x)}, \\ &= N(x)/(d+x)(2d+x)q(x)\end{aligned}$$

where

$$N(x) = 3d + 2d^2(n-cd) + 2[1 - 2c(1-2c)d^2]x - Hx^2.$$

Since $N(-d) = dq(-d) > 0$, $N(-2d) = -dq(-2d) < 0$ and $N(-\infty) = +\infty$, the quadratic $N(x)$ is positive for $x > -d$ and in particular for $x > 0$. It follows that $\omega(x)$ is an increasing function for $0 < x < b$. The second part of the hypothesis (iv)$'$ is also satisfied, since if $c < 0$ or $c > 1/2$ then

$$a + b = \frac{n - cd}{c(1-2c)} > -d/c > -2d.$$

Thus we have a new proof of Theorem 1 when the degenerate critical point is at infinity.

REFERENCES

[1] A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier, *Theory of bifurcations of dynamic systems on a plane* (Halsted Press, New York - Toronto, 1973).

[2] W.A. Coppel, 'A survey of quadratic systems', *J. Differential Equations* **2** (1966), 293–304.

[3] W.A. Coppel, 'Some quadratic systems with at most one limit cycle', *Dynamics Reported* (to appear).

[4] Zhang Zhifen (Chan Chi-fen), 'On the uniqueness of limit cycles of certain equations of nonlinear oscillations', *Dokl. Akad. Nauk SSSR* **119** (1958), 659–662. [Russian] .

[5] Zhang Zhifen, 'Proof of the uniqueness theorem of limit cycles of generalized Liénard equations', *Applicable Anal.* **23** (1986), 63–76.

Department of Mathematics, I.A.S.
Australian National University
G.P.O. Box 4, Canberra, A.C.T. 2601
Australia