Let \( f(X, T_1, \ldots, T_m) \) be a polynomial over an algebraic number field \( K \) of finite degree. In his paper [2], T. Kojima proved

**Theorem.** Let \( K = \mathbb{Q} \). If for every \( m \) integers \( t_1, \ldots, t_m \), there exists an \( r \in K \) such that \( f(r, t_1, \ldots, t_m) = 0 \), then there exists a rational function \( g(T_1, \ldots, T_m) \) over \( \mathbb{Q} \) such that

\[
  f(g(T_1, \ldots, T_m), T_1, \ldots, T) = 0.
\]

Later, A. Schinzel [6] proved

**Theorem.** If for every \( m \) arithmetic progressions \( P_1, \ldots, P_m \) in \( \mathbb{Z} \) there exist integers \( t_i \in P_i \) \((i \leq m)\) and an \( r \in K \) such that \( f(r, t_1, \ldots, t_m) = 0 \) then there exists a rational function \( g(T_1, \ldots, T_m) \) over \( K \) such that

\[
  f(g(T_1, \ldots, T_m), T_1, \ldots, T) = 0.
\]

In his thesis [7], S. Tung applied these theorems to solve some decidability and definability problems. In this paper, we are concerned with geometric progressions of values of \( T_1, \ldots, T_m \). We prove

**Theorem 1.** Assume that there exists \( a_1, \ldots, a_m \in K \) other than 0 and roots of unity such that for any \( m \) integers \( t_1, \ldots, t_m \), there exists an \( r \in K \) with \( f(r, a_1^{t_1}, \ldots, a_m^{t_m}) = 0 \). Then there exist a rational function \( g(T_1, \ldots, T_m) \) over \( K \) and \( m \) integers \( k_1, \ldots, k_m \) not more than \( k \) such that

\[
  f(g(T_1, \ldots, T_m), T_1^{k_1}, \ldots, T_m^{k_m}) = 0
\]

where \( k \) is the \( X \)-degree of \( f(X, T_1, \ldots, T_m) \).

§ 1.

In case of \( m = 1 \), Theorem 1 is an easy consequence from Theorem of P. Roquette (Theorem 2.1 [4]) as follows.
Let $\omega \in \mathbb{N} - N$ be a nonstandard natural number which is divisible by all natural number where $\mathbb{N}$ is an enlargement of $N$. By the assumption of Theorem 1, there exists a $\delta \in \mathbb{K}$ such that

$$f(\delta, a^\omega) = 0.$$  

Let $k_i = [K(\delta, a^\omega); K(a^\omega)]$. Since the $X$-degree of $f(X, T)$ is $k$, 

$$k_i \leq k.$$  

According to Theorem 2.1 in [4], we have

**Theorem 2.** For each natural number $n$, there is one and only one extension $F_n = K(a^{\omega/n})$ of $K(a^\omega)$ within $\mathbb{K}$ such that 

$$[F_n; K(a^\omega)] = n$$

where $\mathbb{K}$ is an enlargement of $K$.

Hence, $K(\delta, a^\omega) = K(a^{\omega/k_i})$. Therefore there exists a rational function $g(T)$ over $T$ such that $\delta = g(a^{\omega/k_i})$. Now we have 

$$f(g(a^{\omega/k_i}), a^\omega) = 0.$$  

Since $a^{\omega/k_i}$ is transcendental over $K$, 

$$f(g(T), T^{k_i}) = 0$$  

as contended.

§2.

In this section we prove Theorem 1 for the case $m = 2$. To prove it, we need iterated enlargements. Iterated enlargements are very useful method but sometime they may cause confusion. So first we discuss basic properties of iterated enlargements. Let $\mathbb{K}$ be an enlargement of $K$. We consider the structure $(\mathbb{K}, K)$ and its enlargement $*(\mathbb{K}, K) = (*\mathbb{K}, *K)$. Then $*\mathbb{K}$ is an elementary extension of $\mathbb{K}$ but not an enlargement of $\mathbb{K}$. By Theorem of Roquette, for each $n \in \mathbb{N}$ and $a \in K$ other than 0 and roots of unity, the following statement is valid for $(\mathbb{K}, K)$;

"For each $\omega \in \mathbb{N} - N$, there is one and only one extension $F_n$ of $K(a^\omega)$ within $\mathbb{K}$ such that $[F_n; K(a^\omega)] = n$."  

By nonstandard principle, the above statement holds for $(*\mathbb{K}, *K)$;
"For each \( \omega \in {^*N} - {^*K} \), there is one and only one extension \( F_n \) of \( L \) within \( {^*K} \) such that \([F_n; L] = n\)."

where \( L = \{ h(a^\omega) | h(X) \in ^*\langle K(X) \rangle \} \). It should be noted that the rational function field over \(^*K\) in the sense of the enlargement generated by \( a^\omega \) must be \( L \), not \(^*K(a^\omega)\).

**Remark.** \(^*K\) is an enlargement of \(^*N\), but Theorem 2 (replacing \(^*K\) and \( K \) by \(^*K\) and \(^*N\) respectively) does not hold, because \(^*N\) is not an end extension of \(^*N\), namely there exist \( c \in {^*N} - {^*N} \) and \( d \in {^*N} \) with \( c < d \). In fact, let \( c \in {^*N} \) be an element which satisfies the set of formulas \( T = \{ c < d | d \in {^*N} - N \} \cup \{ n < c | n \in N \} \). Since any finite subset of \( T \) is satisfiable and \(^*N\) is an enlargement of \(^*N\), such \( c \) exists. On the other hand, \(^*N\) is an end extension of \( N \), so \(^*N\) is also an end extension of \(^*N\), therefore \(^*K\) is not an enlargement of \(^*K\).

The following Lemma 1 has been proved in [4] but we include its proof for the convenience of the reader.

**Lemma 1.** Let \( M \) be any field. Then \(^*M(X)\) is relatively algebraically closed in \(*M(X)*\).

**Proof.** Let \( u(X)/v(X) \) be any element of \(^*M(X)\) - \(^*M(X)\) where \( u(X), v(X) \in ^*\langle M(X) \rangle \) and g.c.d. \((u(X), v(X)) = 1 \) and assume that \( u(X)/v(X) \) is algebraic over \(^*M(X)\). Then there exist \( c_0, c_1, \ldots, c_n \in ^*\langle M(X) \rangle \) with \( c_0 \neq 0 \) and \( c_0(u/v)^n + c_1(u/v)^{n-1} + \cdots + c_n = 0 \). Since \( u/v \in ^*\langle M(X) \rangle \), the degree of \( u \) or \( v \) is infinitely large. We may assume without loss of generality that the degree of \( v \) is infinitely large. Then

\[
   c_0u^n + c_1u^{n-1}v + \cdots + c_nv^n = 0
\]

\[
   c_0u^n \equiv 0 \pmod{(v)}. \]

Since g.c.d. \((u, v) = 1\),

\[
   c_0 \equiv 0 \pmod{(v)}. \]

Since the degree of \( v \) is infinitely large and the degree of \( c_0 \) is finite, \( c_0 = 0 \). This is a contradiction.

**Lemma 2.** Let \( a \in K \) be not 0 nor roots of unity and \( \omega \in {^*N} - {^*N} \) be divisible by all natural number. Then \(^*K(a^\omega/n)\) is the unique extension of \(^*K(a^\omega)\) of degree \( n \) within \(^*K\).

**Proof.** Let \( x \in ^*K \) be algebraic over \(^*K(a^\omega)\) of degree \( n \). Then \( x \in L(a^\omega/n) \) because \( L(a^\omega/n) \) is the unique extension of \( L \) of degree \( n \) within \(^*K\).
and \(*K(a^n)\) is relatively algebraically closed in \(L = \{h(a^n) | h(X) \in *(K(X))\}\) by Lemma 1.

Again by Lemma 1, \(*K(a^{\omega/n})\) is relatively algebraically closed in \(L(a^{\omega/n}) = \{h(a^{\omega/n}) | h(X) \in *(K(X))\}\). Hence \(x \in *K(a^{\omega/n})\), as contended.

Let \(\omega \in *\mathbb{N} - \mathbb{N}\) and \(\mu \in *\mathbb{N} - \mathbb{N}\) be divisible by all natural numbers. By the assumption of Theorem 1, there exists a \(\delta \in *\mathbb{K}\) with

\[
f(\delta, a_{\omega}, a_\mu) = 0.
\]

Since \(a_\mu \in *K\), \(\delta\) is algebraic over \(*K(a_\mu)\) of degree \(k_1 \leq k\). Hence by Lemma 2, \(\delta \in *K(a_{\mu/k_1})\). Let \(F\) be the relative algebraic closure of \(K(a_{\mu/k_1})\) within \(*K\). Then \(\delta \in F(a_{\mu/k_1})\) because \(F(a_{\mu/k_1})\) is relatively algebraically closed in \(*K(a_{\mu/k_1})\). By Theorem 2, \(K(a_{\mu/k_1})\) has the unique extension \(K(a_{\mu/k_1})\) of degree \(n\) within \(F\). Since \(a_{\mu/k_1}\) is transcendental over \(F\), \(K(a_{\mu/k_1}, a_\mu)\) has the unique extension \(K(a_{\mu/k_1}, a_{\mu/k_1})\) of degree \(n\) within \(F(a_{\mu/k_1})\).

Let \(k_2 = [K(\delta, a_{\mu/k_1}, a_{\mu/k_1}); K(a_{\mu/k_1}, a_{\mu/k_1})]\). Then \(k_2 \leq k\) and

\[
K(\delta, a_{\mu/k_1}, a_{\mu/k_1}) = K(a_{\mu/k_1}, a_{\mu/k_1}).
\]

Hence there exists a rational function \(g(T_1, T_2) \in K(T_1, T_2)\) such that

\[
f(g(a_{\mu/k_1}, a_{\mu/k_2}), a_{\mu}, a_\mu) = 0.
\]

Since \(a_{\mu/k_1} \in *K - *K\) and \(a_{\mu/k_2} \in *K - K\) are algebraically independent over \(K\),

\[
f(g(T_1, T_2), T_{\mu/k_1}, T_{\mu/k_2}) = 0.
\]
§ 3.

Proof of Theorem for \( m > 2 \) is essentially the same as that in Section 2. By induction on \( i \in \mathbb{N} \), we define iterated enlargements \( K_i = (\ldots,*2,K,\ldots,*K) \) as follows. Let \( K_0 = \langle K \rangle \). \( K_{i+1} \) is an enlargement of \( (K_i,K) = (\ldots,*2,K,\ldots,*K,\ldots,K,K) \), i.e. \( K_{i+1} = (\ldots,*i+1(K),K) = (\ldots,*i+1K) \). Let \( \omega_j \in \mathbb{N} - \mathbb{N} \) be divisible by all natural numbers. Let \( \delta \in \langle \ldots,*K \rangle \) satisfy

\[
f(\delta, a_1^{\omega_1}, a_2^{\omega_2}, \ldots, a_m^{\omega_m}) = 0.
\]

Then by the same way as in Section 2, there exist natural numbers \( k_1, k_2, \ldots, k_m \) not more than \( k \) such that \( \delta \in K(a_1^{a_1/k_1}, \ldots, a_m^{a_m/k_m}) \). Since \( a_1^{a_1/k_1}, \ldots, a_m^{a_m/k_m} \) are algebraically independent over \( K \), there is a rational function \( g(T_1, \ldots, T_m) \in K(T_1, \ldots, T_m) \) such that

\[
f(g(T_1, \ldots, T_m), T_1^{a_1}, \ldots, T_m^{a_m}) = 0.
\]

References