THE FREE PRODUCT OF SKEW FIELDS

Dedicated to the memory of Hanna Neumann

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1. Introduction

In a recent paper [3] it was shown that the free product $K * L$ of two fields (possibly skew) can be embedded in a field, and moreover, this latter can be chosen to be the 'universal field of fractions' of $K * L$ (cf. [4, 5]). This opens up the prospect of doing for skew fields what the Neumanns and others have done for groups; indeed some sample applications were given in [3]. We pursue this topic here a little further: our main results state (i) every countably generated field can be embedded in a 2-generator field, (ii) in a free product of rings over a field k, any element algebraic over k is conjugate to an element in one of the factors, (iii) any field can be embedded in a field with nth roots for each n. These results are all analogous to well known results in group theory (cf. [8]), and although the proofs are not just a translation of the group case, the latter is of some help. Thus (ii), (iii) follow fairly easily, but they lead to other problems, still open, by replacing 'free product of rings' in (ii) by 'field product of fields, and in (iii) replacing 'nth roots' by 'roots of any equation'. On the other hand, (i) is less immediate, since 'field products' need to be used in the proof, and their manipulation requires some more technical lemmas.

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2. The field product

Throughout, all rings have a unit element 1, preserved by homomorphisms and inherited by subrings, and fields are not necessarily commutative; occasionally we use the prefix 'skew' for emphasis. If K is any ring, by a *K-ring* we understand a ring R with a canonical homomorphism $K \to R$; since R has a 1, this is equivalent to imposing on R a K-bimodule structure satisfying $x(yz) = (xy)z$, for x, y, z in R or K. Given such a structure, the homomorphism is obtained by mapping $\alpha \mapsto \alpha.1$, where $\alpha \in K$ and 1 is the unit element of R. A K-ring is *faithful* if the map

$K \to R$ is injective; we note that if K is a field, any non-zero K-ring is faithful.

Let k be a field and (R_λ) a family of non-zero (and hence faithful) k-rings. In each R_λ we choose a right k-basis including 1; this may be denoted by $B'_\lambda = \{1\} \cup B_\lambda$. We now form the words in $\bigcup B_\lambda$; such a word is called *non-interacting* if neighbouring factors come from different sets B_λ. The set of all non-interacting words (including the empty word to represent 1) forms a right k-basis for the free product of the R_λ over k (see [1, 2]). This free product will be written ${}_k^*R_\lambda$, or if only two factors R, S are present, $R {}_k^* S$.

Although we shall have occasion to deal with the free product directly, most of the time we are interested in the fields that can be formed from it. Thus let (K_λ) be a family of fields containing k as a subfield and let $R = {}_k^*K_\lambda$ be their free product; from [2] we know that R is a fir. We recall that a ring R is called a *fir* (= free ideal ring) if every (right or left) ideal of R is free as R-module, of unique rank. If every finitely generated right (or equivalently, left) ideal is free, of unique rank, R is called a *semifir*. In [3] it was shown that every fir is embeddable in a field; we therefore have a field L containing all the K_λ. However, a minimal field containing all the K_λ may not be unique and we shall need the more precise description of the universal field of fractions [4, 5], which is uniquely determined. We briefly summarize the construction from [4], which more generally, provides a universal field of fractions for any semifir.

Let R be any ring; by an *R-field* we understand a field K which is an R-ring. Such an R-field is called *epic* (*) if the canonical map $R \to K$ is a ring-epimorphism. This is equivalent to the condition that K be generated, as a field, by the image of R. A *specialization* of epic R-fields K, L is an R-ring homomorphism $f{:}K_0 \to L$ from an R-subring K_0 of K to L such that $xf \neq 0$ implies $x^{-1} \in K_0$. It follows from this that K_0 is a *local ring* (i.e. its non-units form an ideal, cf. [4]); in fact $\ker f$ is the maximal ideal and $K_0/\ker f \cong L$. Two specializations are equal if they agree on a common subring K_0 and the common restriction is a specialization. The epic R-fields and specializations can be shown to form a category $\mathscr{F}_R$; an initial object in $\mathscr{F}_R$, if one exists, is called a universal *R-field*, or in case the canonical map is injective, a *universal field of fractions* for R.

For a commutative ring R, the epic R-fields are determined up to isomorphism by the kernels of the canonical maps. The collection of these kernels, the prime ideals of R, forms a category equivalent to $\mathscr{F}_R$ (with inclusions as maps). This is usually written spec R and called the *prime spectrum* of R. In the general case, an epic R-field K is not determined by the elements of R that map to zero, but by the set of square matrices over R that become singular over K. Thus the place of the prime ideal is taken by a collection of square matrices satisfying certain conditions; this is called a *prime matrix ideal*, and it is possible to write

(*) This differs slightly from the usage in [4]: what are called R-fields there are called epic R-fields here.

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down a brief set of conditions characterizing prime matrix ideals [4]. The collection of prime matrix ideals of a general ring R is written field-spec R; it is equivalent, as a category, to $\mathscr{F}_R$. In particular, it is possible to write down conditions for field-spec R to be non-empty. We shall not give the general conditions here, as they are not needed (see [4], chapter 7 for details), but only note one special case.

A matrix A over a ring R is said to be *full* if it is square, say $n \times n$, and cannot be written as $A = PQ$ where P is $n \times r$, Q is $r \times n$ and $r < n$. Any homomorphism maps a non-full matrix to a non-full matrix; if it also maps every full matrix to a full matrix it is called *honest*. An honest homomorphism is necessarily injective. We recall the following result from ([4], page 283):

In a semifir R, the set of all non-full matrices is a prime matrix ideal. It is therefore the least prime matrix ideal, the corresponding epic R-field U is the universal field of fractions of R and the natural injection $R \to U$ is honest. If Φ is the set of all full matrices over R, then U may be obtained as the universal R-*ring over which all matrices of Φ are invertible, briefly, $U = R_\Phi$ is the universal Φ-inverting ring* ([4], page 285).

When there is an epic R-field U in which every full matrix over R is invertible (as here) we shall call U the *fully inverting ring* for R; this is necessarily the universal field of fractions of R, and the result quoted above tells us that it exists for any semifir. Our first result describes the extension of homomorphisms to fully inverting rings:

PROPOSITION 2.1. *Any honest homomorphism between rings R, R' wth fully inverting rings U, U' extends uniquely to a homomorphism between U and U'; in particular, any isomorphism between R and R' extends to a unique isomorphism between U and U'.*

PROOF. Let α be an honest homomorphism from R to R'; α maps the set of full matrices over R to the set of full matrices over R', and hence extends uniquely to a homomorphism between their fully inverting rings.

If (K_λ) is a family of fields containing a common subfield k, then their free product $R = {}_k^{*}K_\lambda$ is a semifir [2] and so has a universal field of fractions. This will be called the *field product* of the K_λ and written ${}_k^{\circ}K_\lambda$ or $K_k^{\circ}L$ in the case of only two factors. Clearly ' $\circ$ ' is a bifunctor on the category of fields and homomorphisms. This is an instance where the non-commutative case runs more smoothly than the commutative case. The 'free product' (i.e. coproduct) of two commutative k-fields K, L in the category of commutative k-algebras is given by their tensor product $K \otimes L$. This need not be an integral domain and so need not have a field of fractions, and even though an epic $(K \otimes L)$-field exists, there may be no universal $(K \otimes L)$-field.

We shall also need the fact that the universal field construction and the free product are commuting operations.

THEOREM 2.2. *Let R and S be k-rings (where k is a field) with universal fields of fractions $\bar{R}$, $\bar{S}$. Then the free product R_k^*S has a universal field of fractions isomorphic to the field product $\bar{R}_k^\circ\bar{S}$.*

PROOF. Clearly $U = \bar{R}_R^\circ\bar{S}$ is a field of fractions of R_k^*S; we complete the proof by showing that it is universal. Given any epic $(R*S)$-field K, we have induced mappings $R \to K$, $S \to K$, obtained by composing $R*S \to K$ with the natural inclusions. Hence there are specializations $\bar{R} \to K$, $\bar{S} \to K$ which can be combined to a specialization $\bar{R} \circ \bar{S} \to K$, necessarily unique, because its values on $R*S$ are prescribed. This shows that U is the universal field of fractions of $R*S$, as claimed.

We observe that this result applies e.g. when R, S are semifirs (containing a field k).

3. Subfields of field products

Let k be a commutative field and $k\langle X\rangle$ the free k-algebra on a set X. This algebra is a fir and hence has a universal field of fractions [4]. If K is a field generated over a central subfield k by a set X, we shall say that *X generates K freely over k* or that X is a *free generating set* of K over k if K is the universal field of fractions of the free algebra $k\langle X\rangle$. In this case K itself may also be called a *free field* over k; thus the free field on a single generator x over k is just the rational function field $k(x)$, but for more than one free generator the free field will be non-commutative. Given any field with a central subfield k, by a *free subset* over k we understand a subset Y of the field such that the subfield generated by Y is freely generated by Y over k.

We note that a subfield of a purely transcendental (i.e. free commutative) extension need not be free when the transcendence degree exceeds 1 (in degree 1 we have Lüroth's theorem). In the general case little is known about subfields of fields, but it seems not unreasonable to conjecture that every subfield of a free field (over a central subfield k) is again free over k. The proof would probably require a closer analysis of the matrix form for the elements of K. Such an attempt, if successful, may also provide techniques for the analysis of field products.

In this section we shall show that every countable field can be embedded in a 2-generator field; this is the analogue of a theorem of Neumann ([8], Theorem 20.7) for groups. As in that case, one may ask whether there is a countable field, or a countably generated field over k, containing a copy of every countable field (of a given characteristic), and as for groups, the answer is 'no'. To see this (*) let us denote for any field K, by $\mathscr{S}(K)$ the set of isomorphism types of finitely generated subgroups of K^*, the group of non-zero elements of K. If K is countable, then so is $\mathscr{S}(K)$. Now Smith [11] has shown that there are $\mathfrak{c} = 2^{\aleph_0}$ isomorphism types of finitely generated orderable groups, and since every countable ordered

(*) I am indebted to A. Macintyre for this proof.

group can be embedded in a countable field (of prescribed characteristic), it follows that there are $\mathfrak{c}$ distinct sets $\mathscr{S}(K)$ as K runs over all countable fields of any given characteristic. Therefore these fields cannot all be embedded in a 2-generator field.

Let P, Q be a subrings of a ring R and suppose that k is a common subring of P and Q. Then there exists a k-bimodule homomorphism

$$P \otimes_k Q \to R. \tag{2}$$

If the mapping (2) is injective, P and Q are said to be *linearly disjoint* in R over k. We observe that this property is not generally symmetric in P and Q. With this notion we obtain the following analogue of a lemma of Neumann ([9], cf. [8]), implicitly used in [3].

LEMMA 3.1. *Let $P = {\ast_K} R_\lambda$ be the free product of a family of K-rings, where K is a field. Given a subring S_λ of each R_λ and a subfield k of K where $k \subseteq S_\lambda$ (for all λ), suppose that for each λ, the pair S_λ, K is linearly disjoint in P over k. Then the subring Q of P generated by the S_λ is their free product over k.*

PROOF. Let $B'_\lambda = \{1\} \cup B_\lambda$ be a right k-basis for S_λ (adapted to the subspace k), then B'_λ is still right K-independent, by hypothesis, and can therefore be enlarged to a right K-basis $A'_\lambda = \{1\} \cup A_\lambda$ of R_λ (still adapted to the subspace K). Now the non-interacting words in $\bigcup A_\lambda$ form a right K-basis for P; hence the non-interacting words in $\bigcup B_\lambda$ are right k-independent and they span Q. Therefore Q is indeed the free product of the S_λ over k.

We note that the sufficient condition given in this lemma is asymmetric, hence we obtain another sufficient condition by requiring all the pairs K, S_λ (in that order) to be linearly disjoint over k.

Two further results are needed, one on field products, and one on free sets.

LEMMA 3.2. *Let K be a field with a subfield k, and let P be the field product of K and $k(x)$, where x is an indeterminate centralizing k. Then the subfield Q of P generated by the fields $K_i = x^{-i}Kx^i$ ($i \in \mathbf{Z}$) is their field product over k.*

PROOF. Take a family of copies of K indexed by $\mathbf{Z}$, say (K_i), and denote by U their field product over k. By the universal property of U it follows that Q is a specialization of U. From the universal property of $P = K \circ k(x)$, this specialization will be an isomorphism whenever there is *some* K-ring L containing an element $y \neq 0$ such that the specialization of U in L which maps $K_i \to y^iKy^{-i}$ is an embedding. Such an L is easily constructed:

$$\begin{tikzcd} U \arrow[d] \arrow[dr] & \\ P \arrow[r] & L = U(y;\alpha) \end{tikzcd}$$

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The mapping which takes K_i to K_{i+1} is an automorphism of $*K_i$ which extends to an automorphism α, say, of $U = \circ K_i$. We form the skew polynomial ring $U[y;\alpha]$ with commutation rule $ay = ya^{\alpha}$. This is an *Ore* ring and so has a field of fractions $L = U(y,\alpha)$ which has the desired properties.

LEMMA 3.3. *Let E be the field generated by a family (e_i) of elements over a central subfield k, and U a field freely generated by a family (u_i) over k, then the elements $u_i + e_i$ form a free set in the field product of U and E over k.*

PROOF. The field product $G = E \circ_k U$ has the following universal property: given any E-field F and any family (f_i) of elements of F, there is a unique specialization from G to F (over E, with domain generated by E and the f_i) which maps u_i to f_i. In particular, there are specializations from G to itself which map u_i to $u_i + e_i$ (resp. to $u_i - e_i$). On composing these mappings (in either order) we obtain the identity mapping, hence they are inverse to each other, and so are automorphisms. It follows that the $u_i + e_i$, like the u_i form a free set.

We can now prove our main result.

THEOREM 3.4. *Let E be a field, countably generated over a central subfield k. Then E can be embedded in a field on two generators over k.*

In essence the proof runs as follows: Suppose that E is generated by (e_i) $(i = 0, 1, \cdots)$, where $e_0 = 0$. We construct an extension field L generated over E by elements x, y, z satisfying

$$y^{-i}xy^i = z^{-i}xz^i + e_i.$$

Hence L is in fact generated by x, y, z alone. If we now adjoin t such that $y = txt^{-1}$, $z = t^{-1}xt$, the resulting field is generated by x and t.

To prove Theorem 3.4, let F_1 be the free field on x, y over k; it has a subfield U, generated by $u_i = y^{-i}xy^i (i = 0, 1, \cdots)$, freely by Lemma 3.2, and similarly, let F_2 be the free field on x, z over k, with subfield V freely generated by $v_i = z^{-i}xz^i$ $(i = 0, 1, \cdots)$.

Let K be the field product of E and F_1 over $k : K = E \circ F_1$. It has a subfield W generated by $w_i = u_i + e_i (i \geqq 0)$, freely by Lemma 3.3. We note that $w_0 = u_0 + e_0 = x_0 = x$, so K is generated by x, y and the $w_i (i \geqq 1)$ over k.

Let L be the field product of K and F_2, amalgamating W and V along the isomorphism $w_i \leftrightarrow v_i$. We note that $w_0 = x = v_0$ and that L is generated by x, y, z and the w_i, or x, y, z and the v_i, or simply by x, y, z. Now L contains the isomorphic subfields $k\langle x, y\rangle, k\langle z, x\rangle$, hence we can adjoin t to L such that $t^{-1}xt = z, t^{-1}yt = x$ (see [3]), Theorem 6.3). It follows that $L(t)$ so defined is generated by x and t over k. This completes the proof of Theorem 3.4.

As in [8] we have the

COROLLARY. *Every field can be embedded in a field L such that every countably generated subfield of L is contained in a 2-generator subfield of L.*

PROOF. Let E be the given field and E_λ a typical countably generated subfield (over a central subfield k, which could of course be the prime field), then there is a 2-generator field L_λ containing E_λ. Form the field product M_λ of E and L_λ over E_λ; if we do this for each countably generated subfield of E, we get a family (M_λ) of fields, all containing E. Form their field product E' amalgamating E; then in E' every countably generated subfield of E is contained in a 2-generator subfield of E', namely E_λ is contained in L_λ. Now repeat the process that led from E to E':

$$E \subseteq E' \subseteq E'' \subseteq \cdots \subseteq E^{\omega} \subseteq E^{\omega+1} \subseteq \cdots \subseteq E^{\nu},$$

where $E^\alpha = \bigcup \{E_\beta \mid \beta < \alpha\}$ at a limit ordinal α, and where ν is the first uncountable ordinal. Then E^ν is a field in which every countably generated subfield is contained in some $E^\alpha (\alpha < \nu)$ and hence in some 2-generator subfield of $E^{\alpha+1} \subseteq E^\nu$. This establishes the corollary.

It is not hard to determine the algebraic elements in a free product. Let $P = {}^*R_\lambda$ be a free product of a family of integral domains R_λ over a field k, then each R_λ is a faithfully flat inert extension of k, and hence, by Theorem 2.2 of [2], III, P is an integral domain. Suppose that $a \in P$ is right algebraic over k, i.e. it satisfies an equation

$$a^n\gamma_0 + a^{n-1}\gamma_1 + \cdots + \gamma_n = 0 \qquad (\gamma_i \in k, \text{ not all } 0). \tag{3}$$

Since P is an integral domain, we may assume that $\gamma_n \neq 0$. It follows that

$$a(a^{n-1}\gamma_0 + \cdots + \gamma_{n-1})(-\gamma_n^{-1}) = 1,$$

and hence a is a unit in P. Now any unit in P is a product of units in the factors R_λ(Theorem 2.2 of [2], III). If a is not in one of the R_λ, then by taking a suitable conjugate, we ensure that $h(a^r) = rh(a)$, where h is the height defined in [2], III. This clearly contradicts (3), hence a lies in some R_λ and we have proved

THEOREM 3.5. *Let $P = {}^*R_\lambda$ be a free product of (not necessarily commutative) integral domains R_λ over a field k. Then any element right (or left) algebraic over k is conjugate to an element in one of the factors.*

Of course it would be more interesting (and presumably also more difficult) to establish the analogue for field products. The conclusion of Theorem 3.5 fails when different subfields are amalgamated. This is shown by the example corresponding to Neumann's [8]: Let k be any commutative field and form the fields $K_1 = k(x, y)$, $K_2 = k(y, z)$, $K_3 = k(z, x)$ with defining relations $y^{-1}xy = x^{-1}$, $z^{-1}yz = y^{-1}$, $x^{-1}zx = z^{-1}$; it is clear how to construct such fields as fields of fractions of suitable skew polynomial rings. Their free product P exists, with amalgamations $K_{12} = k(y)$, $K_{23} = k(z)$, $K_{31} = k(x)$; this follows as for groups. However, in P, xyz is an element of order two: $xyz = yx^{-1}z = yz^{-1}x^{-1} = z^{-1}y^{-1}x^{-1}$. Thus P is not even an integral domain.

4. Algebraic extensions of skew fields

In order to be able to build a satisfactory theory of algebraic field extensions it is necessary to solve the following basic

PROBLEM. *Given any field K with a central subfield k algebraically closed in K find an extension field E of K such that any non-constant element p of $K_k^* k[x]$ vanishes for some value of x in E.*

Of course it is only necessary to find an extension in which a single equation has a root; then the familiar process will provide an extension in which all equations have solutions. However, there is no guarantee that such an extension is unique, nor that all its elements are algebraic over K. Now recently, Robinson [10] has outlined a construction which leads to the notion of an algebraically closed skew field; various people have observed that the class of algebraically closed fields constructed in this way constitutes precisely the class of elementary subfields of the homogeneous universal fields constructed by the methods of Jónsson [6, 7]. Both Jónsson's and Robinson's constructions depend on the fact, proved in [3], that skew fields possess the amalgamation property.

At present we are unable to answer the above problem except in the following special case.

THEOREM 4.1. *Let K be a field and $p \in K_k^* k[x]$ a non-constant polynomial, whose coefficients all lie in a commutative subfield F of K. Then there is an extension field E of K in which $p = 0$ has a root.*

PROOF. By hypothesis, $p \in F[x]$ and there is a commutative field G containing F in which p has a zero (e.g. the algebraic closure of F). Now take E to be the field product of K and G over F, then p has a zero in E.

By induction we can extend K to a field in which every polynomial with coefficient in some commutative subfield of K has a zero; in particular, in such a field every element has nth roots, for every integer n.

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