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Flops and spherical functors

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ABSTRACT

We study derived categories of Gorenstein varieties X and X^+ connected by a flop. We assume that the flopping contractions $f\colon X \to Y$, $f^+\colon X^+ \to Y$ have fibers of dimension bounded by one and Y has canonical hypersurface singularities of multiplicity two. We consider the fiber product $W = X \times_Y X^+$ with projections $p\colon W \to X$, $p^+\colon W \to X^+$ and prove that the flop functors $F = Rp^+_* Lp^*\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X^+)$, $F^+ = Rp_* Lp^{+*}\colon \mathcal{D}^b(X^+) \to \mathcal{D}^b(X)$ are equivalences, inverse to those constructed by Van den Bergh. The composite $F^+ \circ F\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ is a non-trivial auto-equivalence. When variety Y is affine, we present $F^+ \circ F$ as the spherical cotwist of a spherical couple (Ψ^*, Ψ) which involves a spherical functor Ψ constructed by deriving the inclusion of the null category $\mathscr{A}_f$ of sheaves $\mathcal{F} \in \mathrm{Coh}(X)$ with $Rf_*(\mathcal{F}) = 0$ into $\mathrm{Coh}(X)$. We construct a spherical pair $(\mathcal{D}^b(X), \mathcal{D}^b(X^+))$ in the quotient $\mathcal{D}^b(W)/\mathcal{K}^b$, where $\mathcal{K}^b$ is the common kernel of the derived push-forwards for the projections to X and X^+, thus implementing in geometric terms a schober for the flop. A technical innovation of the paper is the $L^1 f^* f_*$ vanishing for Van den Bergh's projective generator. We construct a projective generator in the null category and prove that its endomorphism algebra is the contraction algebra.

1. Introduction

A homological interpretation of the minimal model program (MMP) in birational geometry was proposed in [BO95, BO02]. The basic idea is that MMP is about 'minimisation' of the derived category $\mathcal{D}^b(X)$ of coherent sheaves on a variety X for varieties in a given birational class. More precisely, it is expected that if X allows a divisorial contraction or a flip $X \dashrightarrow Y$, then $\mathcal{D}^b(X)$ has a semi-orthogonal decomposition (SOD) with one component of SOD equivalent to $\mathcal{D}^b(Y)$. Thus, minimizing the birational model should have the categorical meaning of chopping off semi-orthogonal factors of the derived category. A minimal model is expected to be a representative in the birational class of a variety whose derived category does not allow semi-orthogonal factors equivalent to the derived category of a variety birationally equivalent to X.

As MMP in dimension greater than two deals with singular varieties, the right choice of the derived category to consider also matters. In particular, for $\mathbb{Q}$-Gorenstein varieties the derived category of a suitable stack is relevant (cf. [Kaw02, Kaw09]). Minimal models are not unique, and MMP considers birational maps, called flops, that link various minimal models. Conjecturally, flops induce derived equivalences [BO95, Kaw02, BO02].

There is substantial evidence in favour of this conjectural picture, starting from the original paper [BO95], where various instances of flops were proved to induce derived equivalences, and for simple higher-dimensional flips, the required SOD was constructed.

We expect that the whole zoology of categories, functors and natural transformations relevant to MMP should be governed by interesting hidden homotopy types, maps and higher homotopies. The present work can be considered as taking steps in this direction for categories and functors invoked by flops.

It was mentioned by the authors of [BO95] that the functor that provides an equivalence $\mathcal{D}^b(X) \to \mathcal{D}^b(X^+)$, where X and X^+ are connected by a flop, when composed with the analogous functor in the opposite direction $\mathcal{D}^b(X^+) \to \mathcal{D}^b(X)$ is not the identity but produces a non-trivial auto-equivalence of $\mathcal{D}^b(X)$ (nowadays called a flop–flop functor). For Atiyah flop, the functor is given by what is now known as the spherical twist with respect to the spherical object $\mathcal{O}_C(-1)$, where C is the (rational) exceptional curve. It was probably one of the first appearances of the spherical twists. (However, we should also mention here the work of Mukai [Muk87] and Kuleshov [Kul90] who used the action of spherical twists in their non-derived version to describe moduli of sheaves on K3 surfaces. A quick generalisation to the Calabi–Yau/derived case was understood by the authors of [BO95] in those days of *sturm und drang* on derived categories in Moscow in the early 1990s). Kontsevich suggested that the spherical twist of Atiyah flop should be transferred by mirror symmetry into the equivalence of the Fukaya category induced by the Dehn twist along a vanishing cycle. Properties of spherical twists with respect to spherical objects were later scrutinised by Seidel and Thomas in [ST01] and for more general spherical functors by Anno and Logvinenko in [AL17].

Spherical (co)twist is a unification tool for various non-trivial auto-equivalences of $\mathcal{D}^b(X)$ (cf. [Add16]) such as tensor products with line bundles, twists around spherical objects [ST01], EZ-twists [Hor05] or window shifts [DS14].

The homotopy meaning of spherical (co)twists can be read off from their interpretation via schobers, i.e. categorifications of perverse sheaves on stratified topological spaces, suggested by Kapranov and Schechtman [KS14]. The homotopy type of the underlying stratification (the punctured disc for the case of one spherical functor) encodes the algebra of functors and natural transformations in the schober. The schober on the punctured disc has one of possible incarnations via a spherical pair, i.e. a pair of admissible subcategories in a triangulated category satisfying conditions that imply a spherical functor between them.

We study functors and natural transformations for flops with dimension of fibers of the flopping contractions bounded by one. We construct the spherical pair $(\mathcal{D}^b(X), \mathcal{D}^b(X^+))$ in the appropriate quotient of the derived category of $X \times_Y X^+$. The orthogonal complement to $\mathcal{D}^b(X)$ in that quotient is the bounded derived category of the abelian null category

$$\mathscr{A}_f = \{E \in \mathrm{Coh}(X) \mid Rf_*E = 0\}. \tag{1}$$

By deriving the embedding $\mathcal{A}_f \to \mathrm{Coh}(X)$ we obtain the spherical functor $\Psi \colon \mathcal{D}^b(\mathscr{A}_f) \to \mathcal{D}^b(X)$. The spherical cotwist of the *spherical couple* (see Appendix C) (Ψ^*, Ψ) is the flop–flop functor.

We lift our functors and natural transformations to bicategories **Bimod** and **FM** (see Appendix C). We systematically consider 2-categorical adjunctions instead of the usual adjunctions of functors, scrutinise the uniqueness of adjoints and of associated twists and cotwists, i.e. the cones of adjunction units and counits.

Now we describe our results in more detail.

The flop functor and Van den Bergh's functor

We consider a flopping contraction $f : X \to Y$ and its flop $f^+ : X^+ \to Y$ with dimension of fibers bounded by one. Exceptional loci of f and f^+ are assumed to have codimension greater than one in X, respectively in X^+, whereas varieties X, X^+ and Y are assumed to be Gorenstein and Y is assumed to have canonical hypersurface singularities of multiplicity two.

Consider the diagram for the fiber product:

$$
\begin{array}{ccc}
 & X \times_Y X^+ & \\
{}^{p}\swarrow & & \searrow{}^{p^+} \\
X & & X^+ \\
{}_{f}\searrow & & \swarrow{}_{f^+} \\
 & Y &
\end{array}
$$
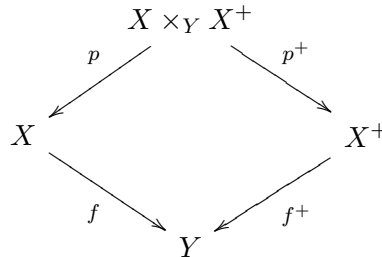
where p and p^+ denote the projections to the factors. The flop functor is

$$F = Rp_*^+ Lp^* : \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(X^+).$$

Note that this functor does not necessarily induce an equivalence if the dimension of fibers of f is greater than one (see [Nam04]), which means that the functor needs an adjustment for fibers of higher dimension.

The above assumptions on the flopping contraction are those adopted by Van den Bergh in [VdB04]. We use also some techniques borrowed from his work. Van den Bergh constructed an equivalence $\Sigma : \mathcal{D}^b(X) \xrightarrow{\simeq} \mathcal{D}^b(X^+)$, which is given via the identification of both categories with the derived category of modules over a sheaf of non-commutative algebras on affine Y. We first give a new interpretation for functor Σ. To this end, we identify $\mathcal{D}^b(X)$ with $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$, the homotopy category of complexes of projective objects $\mathscr{P}_{-1}$ in the heart ${}^{-1}\mathrm{Per}(X/Y)$ of the perverse t-structure, introduced by Bridgeland in [Bri02]. We show that Σ can be defined as the functor $(f^{+*}f_*(-))^{\vee\vee}$ applied term-wise to complexes of objects in $\mathscr{P}_{-1}$. We give an independent proof that the flop functor is an equivalence and that it can be extended to an equivalence Σ_{qc} between unbounded categories of quasi-coherent sheaves (Proposition 4.6).

Then we show that the flop functor is given in the same way by the term-wise application of the functor $f^{+*}f_*(-)$. To this end, we need one of the technical innovations of our paper, the $L^1 f^* f_*$-*vanishing* for objects in $\mathscr{P}_{-1}$. We prove (Lemma 3.4) that if $f : X \to Y$ is a flopping contraction satisfying the above conditions, then $L^1 f^* f_* \mathcal{M}$ is zero, for any $\mathcal{M} \in \mathscr{P}_{-1}$. The proof of the vanishing is based on a local presentation of Y as a divisor in a smooth variety $\mathcal{Y}$. Note that, since Y is singular, object $Lf^* f_* \mathcal{M}$ in general has infinitely many non-zero cohomology sheaves, more precisely, they satisfy 2-periodicity (see § 3.5), which is reminiscent of matrix factorisations [Buc86, Orl04].

We consider a divisorial embedding $i : Y \to \mathcal{Y}$ into a smooth $\mathcal{Y}$ together with $g = i \circ f$, $g^+ = i \circ f^+$. If the base Y of the flopping contraction is affine, then the flop functor F and Van den Bergh's equivalence $\Sigma_{qc} : \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(X^+)$ fit into a functorial exact triangle (Proposition 4.13):

$$\Sigma_{qc}[1] \to Lg^{+*} Rg_* \to F \to \Sigma_{qc}[2]. \tag{2}$$

As both $Lg^{+*} Rg_*$ and Σ_{qc} take $\mathcal{D}^b(X)$ to $\mathcal{D}^b(X^+)$, this allows us to conclude that the flop functor also preserves the boundedness of the derived categories.

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Theorem 4.21 states that Σ_{qc} is actually the inverse of the opposite flop functor

$$F^+ = Rp_* Lp^{+*}\colon \mathcal{D}_{qc}(X^+) \to \mathcal{D}_{qc}(X).$$

This implies that flop functors F and F^+ yield (not mutually inverse) equivalences between $\mathcal{D}^b(X)$ and $\mathcal{D}^b(X^+)$. Following an argument of Chen [Che02], we generalise this statement to the case when base Y is quasi-projective (see § 4.6).

In addition, we show that the composite $F^+ F$, the 'flop–flop' functor, is the term-wise extension of $f^* f_*(-)|_{\mathscr{P}_{-1}}$.

The null category and spherical functor Ψ

One of the original motivations for our work was to recover the importance of the null category for f as in (1). Category $\mathscr{A}_f$ admits a projective generator $\mathcal{P}$, which we obtain from Van den Bergh's projective generator $\mathcal{M}$ for the perverse heart $^{-1}\mathrm{Per}(X/Y)$ by a projection to the null category (see Proposition 2.4). Deriving the embedding $\mathscr{A}_f \to \mathrm{Coh}(X)$ gives us functor

$$\Psi\colon \mathcal{D}^b(\mathscr{A}_f) \to \mathcal{D}^b(X).$$

We prove that, for a flopping contraction $f\colon X \to Y$ with affine Y, functor Ψ is spherical. The flop–flop functor $F^+ F$ is the spherical cotwist of the spherical couple (Ψ^*, Ψ), i.e. the $\Psi^* \dashv \Psi$ adjunction unit fits into a functorial exact triangle:

$$F^+ F \to \mathrm{Id}_{\mathcal{D}^b(X)} \to \Psi\Psi^* \to F^+ F[1].$$

We also show that the spherical twist $\mathcal{D}^b(\mathscr{A}_f) \to \mathcal{D}^b(\mathscr{A}_f)$ of the couple (Ψ^*, Ψ) is the shifted identity functor $\mathrm{Id}_{\mathcal{D}^b(\mathscr{A}_f)}[4]$ (see Corollary 5.18).

Spherical pairs

We assume again $f\colon X \to Y$ to be a flopping contraction with affine Y. Functor p^* induces an isomorphism of endomorphism algebras of a projective generator $\mathcal{P} \in \mathscr{A}_f$ and of $p^*\mathcal{P} \in \mathrm{Coh}(X \times_Y X^+)$. Moreover, $\mathrm{Ext}^i_{X \times_Y X^+}(p^*\mathcal{P}, p^*\mathcal{P}) = 0$, for $i > 0$ (see Proposition 5.3). Thus, we construct a fully faithful functor $\mathcal{D}(\mathscr{A}_f) \to \mathcal{D}_{qc}(X \times_Y X^+)$.

We consider the 'common kernel subcategories':

$$\mathcal{K}^b = \{E \in \mathcal{D}^b(X \times_Y X^+) \,|\, Rp_*(E) = 0, \, Rp_*^+(E) = 0\},$$
$$\mathcal{K}^- = \{E \in \mathcal{D}^-(X \times_Y X^+) \,|\, Rp_*(E) = 0, \, Rp_*^+(E) = 0\}.$$

The composite $\mathcal{D}^b(X) \xrightarrow{Lp^*} \mathcal{D}^-(X \times_Y X^+) \to \mathcal{D}^-(X \times_Y X^+)/\mathcal{K}^-$ (not Lp^* itself) factors via $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$, thus inducing a fully faithful functor (Proposition 5.8)

$$\widetilde{Lp}^*\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b.$$

We prove the existence of SODs (see [Bon89]):

$$\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b = \langle \mathcal{D}^b(\mathscr{A}_{f^+}), \quad \widetilde{Lp}^*\mathcal{D}^b(X)\rangle = \langle \widetilde{Lp}^*\mathcal{D}^b(X), \mathcal{D}^b(\mathscr{A}_f)\rangle. \qquad (3)$$

As a result, we obtain a geometric description of the category $\mathcal{D}^b(\mathscr{A}_{f^+})$:

$$\mathcal{D}^b(\mathscr{A}_{f^+}) \simeq \{E \in \mathcal{D}^b(X \times_Y X^+) \,|\, Rp_*(E) = 0\}/\mathcal{K}^b.$$

By exchanging the roles of X and X^+ in (3) we obtain SODs

$$\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b = \langle \mathcal{D}^b(\mathscr{A}_f), \quad \widetilde{Lp}^{+*}\mathcal{D}^b(X^+)\rangle = \langle \widetilde{Lp}^{+*}\mathcal{D}^b(X^+), \mathcal{D}^b(\mathscr{A}_{f^+})\rangle. \qquad (4)$$

Together decompositions (3) and (4) provide us with a geometric description of *4-periodical SODs* (see Proposition B.3), whose relation to spherical functors was basically discovered by

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Halpern-Leistner and Shipman [HS16] (we thank Kapranov for explanations on this). Two pairs of subcategories $(\mathcal{D}^b(X), \mathcal{D}^b(X^+))$, $(\mathcal{D}^b(\mathscr{A}_f), \mathcal{D}^b(\mathscr{A}_{f+}))$ are *spherical pairs* [KS14] (see Theorem 5.17). The corresponding spherical functor for the second spherical pair is Ψ.

The contraction algebra

Assume the base Y of the flopping contraction f to be the spectrum of a complete Noetherian local ring. Then the reduced fiber of f over the unique closed point of Y is a union of n smooth irreducible rational curves $C_1, \ldots, C_n$ (see Theorem D.1). The category $^{-1}\mathrm{Per}(X/Y)$ has $n+1$ irreducible projective objects $\mathcal{M}_0, \ldots, \mathcal{M}_n$, with $\mathcal{M}_0 \simeq \mathcal{O}_X$ (see [VdB04]). $\mathcal{M} = \bigoplus_{i=0}^n \mathcal{M}_i$ is a projective generator for $^{-1}\mathrm{Per}(X/Y)$.

We prove that the endomorphism algebra

$$A_P = \mathrm{Hom}_{\mathscr{A}_f}(\mathcal{P}, \mathcal{P})$$

of the corresponding projective generator for $\mathscr{A}_f$, $\mathcal{P} = \ker(f^* f_* \mathcal{M} \to \mathcal{M})$, is isomorphic to the *contraction algebra* introduced in [IW14], which is defined as the quotient of $\mathrm{Hom}_X(\mathcal{M}, \mathcal{M})$ by the ideal of morphisms that factor via direct sums of copies of $\mathcal{O}_X$ (Theorem 6.2). This theorem relates our work to the results of Donovan and Wemyss [DW16, DW19], where contraction algebras appear in the context of non-commutative deformations.

Appendices

There are five appendices attached to the main body of the paper.

Appendix A. This appendix is an extract of some properties of the functor $f^!$, right dual to Rf_*, and the Grothendieck duality.

Appendix B. In this appendix we introduce functorial exact triangles and use them to define spherical functors. We recall after [KS14] the notion of a spherical pair and the associated spherical functor. 4-periodical SODs introduced by Halpern-Leistner and Shipman [HS16] produce spherical pairs.

Appendix C. In this appendix we discuss a bicategory $\mathcal{C}$ and a pair of 1-morphisms $s \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $r \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ that fit into a 2-categorical adjunction $(s, r, \eta, \varepsilon)$. When $\mathcal{C}$ is 1-triangulated (meaning the categories of 1-morphisms are triangulated), we define the twist $t_s \in \mathrm{Hom}_{\mathcal{C}}(B, B)$ and the cotwist $c_s \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ as the cones of the counit $\varepsilon \colon sr \to \mathrm{Id}_B$ and the unit $\eta \colon \mathrm{Id}_A \to rs$. By using pseudo-functors and 2-categorical equivalences we show that the twist and cotwist are in a suitable sense invariant under replacing A and B by equivalent objects. If t_s and c_s are invertible in the 2-categorical sense, we say that $(s, r, \eta, \varepsilon)$ is a *spherical couple*.

We lift exact functors between triangulated categories to 1-morphisms in appropriate 1-triangulated bicategories. The first bicategory is **Bimod** whose objects are DG algebras and categories of 1-morphisms are defined as the derived categories of DG bimodules. The second one is the Fourier–Mukai bicategory **FM** of schemes and the derived categories of coherent sheaves on their products as categories of 1-morphisms. Both bicategories admit 2-functors to the bicategory **Cat** of categories, functors and natural transformations. More precisely, we have $\Phi \colon \mathbf{Bimod} \to \mathbf{Cat}$, $\Phi(A) = \mathcal{D}(A)$ and $\Xi \colon \mathbf{FM} \to \mathbf{Cat}$, $\Xi(X) = \mathcal{D}_{\mathrm{qc}}(X)$.

The above theory of 2-categorical adjunctions ensures that once a lift of an exact functor $\mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(Y)$ to a 1-morphism in **Bimod** or **FM** admitting an adjoint is fixed, we obtain essentially unique exact functors corresponding to the twist and the cotwist.

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We describe 2-categorical adjunctions in **Bimod** using formulae for adjoint bimodules as in [AL21]. For $\mathcal{E} \in \mathcal{D}_{qc}(X \times Y)$ we discuss functors adjoint to $\Xi_\mathcal{E} \colon \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(Y)$ and the conditions under which they are FM functors. Results of [LS16] allow us to transfer between **Bimod** and **FM** via fixing compact generators for $\mathcal{D}_{qc}(X)$ and $\mathcal{D}_{qc}(Y)$. We use the 2-categorical adjunctions in **Bimod** to construct functorial exact triangles for $\Xi_\mathcal{E}$ and its adjoints. We check that these triangles are, up to isomorphism, independent of the choice of compact generators. In particular, given a morphism $f \colon X \to Y$, we discuss the lift of Rf_*, its adjoints, and the adjunction (co)units to **Bimod** and **FM**. We also describe a 2-morphism in **Bimod** whose image under Φ is the base-change morphism.

Appendix D. This appendix is devoted to the description of the reduced fiber of a flopping contraction $f \colon X \to Y$ with fibers of dimension bounded by one over a closed point of Y.

Appendix E. In this appendix we show that cohomology of an appropriate complex allow us to calculate morphisms in the derived category of an abelian category between bounded above complexes with bounded cohomology.

Notation

We denote by k an algebraically closed field of characteristic zero. For a Noetherian k scheme X, by $\mathrm{Coh}(X)$, respectively $\mathrm{QCoh}(X)$, we denote the category of coherent, respectively quasi-coherent, sheaves on X.

For an abelian category $\mathcal{A}$, we denote by $\mathcal{D}^b(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ the bounded and unbounded derived categories of $\mathcal{A}$. We write $\mathcal{D}^b(X) = \mathcal{D}^b(\mathrm{Coh}(X))$, $\mathcal{D}_{qc}(X) = \mathcal{D}(\mathrm{QCoh}(X))$.

For an abelian category $\mathcal{A}$, by $\mathrm{Perf}(\mathcal{A})$ we denote the full subcategory of $\mathcal{D}^b(\mathcal{A})$ of objects that are quasi-isomorphic to finite complexes of projective objects in $\mathcal{A}$. For a scheme X, by $\mathrm{Perf}(X)$ we denote the category of perfect complexes on X, i.e. objects of $\mathcal{D}^b(X)$ locally quasi-isomorphic to finite complexes of locally free sheaves.

For a k-algebra A, we denote by Mod–A the abelian category of right A modules.

For a t-structure $(\mathcal{T}_{\leqslant 0}, \mathcal{T}_{\geqslant 0})$ on a triangulated category $\mathcal{T}$ with heart $\mathcal{A} = \mathcal{T}_{\leqslant 0} \cap \mathcal{T}_{\geqslant 0}$ and an object $T \in \mathcal{T}$, we denote by $\mathcal{H}_\mathcal{A}^i(T)$ the ith cohomology of T with respect to the t-structure $(\mathcal{T}_{\leqslant 0}, \mathcal{T}_{\geqslant 0})$. The truncation functors are denoted by $\tau_{\geqslant i}^\mathcal{A}$ and $\tau_{\leqslant i}^\mathcal{A}$. If $\mathcal{T} = \mathcal{D}_{qc}(X)$ or $\mathcal{D}^b(X)$ with the standard t-structure with heart $\mathrm{QCoh}(X)$, respectively $\mathrm{Coh}(X)$, we shorten the notation to $\mathcal{H}_X^i(T)$, $\tau_{\geqslant i}^X$ and $\tau_{\leqslant i}^X$ respectively.

Assumptions

Throughout the paper we assume X and Y to be normal varieties over k. We usually work under one of the following assumptions on a morphism $f \colon X \to Y$.

(a) We assume $f \colon X \to Y$ is a proper morphism with **d**imension of fibers bounded by one and such that $Rf_*\mathcal{O}_X \simeq \mathcal{O}_Y$.

(b) We assume $f \colon X \to Y$ is a projective birational morphism with dimension of fibers bounded by one between quasi-**p**rojective Gorenstein varieties of dimension $n \geqslant 3$. The exceptional locus of f is of codimension greater than one in X. Variety Y has canonical hypersurface singularities of multiplicity two.

(c) We assume the same as in assumption (p) and we further assume that variety Y is **a**ffine and is embedded as a principal divisor into a smooth variety $\mathcal{Y}$ of dimension $n + 1$.

(d) We assume the same as in assumption (p) with an extra assumption that $Y = \mathrm{Spec}\, R$, where R is a **c**omplete local k-algebra.

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Let f satisfy assumption (p). As Y has canonical singularities, $Rf_*\mathcal{O}_X \simeq \mathcal{O}_Y$ (cf. [Elk81]), i.e. assumption (d) is satisfied when Y has rational singularities and $f\colon X \to Y$ is a smooth birational resolution.

2. The null category $\mathscr{A}_f$

Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes. In this section we introduce the *null category* of f:

$$\mathscr{A}_f = \{E \in \mathrm{Coh}(X) \,|\, Rf_*(E) = 0\}. \tag{5}$$

Under the assumption that the dimension of fibers of f is bounded by one, Y is affine and $Rf_*\mathcal{O}_X = \mathcal{O}_Y$, we construct a projective generator $\mathcal{P}$ for $\mathscr{A}_f$. We also study the behaviour of the null category under decomposition $f = h \circ g$ and under restriction to fibers.

2.1 The triangulated and abelian null category of a morphism of schemes

For a proper morphism of Noetherian schemes $f\colon X \to Y$, we consider the triangulated null category $\mathcal{C}_f$ defined as the kernel of functor Rf_* restricted to $\mathcal{D}^b(X)$:

$$\mathcal{C}_f = \{E^{\textbf{·}} \in \mathcal{D}^b(X) \,|\, Rf_*(E^{\textbf{·}}) = 0\}. \tag{6}$$

Similarly, we define $\mathcal{C}_f^- \subset \mathcal{D}^-(X)$ and $\mathcal{C}_{f\,\mathrm{qc}} \subset \mathcal{D}_{\mathrm{qc}}(X)$. Denote by $\iota_{f_*}\colon \mathcal{C}_{f\,\mathrm{qc}} \to \mathcal{D}_{\mathrm{qc}}(X)$ the inclusion functor. As $Lf^*\colon \mathcal{D}_{\mathrm{qc}}(Y) \to \mathcal{D}_{\mathrm{qc}}(X)$ is fully faithful with right adjoint Rf_*, $\mathcal{D}_{\mathrm{qc}}(X)$ admits a SOD $\mathcal{D}_{\mathrm{qc}}(X) = \langle \mathcal{C}_{f\,\mathrm{qc}}, Lf^*\mathcal{D}_{\mathrm{qc}}(Y)\rangle$, [Bon89, Lemma 3.1]. In particular, ι_{f_*} admits a left adjoint $\iota_f^*\colon \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{C}_{f\,\mathrm{qc}}$, which fits into a functorial exact triangle (see (B.1)):

$$Lf^*Rf_* \to \mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(X)} \to \iota_{f_*}\iota_f^* \to Lf^*Rf_*[1]. \tag{7}$$

For the null category of f as in (5), we have $\mathscr{A}_f = \mathrm{Coh}(X) \cap \mathcal{C}_f$.

LEMMA 2.1. *Let $f\colon X \to Y$ be a proper morphism with dimension of fibers bounded by one. Then category $\mathscr{A}_f$ is abelian. The embedding functor $\mathscr{A}_f \to \mathrm{Coh}(X)$ is exact and fully faithful. Its image is closed under extensions.*

Proof. Let A, B be objects in $\mathscr{A}_f$ and let $\varphi \in \mathrm{Hom}_X(A, B)$. Consider the kernel K, the cokernel C, and the image I of φ. As $Rf_*A \simeq 0 \simeq Rf_*B$, the long exact sequences of higher derived functors for f_* applied to short exact sequences on X

$$0 \to K \to A \to I \to 0, \quad 0 \to I \to B \to C \to 0,$$

together with vanishing of $R^i f_*$, for $i > 1$, imply that first I, hence K and C lie in $\mathscr{A}_f$. That $\mathscr{A}_f$ is closed under extensions is obvious. $\qquad\square$

It follows that $\mathrm{Ext}^1_X(A, B) = \mathrm{Ext}^1_{\mathscr{A}_f}(A, B)$, for $A, B \in \mathscr{A}_f$.

LEMMA 2.2 [Bri02, Lemma 3.1]. *Let X and Y be Noetherian schemes and $f\colon X \to Y$ a proper morphism with fibers of dimension bounded by one. Then $E \in \mathcal{C}_f$ if and only if $\mathcal{H}^i_X(E) \in \mathscr{A}_f$, for all $i \in \mathbb{Z}$. In particular, $\mathscr{A}_f \subset \mathcal{C}_f$ is the heart of a bounded t-structure.*

Remark 2.3. The restriction of the standard t-structure on $\mathcal{D}_{\mathrm{qc}}(X)$ to

$$\mathcal{C}_{f\,\mathrm{qc}} := \{E^{\textbf{·}} \in \mathcal{D}_{\mathrm{qc}}(X) \,|\, Rf_*(E^{\textbf{·}}) = 0\}$$

was considered by Bridgeland [Bri02] for a birational morphism of projective varieties satisfying assumption (d) which is easily generalised to a projective morphism $f\colon X \to Y$ satisfying assumption (d). In this case, category $\mathcal{D}_{\mathrm{qc}}(X)$ admits a recollement [BBD82] or, equivalently, an

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admissible subcategory, $\mathcal{C}_{f\,\mathrm{qc}}$ (see [Bon89]). The Verdier quotient $\mathcal{D}_{\mathrm{qc}}(X)/\mathcal{C}_{f\,\mathrm{qc}}$ is identified with $\mathcal{D}_{\mathrm{qc}}(Y)$. The t-structure on $\mathcal{C}_{f\,\mathrm{qc}}$ with heart $\mathscr{A}_f[-p]$ can be glued with the standard t-structure on $\mathcal{D}_{\mathrm{qc}}(Y)$ (see [BBD82]). As functor $f^!$ is needed for the gluing of t-structures, one has to consider unbounded derived categories of quasi-coherent sheaves on X and Y. The heart of the resulting t-structure is the category ${}^p\mathrm{Per}_{\mathrm{qc}}(X/Y)$ of perverse sheaves. If $p=-1$ or $p=0$, the t-structures with hearts ${}^p\mathrm{Per}_{\mathrm{qc}}(X/Y)$ can also be obtained by the tilting in torsion pairs $(\mathcal{T}_{-1}, \mathcal{F}_{-1})$, $(\mathcal{T}_0, \mathcal{F}_0)$ in $\mathrm{QCoh(X)}$ (see [VdB04]), where

$$\mathcal{T}_0 = \{T \in \mathrm{Coh}(X) \,|\, R^1 f_*(T) = 0\}, \tag{8}$$

$$\mathcal{F}_0 = \{E \in \mathrm{Coh}(X) \,|\, f_*(E) = 0, \, \mathrm{Hom}(\mathscr{A}_f, E) = 0\}, \tag{9}$$

$$\mathcal{T}_{-1} = \{T \in \mathrm{Coh}(X) \,|\, R^1 f_*(T) = 0, \, \mathrm{Hom}(T, \mathscr{A}_f) = 0\}, \tag{10}$$

$$\mathcal{F}_{-1} = \{E \in \mathrm{Coh}(X) \,|\, f_*(E) = 0\}. \tag{11}$$

By restricting the torsion pairs to $\mathrm{Coh}(X)$ one can define abelian categories ${}^p\mathrm{Per}(X/Y)$ as subcategories of $\mathcal{D}^b(X)$. Thus, for $p = -1, 0$,

$${}^p\mathrm{Per}(X/Y) := \{E \in \mathcal{D}^b(X) \,|\, \mathcal{H}^0(E) \in \mathcal{T}_p, \, \mathcal{H}^{-1}(E) \in \mathcal{F}_p \text{ and } \mathcal{H}^i(E) = 0, \text{ for } i \neq -1, 0\}.$$

2.2 A projective generator for $\mathscr{A}_f$

Let $f \colon X \to Y$ satisfy assumption (d) and assume Y is affine. By [VdB04, Proposition 3.2.5], there exists a vector bundle $\mathcal{M}$ on X which is a projective generator for ${}^{-1}\mathrm{Per}(X/Y)$. Consider

$$\mathcal{P} := \mathcal{H}_X^{-1} \iota_f^* \mathcal{M}. \tag{12}$$

It is an object in $\mathscr{A}_f$, which, by abuse of notation, we identify with its image under ι_{f*}.

PROPOSITION 2.4. *Sheaf $\mathcal{P}$ is a projective generator for the category $\mathscr{A}_f$.*

Proof. As the t-structure on $\mathcal{C}_{f\,\mathrm{qc}}$ is defined as the restriction of the standard t-structure on $\mathcal{D}_{\mathrm{qc}}(X)$, the functor $\iota_{f*} \colon \mathcal{C}_{f\,\mathrm{qc}} \to \mathcal{D}_{\mathrm{qc}}(X)$ is exact, in particular, it commutes with the truncation functors $\tau_{\geqslant i}$. This implies isomorphism of functors

$$\tau_{\geqslant -1}^X \iota_{f*} \simeq \iota_{f*} \tau_{\geqslant -1}^{\mathscr{A}_f}.$$

It follows that $\tau_{\geqslant -1}^X \iota_{f*} \iota_f^* \mathcal{M}$ is isomorphic to $\iota_{f*} \circ {}^{-1}\iota_f^* \mathcal{M}$, where

$${}^{-1}\iota_f^* := \mathcal{H}_{\mathscr{A}_f}^{-1} \circ \iota_f^* \colon {}^{-1}\mathrm{Per}(X/Y) \to \mathscr{A}_f[1]$$

is the left adjoint functor to the inclusion $\mathscr{A}_f[1] \to {}^{-1}\mathrm{Per}(X/Y)$. By Lemma 2.5, $\mathcal{P} \in \mathscr{A}_f$ is a projective generator. $\square$

LEMMA 2.5. *Let $\mathcal{D}_0$ and $\mathcal{D}$ be triangulated categories with t-structures with hearts $\mathcal{A}_0 \subset \mathcal{D}_0$ and $\mathcal{A} \subset \mathcal{D}$, and $i_* \colon \mathcal{D}_0 \to \mathcal{D}$ a t-exact functor with left adjoint i^*. If $\mathcal{M} \in \mathcal{A}$ is projective, then $\mathcal{P} := \mathcal{H}^0(i^* \mathcal{M}) \in \mathcal{A}_0$ is projective. Moreover, if $\mathcal{M} \in \mathcal{A}$ is a projective generator and i_* is fully faithful, then $\mathcal{P} \in \mathcal{A}_0$ is a projective generator.*

Proof. Let E be an object in $\mathcal{A}_0$. Then $i_* E \in \mathcal{A}$, hence $0 = \mathrm{Ext}_{\mathcal{D}}^1(\mathcal{M}, i_* E) \simeq \mathrm{Ext}_{\mathcal{D}_0}^1(i^* \mathcal{M}, E)$. As functor i^* is left adjoint to a t-exact functor, it is right t-exact, i.e. $i^* \mathcal{M} \in \mathcal{D}_0^{\leqslant 0}$. Thus, we have an exact triangle

$$\tau_{\leqslant -1} i^* \mathcal{M} \to i^* \mathcal{M} \to \mathcal{P} \to \tau_{\leqslant -1} i^* \mathcal{M}[1]. \tag{13}$$

As, for degree reasons, $\mathrm{Hom}_{\mathcal{D}_0}(\tau_{\leqslant -1} i^* M, E) = 0$, by applying $\mathrm{Hom}_{\mathcal{D}_0}(-, E)$ to (13), we get that $\mathrm{Ext}_{\mathcal{D}_0}^1(\mathcal{P}, E) = 0$, i.e. $\mathcal{P} \in \mathcal{A}_0$ is projective.

If $\mathcal{M} \in \mathcal{A}$ is a projective generator and i_* is fully faithful, then, for any non-zero $E \in \mathcal{A}_0$, object $i_* E$ is non-zero, hence $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, i_* E) \simeq \mathrm{Hom}_{\mathcal{D}_0}(i^* \mathcal{M}, E) \neq 0$. It follows from the long exact sequence obtained by applying $\mathrm{Hom}_{\mathcal{D}_0}(-, E)$ to (13) that $\mathrm{Hom}_{\mathcal{D}_0}(\mathcal{P}, E) \neq 0$, i.e. $\mathcal{P}$ is a projective generator for $\mathcal{A}_0$. $\square$

Remark 2.6. If morphism f satisfies assumption (d) and Y is affine, the category ${}^0\mathrm{Per}(X/Y)$ has a projective generator $\mathcal{N} = \mathcal{M}^\vee$ (see [VdB04, Proposition 3.2.5]). An argument analogous to that in the proof of Proposition 2.4 shows that $\mathcal{P}_\mathcal{N} := \mathcal{H}_X^0 \iota_f^* \mathcal{N}$ is a projective generator for $\mathscr{A}_f$.

For a sheaf F with $R^1 f_* F = 0$, the cohomology sequence of triangle (7) applied to F gives an exact sequence

$$0 \to \mathcal{H}_X^{-1}(\iota_{f_*} \iota_f^* F) \to f^* f_* F \to F \to \mathcal{H}_X^0(\iota_{f_*} \iota_f^* F) \to 0. \tag{14}$$

As by [VdB04, Lemma 3.1.3] morphism $f^* f_* \mathcal{M} \to \mathcal{M}$ is surjective, sequence

$$0 \to \mathcal{P} \to f^* f_* \mathcal{M} \to \mathcal{M} \to 0 \tag{15}$$

is exact in $\mathrm{Coh}(X)$.

Let $f \colon X \to Y$ satisfy assumption (d) and $Y = \mathrm{Spec}\, R$ be a spectrum of a complete Noetherian local ring R. The reduced fiber $C_{\mathrm{red}} = \bigcup_{i=1}^n C_i$ of f over the unique closed point $y \in Y$ is a tree of rational curves (see a more precise statement in Theorem D.1).

The Picard group of X is isomorphic to $\mathbb{Z}^n$, where the isomorphism is given by the degrees of the restriction to irreducible components of C_{red}: $\mathcal{L} \mapsto \deg(\mathcal{L}|_{C_i})_{i=1,\dots,n}$.

Remark 2.7 (Cf. [VdB04, Lemma 3.4.4]). Let $x_i \in C_i \subset X$ be a closed point such that $x_i \notin C_k$, for any $k \neq i$, and $j_i \colon X_i \to \mathcal{X}_i$ a closed embedding of a neighbourhood X_i of x_i in X into a smooth variety $\mathcal{X}_i$. There exists an effective Cartier divisor $\mathcal{D}_i \subset \mathcal{X}_i$ such that scheme-theoretically $\mathcal{D}_i \cap j_{i*} C_i = \{j_{i*} x_i\}$. By pulling back $\mathcal{D}_i$ to X_i, we obtain an effective divisor $D_i \subset X$ such that scheme-theoretically $D_i.C_i = \{x_i\}$ and $D_i.C_k = 0$, for $k \neq i$. We denote by $\iota_{D_i} \colon D_i \to X$ the embedding of D_i into X.

Denote by $\mathcal{L}_i$ the line bundle on X defined by

$$\mathcal{L}_i := \mathcal{O}_X(D_i). \tag{16}$$

For every i, Van den Bergh defined a vector bundle $\mathcal{M}_i$ via the exact sequence

$$0 \to \mathcal{O}_X^{r_i-1} \to \mathcal{M}_i \to \mathcal{L}_i \to 0 \tag{17}$$

corresponding to a choice of $r_i - 1$ generators of $\mathrm{Ext}_X^1(\mathcal{L}_i, \mathcal{O}_X)$ as an R-module. Denote $\mathcal{M}_0 := \mathcal{O}_X$. Then

$$\mathcal{M} = \bigoplus_{i=0}^n \mathcal{M}_i$$

is a projective generator for ${}^{-1}\mathrm{Per}(X/Y)$ (see [VdB04, Proposition 3.5.4]). The dual vector bundle

$$\mathcal{N} = \mathcal{M}^\vee \tag{18}$$

is a projective generator for ${}^0\mathrm{Per}(X/Y)$ (see [VdB04, Proposition 3.2.5]).

We put

$$\mathcal{P}_i = \mathcal{H}_X^{-1} \iota_f^* \mathcal{M}_i. \tag{19}$$

By Proposition 2.4, sheaf $\mathcal{P} = \bigoplus_{i=1}^n \mathcal{P}_i$ is a projective generator for $\mathscr{A}_f$.

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Example. Let X be a smooth threefold, $f\colon X \to Y$ a flopping contraction. Assume $Y = \operatorname{Spec} R$ has rational singularities and $C_{\mathrm{red}} \simeq \mathbb{P}^1 \subset X$. Then the normal bundle $N_{X/C_{\mathrm{red}}}$ is isomorphic either to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(1) \oplus \mathcal{O}(-3)$. Let $D \subset X$ be a divisor such that $D.C_{\mathrm{red}} = 1$. For the first two cases, $\mathcal{M} \simeq \mathcal{O}_X \oplus \mathcal{O}_X(D)$, and it is of higher rank for the third case. If $N_{X/C_{\mathrm{red}}} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, the projective generator $\mathcal{P}$ is $\mathcal{O}(-1)$. When $N_{X/C_{\mathrm{red}}} \simeq \mathcal{O} \oplus \mathcal{O}(-2)$, $\mathcal{P}$ is an n-iterated extension of $\mathcal{O}(-1)$ by $\mathcal{O}(-1)$, where n is the width of C_{red}, see [Rei83] (cf. [Tod07]).

2.3 Basic properties of $\mathscr{A}_f$

Let $f\colon X \to Y$ satisfy assumption (d) and let E be a coherent sheaf on Y. By the derived projection formula, we have $Rf_*Lf^*(E) \simeq E$. As f has fibers of dimension bounded by one, Leray spectral sequence $R^p f_* L^q f^*(E) \Rightarrow \mathcal{H}_X^{p-q} E$ degenerates. Hence, an exact sequence:

$$0 \to R^1 f_* L^1 f^* E \to E \to f_* f^* E \to 0. \tag{20}$$

LEMMA 2.8. *Let $f\colon X \to Y$ and E be as above. Then $R^1 f_* f^* E = 0$ and $R f_* L^i f^* E = 0$, for any $i > 1$. Further suppose that E has no torsion supported at the image $f(\operatorname{Ex} f)$ of the exceptional locus of f. Then $f_* f^*(E) \simeq E$ and sheaf $L^1 f^* E$ is in $\mathscr{A}_f$.*

Proof. The derived projection formula $Rf_* Lf^* E \simeq E$ and Leray spectral sequence $R^p f_* L^q f^*(E) \Rightarrow \mathcal{H}_X^{p-q} E$ imply $R^1 f_* f^* E \simeq 0$, $f_* L^1 f^* E \simeq 0$, $R f_* L^i f^* E \simeq 0$, for any $i > 1$. As $R^1 f_* L^1 f^* E$ is supported on $f(\operatorname{Ex} f)$, the assumption that E has no torsion supported on $f(\operatorname{Ex} f)$ and sequence (20) imply that $R^1 f_* L^1 f^* E$ is zero and $E \simeq f_* f^* E$. $\square$

LEMMA 2.9. *Let $f\colon X \to Y$ and E be as above. If $E = f_* E'$, for some $E' \in \operatorname{Coh}(X)$, then $f_* f^*(E) \simeq E$ and sheaves $L^i f^* E$ are in $\mathscr{A}_f$, for $i > 0$.*

Proof. Sequence (20) implies that morphism $\alpha\colon f_* E' \to f_* f^* f_* E'$ is surjective. Morphism α is the inclusion of a direct summand, because its composition with the canonical map $f_* f^* f_* E' \to f_* E'$ is the identity. Hence, α is an isomorphism. Exact sequence (20) implies that $R^1 f_* L^1 f^* f_* E' = 0$. The rest follows from Lemma 2.8. $\square$

PROPOSITION 2.10. *Let $f\colon X \to Y$ be a proper morphism with dimension of fibers bounded by one. Consider a decomposition for f:*
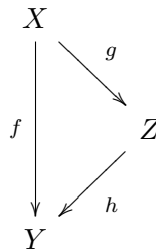
Then, for $E \in \operatorname{Coh}(X)$ with $R^1 f_* E = 0$, we have $R^1 g_* E = 0$. Functor g_* restricts to an exact functor $g_*\colon \mathscr{A}_f \to \mathscr{A}_h$.

Proof. Morphism g is proper by the valuative criterion. Replace Z by the (closed) image of g, if necessary. Then h becomes a proper morphism, cf. [Gro61, Corollaire 5.4.3]. As the dimension of fibers for f is bounded by one, so is the dimension of fibers for h.

As $Rf_*(E)$ is a sheaf on Y, spectral sequence $R^q h_* R^s g_* E \Rightarrow R^{q+s} f_* E$ implies that $Rh_* R^1 g_*(E) = 0$. Sheaf $R^1 g_*(E)$ is supported in the locus of points $z \in Z$ such that the fiber of morphism g over z is one-dimensional. As the null category of a finite morphism is zero,

morphism $h\colon Z \to Y$ restricted to the support of $R^1 g_*(E)$ must have fibers of dimension one. Let $y \in Y$ be a point in $h(\mathrm{Supp}\, R^1 g_*(E))$. Then the fiber of f over y is two-dimensional, which contradicts the assumptions. Thus, $R^1 g_*(E)$ is zero.

For any $E \in \mathscr{A}_f$, the spectral sequence $R^p h_* R^q g_* E \Rightarrow R^{p+q} f_* E = 0$ degenerates. It follows that $g_* E \in \mathscr{A}_h$. As $R^1 g_*(E) = 0$, functor $g_*\colon \mathscr{A}_f \to \mathscr{A}_h$ is exact. $\qquad\square$

PROPOSITION 2.11. *Let* $f\colon X \to Y$ *be a proper morphism with dimension of fibers bounded by one and* $g\colon Z \to Y$ *a morphism of schemes over field* k. *Assume a coherent sheaf* E *on* X *satisfy* $R^l f_* E = 0$, *for* $l \geqslant l_0$, *for some* $l_0 \in \{0, 1\}$. *Then* $R^l \pi_{Z*} \pi_X^* E = 0$, *for* $l \geqslant l_0$, *where* $\pi_X\colon W \to X$ *and* $\pi_Z\colon W \to Z$ *are the projections for* $W = X \times_Y Z$:

$$
\begin{array}{ccc}
X \times_Y Z & \xrightarrow{\ \pi_X\ } & X \\
{\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{\ g\ } & Y
\end{array}
\tag{21}
$$

Proof. The construction of $X \times_Y Z$ and the statement are local on Y, hence we may assume that $Y = \mathrm{Spec}(R)$ is affine. Moreover, the statement is local on Z, so we may further reduce to the case when $Z = \mathrm{Spec}(A)$ is affine. Hence, $g\colon Z \to Y$ is an affine morphism, which implies that π_X is affine.

The commutativity of (21) implies $R(f\pi_X)_* \pi_X^* E \simeq R(g\pi_Z)_* \pi_X^* E$. Morphisms g and π_X are affine, hence we have an isomorphism $R^l f_*(\pi_{X*} \pi_X^* E) \simeq g_* R^l \pi_{Z*}(\pi_X^* E)$, for any $l \geqslant 0$. Morphism g is affine, thus $g_* R^l \pi_{Z*}(\pi_X^* E)$ is zero if and only if $R^l \pi_{Z*}(\pi_X^* E)$ is also zero. Hence, we need to show that $R^l f_*(\pi_{X*} \pi_X^* E) = 0$, for $l \geqslant l_0$.

Morphism π_X is affine, hence $\pi_{X*} \pi_X^* E \simeq E \otimes \pi_{X*}(\mathcal{O}_{X \times_Y Z})$. The base-change morphism (cf. (47)) $f^* g_* \mathcal{O}_Z \to \pi_{X*} \mathcal{O}_{X \times_Y Z}$ is an isomorphism. Indeed, this can be checked locally on X: if $X = \mathrm{Spec}(B)$, then both sheaves $\pi_{X*} \mathcal{O}_{X \times_Y Z}$ and $f^* g_* \mathcal{O}_Z$ correspond to B-module $B \otimes_R A$. Thus,

$$
\pi_{X*} \pi_X^* E \simeq E \otimes f^* g_* \mathcal{O}_Z \simeq \mathcal{H}_X^0(E \otimes^L Lf^* g_* \mathcal{O}_Z).
$$

Derived projection formula $Rf_*(E \otimes^L Lf^* g_* \mathcal{O}_Z) \simeq Rf_*(E) \otimes^L g_* \mathcal{O}_Z$ implies that $Rf_*(E \otimes^L Lf^* g_* \mathcal{O}_Z) \in \mathcal{D}(Y)^{<l_0}$. Morphism f is proper and with dimension of fibers bounded by one, hence $R^p f_*(F) = 0$, for $p > 1$ and for any sheaf F. It follows that spectral sequence

$$
R^p f_* \mathcal{H}_X^q(E \otimes^L Lf^* g_* \mathcal{O}_Z) \Rightarrow R^{p+q} f_*(E \otimes^L Lf^* g_* \mathcal{O}_Z)
$$

degenerates. Therefore, $R^l f_*(\pi_{X*} \pi_X^* E) \simeq R^l f_* \mathcal{H}_X^0(E \otimes Lf^* g_* \mathcal{O}_Z) = 0$, for $l \geqslant l_0$. $\qquad\square$

COROLLARY 2.12. *Let* $f\colon X \to Y$ *and* $g\colon Z \to Y$ *be as in Proposition 2.11. For* $E \in \mathscr{A}_f$, *its pull-back* $\pi_X^* E$ *is an object in* $\mathscr{A}_{\pi_Z}$.

3. $L^1 f^*$ vanishing and 2-periodicity

Let $f\colon X \to Y$ satisfy assumption (d). For $p = -1, 0$, we denote by $\mathscr{P}_p$ the category of locally projective objects in ${}^p\mathrm{Per}(X/Y)$. An object $\mathcal{M}$ belongs to $\mathscr{P}_p$ if there exists an affine open covering $Y = \bigcup Y_i$, inducing $X = \bigcup X_i$ with $X_i = f^{-1}(Y_i)$, such that $\mathcal{M}|_{X_i}$ is projective in ${}^p\mathrm{Per}(X_i/Y_i)$. By [VdB04, Proposition 3.2.6] objects in $\mathscr{P}_{-1}$ and $\mathscr{P}_0$ are locally free sheaves on X.

In this section we discuss a spherical couple associated to a Cartier divisor. Under the assumption that Y has hypersurface singularities we prove that the sheaf $L^1 f^* f_* \mathcal{M}$ is zero, for any

object $\mathcal{M} \in \mathscr{P}_{-1}$. Though technical, this result is crucial for the various description of the flop and flop–flop functors presented in the subsequent sections.

3.1 Cartier divisors and spherical couples

Let X be a quasi-compact, quasi-separated scheme. By an effective Cartier divisor $i \colon D \to X$ we mean a subscheme whose ideal sheaf $\mathcal{I}_D$ is invertible. Here we discuss a 2-categorical adjunction and a spherical couple in the bicategory **FM** (see Appendix C) related to such a divisor.

Denote by $\Gamma \subset D \times X$, $\Gamma^t \subset X \times D$ the graphs of i. Sheaf $\mathcal{O}_\Gamma \in \mathcal{D}_{\mathrm{qc}}(D \times X)$ is an FM kernel for $Ri_* \colon \mathcal{D}_{\mathrm{qc}}(D) \to \mathcal{D}_{\mathrm{qc}}(X)$, and $\mathcal{O}_{\Gamma^t} \in \mathcal{D}_{\mathrm{qc}}(X \times D)$ is an FM kernel for $Li^* \colon \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(D)$.

For a scheme Y, we denote by $\Delta^Y \subset Y \times Y$ the diagonal and, for closed $Z \subset Y$, by $\Delta^{Z,Y}$ the image of Z under the diagonal morphism $\mathrm{diag}^Y \colon Y \to Y \times Y$. For $F \in \mathcal{D}_{\mathrm{qc}}(D \times Y)$, we have

$$F * \mathcal{O}_{\Gamma^t} = (i \times \mathrm{Id}_Y)_* F \in \mathcal{D}_{\mathrm{qc}}(X \times Y). \tag{22}$$

In particular,

$$\mathcal{O}_\Gamma * \mathcal{O}_{\Gamma^t} = (i \times \mathrm{Id}_X)_* \mathcal{O}_\Gamma \simeq \mathcal{O}_{\Delta^{D,X}}.$$

Embedding $\Delta^{D,X} \subset \Delta^X$ gives

$$\eta \colon \mathcal{O}_{\Delta^X} \to \mathcal{O}_{\Delta^{D,X}} \simeq \mathcal{O}_{\Gamma^t} * \mathcal{O}_\Gamma. \tag{23}$$

For any scheme Y, and objects $F \in \mathcal{D}_{\mathrm{qc}}(X \times Y)$, $G \in \mathcal{D}_{\mathrm{qc}}(Y \times X)$, we have

$$F * \mathcal{O}_\Gamma * \mathcal{O}_{\Gamma^t} \simeq (i \times \mathrm{Id}_Y)_*(i \times \mathrm{Id}_Y)^* F, \quad \mathcal{O}_\Gamma * \mathcal{O}_{\Gamma^t} * G \simeq (\mathrm{Id}_Y \times i)_*(\mathrm{Id}_Y \times i)^* G.$$

One checks locally that for $F \in \mathrm{QCoh}(X \times Y)$ and $G \in \mathrm{QCoh}(Y \times X)$ morphisms

$$F \simeq F * \mathcal{O}_{\Delta^X} \xrightarrow{F * \eta} F * \mathcal{O}_\Gamma * \mathcal{O}_{\Gamma^t} \simeq (i \times \mathrm{Id}_Y)_*(i \times \mathrm{Id}_Y)^* F \simeq F \otimes \mathcal{O}_{D \times Y},$$

$$G \simeq \mathcal{O}_{\Delta^X} * G \xrightarrow{\eta * G} \mathcal{O}_\Gamma * \mathcal{O}_{\Gamma^t} * G \simeq (\mathrm{Id}_Y \times i)_*(\mathrm{Id}_Y \times i)^* G \simeq G \otimes \mathcal{O}_{Y \times D}$$

are induced by restriction morphisms $\mathcal{O}_{X \times Y} \to \mathcal{O}_{D \times Y}$ and $\mathcal{O}_{Y \times X} \to \mathcal{O}_{Y \times D}$, respectively.

For any scheme Y and $F \in \mathcal{D}_{\mathrm{qc}}(X \times Y)$, we have

$$F * \mathcal{O}_\Gamma \simeq (i \times \mathrm{Id}_Y)^* F \in \mathcal{D}_{\mathrm{qc}}(D \times Y). \tag{24}$$

Hence,

$$\mathcal{O}_{\Gamma^t} * \mathcal{O}_\Gamma \simeq (i \times \mathrm{Id}_D)^* \mathcal{O}_{\Gamma^t} \simeq (i \times \mathrm{Id}_D)^*(i \times \mathrm{Id}_D)_* \mathcal{O}_{\Delta^D} \in \mathcal{D}_{\mathrm{qc}}(D \times D).$$

Subscheme $D \times D \subset X \times D$ is a Cartier divisor with ideal sheaf $\mathcal{I}_{D \times D}$. Therefore, object $(i \times \mathrm{Id}_D)^*(i \times \mathrm{Id}_D)_* \mathcal{O}_{\Delta^D}$ has two non-zero cohomology sheaves, $\mathcal{O}_{\Delta^D}$ in degree 0 and $\mathcal{O}_{\Delta^D} \otimes \mathcal{I}_{D \times D}|_{D \times D}$ in degree -1. Truncation to the zeroth cohomology gives morphism

$$\varepsilon \colon \mathcal{O}_{\Gamma^t} * \mathcal{O}_\Gamma \simeq (i \times \mathrm{Id}_D)^*(i \times \mathrm{Id}_D)_* \mathcal{O}_{\Delta^D} \to \mathcal{O}_{\Delta^D} \tag{25}$$

Formulas (22) and (24) imply that, for any scheme Y and objects $F \in \mathcal{D}_{\mathrm{qc}}(D \times Y)$ and $G \in \mathcal{D}_{\mathrm{qc}}(Y \times D)$, we have

$$F * \mathcal{O}_{\Gamma^t} * \mathcal{O}_\Gamma \simeq (i \times \mathrm{Id}_Y)^*(i \times \mathrm{Id}_Y)_* F, \quad \mathcal{O}_{\Gamma^t} * \mathcal{O}_\Gamma * G \simeq (\mathrm{Id}_Y \times i)^*(\mathrm{Id}_Y \times i)_* G.$$

One checks locally along D that for $F \in \mathrm{QCoh}(D \times Y)$ and $G \in \mathrm{QCoh}(Y \times D)$ morphisms

$$(i \times \mathrm{Id}_Y)^*(i \times \mathrm{Id}_Y)_* F \simeq F * \mathcal{O}_{\Gamma^t} * \mathcal{O}_\Gamma \xrightarrow{F * \varepsilon} F * \mathcal{O}_{\Delta^D} \simeq F,$$

$$(\mathrm{Id}_Y \times i)^*(\mathrm{Id}_Y \times i)_* G \simeq \mathcal{O}_{\Gamma^t} * \mathcal{O}_\Gamma * G \xrightarrow{\varepsilon * G} \mathcal{O}_{\Delta^D} * G \simeq G$$

are the truncations to the zeroth cohomology.

An early version of formula (26) in the following theorem was first proven in [BO95, Lemma 3.3] under the assumption that both varieties are smooth. The result was also stated without proof in [Ann07, Add16].

THEOREM 3.1. *Let* $i\colon D \to X$ *be the embedding of an effective Cartier divisor. Then* $(\mathcal{O}_{\Gamma^t}, \mathcal{O}_\Gamma, \eta, \varepsilon)$ *is a spherical couple in the bicategory* **FM**. *Object* $\mathrm{diag}_*^D \mathcal{I}_D|_D[2]$ *is the spherical twist and* $\mathrm{diag}_*^X \mathcal{I}_D$ *the spherical cotwist. The associated functorial exact triangles read*

$$\mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(X)} \otimes \mathcal{I}_D \to \mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(X)} \xrightarrow{\eta} Ri_* Li^* \to \mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(X)} \otimes \mathcal{I}_D[1], \tag{26}$$

$$\mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(D)} \otimes \mathcal{I}_D|_D[1] \to Li^* Ri_* \xrightarrow{\varepsilon} \mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(D)} \to \mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(D)} \otimes \mathcal{I}_D|_D[2]. \tag{27}$$

Proof. First, we check that $(\mathcal{O}_{\Gamma^t}, \mathcal{O}_\Gamma, \eta, \varepsilon)$ is a 2-categorical adjunction, i.e. that (C.1) are equal to the identity morphisms.

As $\mathcal{O}_\Gamma \simeq (\mathrm{Id}_D \times i)_* \mathcal{O}_{\Delta^D}$, the composite $\mathcal{O}_\Gamma \xrightarrow{\eta * \mathcal{O}_\Gamma} \mathcal{O}_\Gamma * \mathcal{O}_{\Gamma^t} * \mathcal{O}_\Gamma \xrightarrow{\mathcal{O}_\Gamma * \varepsilon} \mathcal{O}_\Gamma$ reads

$$(\mathrm{Id}_D \times i)_* \mathcal{O}_{\Delta^D} \xrightarrow{\eta * \mathcal{O}_\Gamma} (\mathrm{Id}_D \times i)_* (\mathrm{Id}_D \times i)^* (\mathrm{Id}_D \times i)_* \mathcal{O}_{\Delta^D} \xrightarrow{\mathcal{O}_\Gamma * \varepsilon} (\mathrm{Id}_D \times i)_* \mathcal{O}_{\Delta^D}.$$

Object $(\mathrm{Id}_D \times i)_* \mathcal{O}_{\Delta^D}$ is supported on $D \times D$, hence the first morphism, restriction to $D \times D$, is the identity on the zeroth cohomology. The second morphism is the truncation at zeroth cohomology, hence $(\mathcal{O}_\Gamma * \varepsilon) \circ (\eta * \mathcal{O}_\Gamma) = \mathrm{Id}_{\mathcal{O}_\Gamma}$.

As $\mathcal{O}_{\Gamma^t} \simeq (i \times \mathrm{Id}_D)_* \mathcal{O}_{\Delta^D}$, the composite $\mathcal{O}_{\Gamma^t} \xrightarrow{\mathcal{O}_{\Gamma^t} * \eta} \mathcal{O}_{\Gamma^t} * \mathcal{O}_\Gamma * \mathcal{O}_{\Gamma^t} \xrightarrow{\varepsilon * \mathcal{O}_{\Gamma^t}} \mathcal{O}_{\Gamma^t}$ reads

$$(i \times \mathrm{Id}_D)_* \mathcal{O}_{\Delta^D} \xrightarrow{\mathcal{O}_{\Gamma^t} * \eta} (i \times \mathrm{Id}_D)_* (i \times \mathrm{Id}_D)^* (i \times \mathrm{Id}_D)_* \mathcal{O}_{\Delta^D} \xrightarrow{\varepsilon * \mathcal{O}_{\Gamma^t}} (i \times \mathrm{Id}_D)_* \mathcal{O}_{\Delta^D}.$$

The arguments as above show that $(\varepsilon * \mathcal{O}_{\Gamma^t}) \circ (\mathcal{O}_{\Gamma^t} * \eta)$ is the identity on $\mathcal{O}_{\Gamma^t}$.

Now we calculate the twist and cotwist of the 2-categorical adjunction $(\mathcal{O}_{\Gamma^t}, \mathcal{O}_\Gamma, \eta, \varepsilon)$. Morphism η in (23) is induced by the morphism $\mathcal{O}_X \to \mathcal{O}_D$ with kernel $\mathcal{I}_D$. Hence,

$$\mathrm{diag}_*^X \mathcal{I}_D \to \mathrm{diag}_*^X \mathcal{O}_X \xrightarrow{\eta} \mathrm{diag}_*^X \mathcal{O}_D \to \mathrm{diag}_*^X \mathcal{I}_D[1]$$

is an exact triangle in $\mathcal{D}_{\mathrm{qc}}(X \times X)$. Thus, the cotwist equals $\mathrm{diag}_*^X \mathcal{I}_D$. It is an equivalence whose inverse is $\mathrm{diag}_*^X \mathcal{I}_D^{-1}$. The corresponding functorial exact triangle is (26).

Morphism ε in (25) is induced by the truncation to the zeroth cohomology of the object $(i \times \mathrm{Id}_D)^* (i \times \mathrm{Id}_D)_* \mathcal{O}_{\Delta^D}$. As we noted before, $(i \times \mathrm{Id}_D)^* (i \times \mathrm{Id}_D)_* \mathcal{O}_{\Delta^D}$ has two non-zero cohomology sheaves and the truncation to the zeroth cohomology yields an exact triangle

$$\mathcal{O}_{\Delta^D} \otimes \mathcal{I}_{D \times D}|_{D \times D}[1] \to (i \times \mathrm{Id}_D)^* (i \times \mathrm{Id}_D)_* \mathcal{O}_{\Delta^D} \to \mathcal{O}_{\Delta^D} \to \mathcal{O}_{\Delta^D} \otimes \mathcal{I}_{D \times D}|_{D \times D}[2].$$

The twist

$$\mathcal{O}_{\Delta^D} \otimes \mathcal{I}_{D \times D}|_{D \times D}[1] \simeq \mathrm{diag}_*^D \mathcal{I}_D|_D[1]$$

is an equivalence whose inverse is $\mathrm{diag}_*^D \mathcal{I}_D^{-1}|_D[-1]$. The corresponding functorial exact triangle is (27). □

3.2 $L^1 f^*$ vanishing and consequences

Vanishing of $L^1 f^* f_*(-)$ is local on Y, therefore throughout this section we assume that Y is affine and fix a closed embedding $i\colon Y \to \mathcal{Y}$ of Y as the zero locus of a regular function on a smooth affine $\mathcal{Y}$. As $Y \subset \mathcal{Y}$ is a principal Cartier divisor, the sheaf $\mathcal{I}_Y$ is isomorphic to $\mathcal{O}_\mathcal{Y}$.

The following lemma is one of the key technical points of this paper.

LEMMA 3.2. *Let $f: X \to Y$ satisfy assumption (p) and let $Y = \operatorname{Spec} R$ be a spectrum of a complete Noetherian local ring with hypersurface singularities. Let further $D_i \subset X$ be as in Remark 2.7. Then $L^1 f^* f_* \mathcal{O}_{D_i} \simeq 0$.*

Proof. Let $\iota_i: D_i \to X$ denote the embedding of D_i into X. We denote by h the composite $h = if\iota_i: D_i \to \mathcal{Y}$. Morphism h is finite, hence h_* is exact and, by Grothendieck duality (A.1), we have

$$\mathcal{E}xt^i_{\mathcal{Y}}(Rh_* \mathcal{O}_{D_i}, \mathcal{O}_{\mathcal{Y}}) \simeq Rh_* \mathcal{E}xt^i_{D_i}(\mathcal{O}_{D_i}, h^! \mathcal{O}_{\mathcal{Y}}) \simeq Rh_* \mathcal{H}^i_{D_i}(h^! \mathcal{O}_{\mathcal{Y}}). \tag{28}$$

Divisor D_i is Cartier in a Gorenstein scheme, hence it is Gorenstein itself. As $\mathcal{Y}$ is smooth and it is the spectrum of a complete ring, we have $\omega_{\mathcal{Y}}^{\cdot} \simeq \mathcal{O}_{\mathcal{Y}}[n+1]$. It follows that $h^! \mathcal{O}_{\mathcal{Y}} = h^! \omega_{\mathcal{Y}}^{\cdot}[-n-1] = \omega_{D_i}[-2]$ is a sheaf in homological degree 2. As h_* is exact, $Rh_* \mathcal{H}^i_{D_i}(h^! \mathcal{O}_{\mathcal{Y}}) \simeq h_* \mathcal{H}^i_{D_i}(h^! \mathcal{O}_{\mathcal{Y}})$. Hence, by (28), we have

$$\mathcal{E}xt^i_{\mathcal{Y}}(h_* \mathcal{O}_{D_i}, \mathcal{O}_{\mathcal{Y}}) = 0, \quad \text{for } i \neq 2.$$

As $\mathcal{Y}$ is affine, $\operatorname{Ext}^i_{\mathcal{Y}}(h_* \mathcal{O}_{D_i}, \mathcal{O}_{\mathcal{Y}}) = 0$, for $i \neq 2$. Hence, the projective dimension of $h_* \mathcal{O}_{D_i}$ as an $\mathcal{O}_{\mathcal{Y}}$ module is two. In other words, $h_* \mathcal{O}_{D_i}$ has a locally free resolution

$$0 \to \mathcal{E}_{-2} \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to h_* \mathcal{O}_{D_i} \to 0.$$

Denote by $\mathcal{E}.$ the complex $0 \to \mathcal{E}_{-2} \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to 0$. The cohomology of complex $i_* i^* \mathcal{E}.$ is $\operatorname{Tor}^{\mathcal{Y}}(h_* \mathcal{O}_{D_i}, \mathcal{O}_Y)$. As $Y \subset \mathcal{Y}$ is a Cartier divisor, $\mathcal{O}_Y$ has a resolution of length two on $\mathcal{Y}$, i.e. the zeroth and first Tor groups only can be non-zero. As functor i_* is exact, we conclude that the morphism $i^* \mathcal{E}_{-2} \to i^* \mathcal{E}_{-1}$ is injective.

Morphism f is birational, hence the higher derived functors of f^* are torsion. In particular, the kernel of $f^* i^* \mathcal{E}_{-2} \to f^* i^* \mathcal{E}_{-1}$ is torsion, meaning zero because $f^* i^* \mathcal{E}_{-2}$ is locally free. This shows that morphism $f^* i^* \mathcal{E}_{-2} \to f^* i^* \mathcal{E}_{-1}$ is also injective, thus complex $\widetilde{\mathcal{E}}. := f^* i^* \mathcal{E}.$ has zero cohomology except for $\mathcal{H}^0_X(\widetilde{\mathcal{E}}.)$ and $\mathcal{H}^{-1}_X(\widetilde{\mathcal{E}}.)$.

Let $E \subset X$ be the exceptional locus of f. Assume $L^1 f^* f_* \mathcal{O}_{D_i} \neq 0$. Its support is contained in E. Moreover, because $L^1 f^* f_* \mathcal{O}_{D_i} \in \mathscr{A}_f$ (see Lemma 2.9), the support of this sheaf should contain at least one component of the curve in the fiber over the closed point of Y (see Theorem D.1 for the structure of the fiber). Therefore, there exists a closed point $x \in E \setminus D_i$ in the support of $L^1 f^* f_* \mathcal{O}_{D_i}$.

By Theorem 3.1 for divisor Y in $\mathcal{Y}$, object $i^* \mathcal{E}. \simeq Li^* i_* f_* \mathcal{O}_{D_i}$ fits into an exact triangle

$$f_* \mathcal{O}_{D_i}[1] \to i^* \mathcal{E}. \to f_* \mathcal{O}_{D_i} \to f_* \mathcal{O}_{D_i}[2].$$

By applying functor Lf^* to this triangle, we obtain an exact sequence

$$f^* f_* \mathcal{O}_{D_i} \to \mathcal{H}^{-1}_X(\widetilde{\mathcal{E}}.) \to L^1 f^* f_* \mathcal{O}_{D_i} \to 0.$$

Sheaf $f^* f_* \mathcal{O}_{D_i}$ is supported on the preimage of $f(D_i) \subset Y$, i.e. on the set $E \cup D_i$. Let us restrict our attention to an open affine neighbourhood $\widetilde{i}: \widetilde{X} \hookrightarrow X$ of point x. The support of both sheaves $f^* f_* \mathcal{O}_{D_i}$ and $L^1 f^* f_* \mathcal{O}_{D_i}$ is contained in E. Therefore, the assumptions on the exceptional locus for f imply that $\mathcal{H}^{-1}_X(\widetilde{\mathcal{E}})$ is a non-zero sheaf with support of some codimension $l > 1$.

Let $j: \widetilde{X} \to \mathcal{X}$ be a closed embedding into a smooth variety of dimension m. We have $\omega_{\widetilde{X}} \simeq \widetilde{i}^! f^! \mathcal{O}_Y \simeq \widetilde{i}^! \mathcal{O}_X \simeq \mathcal{O}_{\widetilde{X}}$, hence Grothendieck duality (A.1) gives

$$\mathcal{E}xt^{\cdot}_{\mathcal{X}}(j_* \widetilde{\mathcal{E}}., \omega_{\mathcal{X}}) \simeq j_* \mathcal{E}xt^{\cdot}_{\widetilde{X}}(\widetilde{\mathcal{E}}., \mathcal{O}_{\widetilde{X}}[n-m]).$$

Complex $\widetilde{\mathcal{E}}.$ consists of three locally free sheaves, thus $\mathcal{E}xt^k_X(\widetilde{\mathcal{E}}., \mathcal{O}_X)$ is zero, for $k > 2$. It follows that $\mathcal{E}xt^k_{\mathcal{X}}(j_* \widetilde{\mathcal{E}}., \omega_{\mathcal{X}}) = 0$, for $k > m - n + 2$. By applying functor $\mathcal{H}om^{\cdot}_{\mathcal{X}}(-, \omega_{\mathcal{X}})$ to the exact

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triangle

$$j_* \mathcal{H}_X^{-1}(\widetilde{\mathcal{E}}.)[1] \to j_* \widetilde{\mathcal{E}}. \to j_* \mathcal{H}_X^0(\widetilde{\mathcal{E}}.) \to j_* \mathcal{H}_X^{-1}(\widetilde{\mathcal{E}}.)[2],$$

we obtain an isomorphism

$$\mathcal{E}xt_{\mathcal{X}}^k(j_* \mathcal{H}_X^{-1}(\widetilde{\mathcal{E}}.), \omega_{\mathcal{X}}) \xrightarrow{\simeq} \mathcal{E}xt_{\mathcal{X}}^{k+2}(j_* \mathcal{H}_X^0(\widetilde{\mathcal{E}}.), \omega_{\mathcal{X}}), \tag{29}$$

for any $k \geqslant m - n + 2$. Sheaf $j_* \mathcal{H}_X^{-1}(\widetilde{\mathcal{E}}.)$ is non-zero with support of codimension $l + m - n \geqslant m - n + 2$. It follows that sheaf $\mathcal{E}xt_{\mathcal{X}}^{l+m-n}(j_* \mathcal{H}_X^{-1}(\widetilde{\mathcal{E}}.), \omega_{\mathcal{X}})$ is non-zero with support of codimension $l + m - n$, cf. [HL10, Proposition 1.1.6].

On the other hand, the codimension of the support of $\mathcal{E}xt_{\mathcal{X}}^{l+m-n+2}(j_* \mathcal{H}_X^0(\widetilde{\mathcal{E}}.), \omega_{\mathcal{X}})$ is greater than or equal to $l + m - n + 2$, which contradicts (29). $\qquad\square$

LEMMA 3.3. *Let* $f \colon X \to Y$ *satisfy assumption (d) and let* $Y = \operatorname{Spec} R$ *be a spectrum of a complete Noetherian local ring. Let further* $\mathcal{M}_i$ *be the projective object in* $^{-1}\mathrm{Per}(X/Y)$ *as in (17) and* $\mathcal{N}_i = \mathcal{H}om_X(\mathcal{M}_i, \mathcal{O}_X)$ *the projective object in* $^0\mathrm{Per}(X/Y)$. *Then there exist exact sequences*

$$0 \to \mathcal{O}_X^{r_i} \to \mathcal{M}_i \to \mathcal{O}_{D_i} \to 0, \tag{30}$$

$$0 \to \mathcal{N}_i \to \mathcal{O}_X^{r_i} \to \mathcal{O}_{D_i} \to 0. \tag{31}$$

Proof. The sheaf $\mathcal{M}_i$ is defined as an extension

$$0 \to \mathcal{O}_X^{r_i - 1} \to \mathcal{M}_i \xrightarrow{q} \mathcal{O}_X(D_i) \to 0.$$

For any divisor D_i, one can find a linearly equivalent divisor D_i' such that $D_i \cap D_i'$ is empty. Indeed, let D_i' be a divisor as in Remark 2.7 linearly equivalent to D_i which intersects C_i in a point x_i', not equal to $x_i = D_i \cap C_i$. Then $D_i \cap D_i'$ contains no closed points (as all closed points of X lie on C), hence empty. It follows that $\mathcal{O}_{D_i}(D_i) \simeq \mathcal{O}_{D_i}(D_i') \simeq \mathcal{O}_{D_i}$ and sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D_i) \xrightarrow{r} \mathcal{O}_{D_i} \to 0$$

is exact. As $\mathrm{Ext}_X^1(\mathcal{O}_X, \mathcal{O}_X) \simeq 0$, the snake lemma for the two top rows of the following diagram gives the following commutative diagram in which rows and columns are exact sequences.

$$
\begin{array}{ccccc}
0 & \longrightarrow & \mathcal{O}_{D_i} & \xrightarrow{\simeq} & \mathcal{O}_{D_i} \\
& & \uparrow{\scriptstyle rq} & & \uparrow{\scriptstyle r} \\
\mathcal{O}_X^{r_i-1} & \longrightarrow & \mathcal{M}_i & \xrightarrow{q} & \mathcal{O}_X(D_i) \\
{\scriptstyle \simeq}\uparrow & & \uparrow & & \uparrow \\
\mathcal{O}_X^{r_i-1} & \longrightarrow & \mathcal{O}_X^{r_i} & \longrightarrow & \mathcal{O}_X
\end{array}
$$

It gives exact sequence (30). By considering local homomorphisms into the structure sheaf we obtain sequence (31). $\qquad\square$

LEMMA 3.4. *Let* $f \colon X \to Y$ *satisfy assumption (p) and assume that* Y *has hypersurface singularities. Then the sheaf* $L^1 f^* f_* \mathcal{M}$ *is zero, for any* $\mathcal{M} \in \mathscr{P}_{-1}$.

Proof. The statement is local in Y, therefore we might assume that $Y = \operatorname{Spec} R$ is a spectrum of a complete Noetherian local ring with hypersurface singularities. Then $\mathcal{M}$ is a direct sum of copies of $\mathcal{M}_i$ and $\mathcal{O}_X$. Thus, it suffices to verify the lemma for $\mathcal{M}_i$, as clearly $L^1 f^* \mathcal{O}_Y = 0$.

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As $R^1 f_* \mathcal{O}_X \simeq 0$, applying Rf_* to sequence (30) gives an exact sequence

$$0 \to \mathcal{O}_Y^{r_i} \to f_* \mathcal{M}_i \to f_* \mathcal{O}_{D_i} \to 0.$$

It implies an exact sequence

$$0 \to L^1 f^* f_* \mathcal{M}_i \to L^1 f^* f_* \mathcal{O}_{D_i} \to \mathcal{O}_X^{r_i}.$$

By Lemma 3.2, the sheaf $L^1 f^* f_* \mathcal{O}_{D_i}$ is zero, hence the result. $\qquad\square$

LEMMA 3.5. *Let $f\colon X \to Y$ satisfy assumption (p) and let Y have hypersurface singularities. Let $\mathcal{N}$ be a locally projective object in $^0\mathrm{Per}(X/Y)$. Then $f^* f_* \mathcal{N}$ is torsion-free and sequence*

$$0 \to f^* f_* \mathcal{N} \to \mathcal{N} \to \mathcal{Q} \to 0, \tag{32}$$

is exact for the sheaf $\mathcal{Q} = \mathcal{H}_X^0(\iota_{f*} \iota_f^* \mathcal{N})$, which is locally projective in $\mathscr{A}_f$.

Proof. First, we show that $f^* f_* \mathcal{N}$ is torsion-free. This can be done locally, i.e. we may assume that Y is a spectrum of a complete Noetherian local ring. By [VdB04, Lemma 3.5.2] sheaf $\mathcal{N}$ is a direct sum of copies of $\mathcal{O}_X$ and $\mathcal{N}_i$, where $\mathcal{N}_i$ are vector bundles dual to $\mathcal{M}_i$.

By [VdB04, Lemma 3.1.2], the sheaf $R^1 f_* \mathcal{N}_i$ is zero. By applying f_* to sequence (31), we obtain an exact sequence on Y

$$0 \to f_* \mathcal{N}_i \to \mathcal{O}_Y^{r_i} \to f_* \mathcal{O}_{D_i} \to 0. \tag{33}$$

As $f_* \mathcal{O}_{D_i}$ is a torsion sheaf, $f_* \mathcal{N}_i$ is non-zero. By applying f^* to sequence (33), we obtain

$$0 \to L^1 f^* f_* \mathcal{O}_{D_i} \to f^* f_* \mathcal{N}_i \to \mathcal{O}_X^{r_i}.$$

It implies that the torsion of $f^* f_* \mathcal{N}_i$ is $L^1 f^* f_* \mathcal{O}_{D_i}$, which is zero by Lemma 3.2. Hence, sheaf $f^* f_* \mathcal{N}_i$ is torsion free and the counit morphism $\varepsilon_{\mathcal{N}}\colon f^* f_* \mathcal{N}_i \to \mathcal{N}_i$ is an embedding ($\varepsilon_{\mathcal{N}}$ is non-zero as under the $f^* \dashv f_*$ adjunction it corresponds to the identity endomorphism of $f_* \mathcal{N}_i$).

The cokernel of the morphism $f^* f_* \mathcal{N} \to \mathcal{N}$ is, by the definition of ι_f^*, isomorphic to $\mathcal{H}_X^0(\iota_f^* \mathcal{N})$, hence $\mathcal{Q}$ is an object in $\mathscr{A}_f$. Now let E be any object in $\mathscr{A}_f$. Then E, considered as an object in $\mathcal{D}^b(X)$, lies in $^0\mathrm{Per}(X/Y)$, hence $\mathrm{Ext}_X^1(\mathcal{N}, E) \simeq 0$. Moreover, $f^* \dashv f_*$ adjunction implies that $\mathrm{Hom}_X(f^* f_* \mathcal{N}, E) \simeq 0$. Thus, by applying higher derived functors of $\mathrm{Hom}_X(-, E)$ to sequence (32), we obtain $\mathrm{Ext}_X^1(\mathcal{Q}, E) \simeq 0$, which proves that $\mathcal{Q} \in \mathscr{A}_f$ is indeed a projective object. $\qquad\square$

Remark 3.6. In the complete local case, $\mathcal{P}$ in formula (12) is isomorphic to $\mathcal{Q}$ of Lemma 3.5. Indeed, applying ι_f^* to sequences (30) and (31) and using the fact that $\iota_f^*(\mathcal{O}_X) \simeq 0$, we obtain an isomorphism $\mathcal{H}_X^0(\iota_f^* \mathcal{N}_i) \simeq \mathcal{H}_X^{-1}(\iota_f^* \mathcal{O}_{D_i}) \simeq \mathcal{H}_X^{-1}(\iota_f^* \mathcal{M}_i)$.

The t-structure on $\mathcal{D}^b(X)$ with heart $^{-1}\mathrm{Per}(X/Y)$ is obtained from the standard t-structure by a tilt in the torsion pair $(\mathcal{T}_{-1}, \mathcal{F}_{-1})$ as in (10) and (11), see [VdB04, Lemma 3.1.2]. Hence, for any $F \in \mathcal{D}^b(X)$, sequence

$$0 \to \mathcal{F}_{-1}(\mathcal{H}_X^{i-1}(F))[1] \to \mathcal{H}_{^{-1}\mathrm{Per}}^i(F) \to \mathcal{T}_{-1}(\mathcal{H}_X^i(F)) \to 0 \tag{34}$$

is exact in $\mathrm{Coh}(X)$, where by $\mathcal{T}_{-1}(G)$ and $\mathcal{F}_{-1}(G)$ we denote the torsion and torsion-free part, respectively, of a coherent sheaf G with respect to the torsion pair $(\mathcal{T}_{-1}, \mathcal{F}_{-1})$.

PROPOSITION 3.7. *Let $f\colon X \to Y$ satisfy assumption (p) and assume that Y has hypersurface singularities. Let $\mathcal{M}$ be a projective object in $^{-1}\mathrm{Per}(X/Y)$. Then*

$$\mathcal{H}_{^{-1}\mathrm{Per}}^0(Lf^* f_* \mathcal{M}) = f^* f_* \mathcal{M}.$$

Proof. Lemma 3.8 implies that $f^*f_*\mathcal{M}$ is an object in $\mathrm{Coh}(X) \cap {}^{-1}\mathrm{Per}(X/Y)$, hence $\mathcal{T}_{-1}(f^*f_*\mathcal{M}) = f^*f_*\mathcal{M}$. Moreover, $L^1f^*f_*\mathcal{M} = 0$ by Lemma 3.4. We conclude by sequence (34). □

LEMMA 3.8. *Let $f \colon X \to Y$ satisfy assumption (d) and let E be a coherent sheaf on Y. Then f^*E is an object in ${}^p\mathrm{Per}(X/Y)$, for $p = -1, 0$. If, moreover, E is locally free and Y is affine, then f^*E is an object in $\mathscr{P}_p$, for $p = -1, 0$.*

Proof. We have $\mathrm{Coh}(X) \cap {}^0\mathrm{Per}(X/Y) = \mathcal{T}_0$ and $\mathrm{Coh}(X) \cap {}^{-1}\mathrm{Per}(X/Y) = \mathcal{T}_{-1}$, for categories $\mathcal{T}_0$ and $\mathcal{T}_{-1}$ defined in (8) and (10), respectively.

Let E be a coherent sheaf on Y. By Lemma 2.9, sheaf $R^1f_*f^*E$ is zero, hence f^*E is an object in ${}^0\mathrm{Per}(X/Y)$. From adjunction

$$\mathrm{Hom}_X(f^*E, \mathcal{A}_f) \simeq \mathrm{Hom}_Y(E, f_*\mathcal{A}_f) \simeq 0,$$

it follows that f^*E lies also in ${}^{-1}\mathrm{Per}(X/Y)$.

Now we assume further that E is locally free and Y is affine. Let E' be any object in ${}^p\mathrm{Per}(X/Y)$, for $p = -1, 0$. Then

$$\mathrm{Ext}^1_X(f^*E, E') \simeq \mathrm{Ext}^1_X(Lf^*E, E') \simeq \mathrm{Ext}^1_Y(E, Rf_*E').$$

Because Rf_* is exact for the perverse t-structures on $\mathcal{D}^b(X)$, object Rf_*E' is a coherent sheaf on Y. As E is locally free and Y is affine, $\mathrm{Ext}^1_Y(E, Rf_*E') \simeq 0$, i.e. f^*E is projective in ${}^p\mathrm{Per}(X/Y)$. □

Remark 3.9. For any locally free sheaf $\mathcal{E}$ on Y, we also have an isomorphism $\mathcal{H}^0_{{}^{-1}\mathrm{Per}(X/Y)}(Lf^*\mathcal{E}) = f^*\mathcal{E}$. Indeed, $Lf^*\mathcal{E}$ is isomorphic to $f^*\mathcal{E}$ and the latter is an object in ${}^{-1}\mathrm{Per}(X/Y)$, by Lemma 3.8.

3.3 The equivalence of reflexive sheaves under flopping contractions

Recall that a sheaf F is *normal* when, for any open sets $U \subset V$ such that codimension of $V \setminus U$ in V is greater than one, the restriction morphism $F(V) \to F(U)$ is an isomorphism.

LEMMA 3.10 [Har80, Proposition 1.6]. *Let X be a normal, integral Noetherian scheme. A sheaf F on X is reflexive if and only if it is torsion-free and normal.*

For a scheme X, denote by $\mathrm{Ref}(X)$ the category of reflexive sheaves.

LEMMA 3.11 (Cf. [VdB04, Lemma 4.2.1]). *Let $f \colon X \to Y$ be a projective birational morphism, X, Y normal Noetherian and the exceptional locus of f have codimension > 1 in X. Then the following functors are mutually inverse equivalences:*

$$f_* \colon \mathrm{Ref}(X) \to \mathrm{Ref}(Y), \quad (f^*(-))^{\vee\vee} \colon \mathrm{Ref}(Y) \to \mathrm{Ref}(X).$$

Proof. For a torsion-free sheaf F on X, its push-forward f_*F is also torsion-free. Thus, f_*F will be reflexive if we check that it is normal.

Let G be a torsion-free sheaf on Y which is not normal. There exist open sets $j \colon U \hookrightarrow V$ with the complement of U in codimension greater than one in V such that canonical morphism $G \to j_*j^*G$ is not an isomorphism. Then we have a non-trivial extension

$$0 \to G \to j_*j^*G \to Q \to 0.$$

The question is local, so we can assume Y to be affine. Sheaf Q is supported in codimension greater than one, hence there exists a closed subscheme $Z \subset Y$ of codimension greater than one

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such that sequence

$$0 \to K \to \mathcal{O}_Z^{\oplus k} \to Q \to 0$$

is exact. As G is torsion-free, $\operatorname{Hom}(K, G)$ is zero. Hence, vanishing of $\operatorname{Ext}_Y^1(\mathcal{O}_Z, G)$ implies that $\operatorname{Ext}_Y^1(Q, G)$ is also zero. Thus, in order to show that a torsion-free sheaf G is normal it suffices to check that $\operatorname{Ext}_Y^1(\mathcal{O}_Z, G)$ vanishes, for any closed $Z \subset Y$ of codimension greater than one.

Group $\operatorname{Ext}_Y^1(\mathcal{O}_Z, f_*F)$ is zero if and only if morphism $\operatorname{Hom}(\mathcal{O}_Y, f_*F) \to \operatorname{Hom}(I_Z, f_*F)$ given by short exact sequence

$$0 \to I_Z \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0$$

is an isomorphism. By adjunction, $\operatorname{Hom}_Y(I_Z, f_*F)$ is isomorphic to $\operatorname{Hom}_X(f^*I_Z, F)$. Exceptional locus of f has codimension greater than one, hence f^*I_Z is isomorphic to $\mathcal{O}_X$ in codimension one. As F is reflexive, this implies $\operatorname{Hom}_X(f^*I_Z, F) = \operatorname{Hom}_X(\mathcal{O}_X, F)$.

Thus, functor f_* maps reflexive sheaves on X to reflexive sheaves on Y. Clearly, the same is true about $(f^*(-))^{\vee\vee}$.

Now let F be an object in $\operatorname{Ref}(X)$. Adjunction implies a morphism $\alpha \colon (f^*f_*F)^{\vee\vee} \to F^{\vee\vee} \simeq F$. Both sheaves are reflexive and α is an isomorphism outside exceptional locus of f, which has codimension greater than one. Therefore, α is an isomorphism.

For $G \in \operatorname{Ref}(Y)$, reflexification $f^*G \to (f^*G)^{\vee\vee}$ together with adjunction $G \to f_*f^*G$ give $\beta \colon G \to f_*((f^*G)^{\vee\vee})$. Again, morphism β is an isomorphism in codimension one between reflexive sheaves, hence an isomorphism.

Finally, a morphism between reflexive sheaves is determined by its restriction to any open set with the complement of codimension greater than one. Thus, functors f_* and $(f^*(-))^{\vee\vee}$ give isomorphisms on the groups of morphisms between objects in $\operatorname{Ref}(X)$ and $\operatorname{Ref}(Y)$. $\qquad\square$

Let $f \colon X \to Y$ satisfy assumption (a). For $\mathcal{M}$ an object in $\mathscr{P}_{-1}$, put

$$\mathcal{N}^+ := (f^{+*}f_*\mathcal{M})^{\vee\vee}. \tag{35}$$

By Lemma 3.11, sheaf $\mathcal{N}^+$ belongs to $\operatorname{Ref}(X^+)$.

PROPOSITION 3.12. *Let $f \colon X \to Y$ satisfy assumption (a) and $f^+ \colon X^+ \to Y$ its flop. Then functor*

$$(f^{+*}f_*(-))^{\vee\vee} \colon \operatorname{Ref}(X) \to \operatorname{Ref}(X^+)$$

restricts to equivalences $\mathscr{P}_{-1} \xrightarrow{\simeq} \mathscr{P}_0^+$ and $\mathscr{P}_0 \xrightarrow{\simeq} \mathscr{P}_{-1}^+$.

Proof. By [VdB04, Proposition 3.2.6], categories $\mathscr{P}_{-1}$ and $\mathscr{P}_0$ are subcategories of $\operatorname{Ref}(X)$. By [VdB04, Proposition 4.3.1], subcategories $f_*\mathscr{P}_{-1}$ and $f_*^+\mathscr{P}_0^+$ of $\operatorname{Ref}(Y)$ are equivalent. By Lemma 3.11, the subcategory $(f^{+*}f_*\mathscr{P}_{-1})^{\vee\vee}$ of $\operatorname{Ref}(X^+)$ corresponds to $\mathscr{P}_{-1}$ under equivalences $\operatorname{Ref}(X) \simeq \operatorname{Ref}(Y) \simeq \operatorname{Ref}(X^+)$, hence $\mathscr{P}_0^+ \simeq (f^{+*}f_*\mathscr{P}_{-1})^{\vee\vee}$.

Exchanging the roles of f and f^+, we get equivalence of $\mathscr{P}_{-1}^+$ with $\mathscr{P}_0$. $\qquad\square$

3.4 A divisorial embedding into a smooth scheme and cohomology

We assume that Y is affine and fix a closed embedding $i \colon Y \to \mathcal{Y}$ of Y as the zero locus of a regular function on a smooth affine $\mathcal{Y}$. Denote $g = i \circ f$, i.e. consider the following commutative diagram.

We describe objects $Lg^*g_*\mathcal{M}$ and $Lg^*f_*\mathcal{N}$, for any $\mathcal{M} \in \mathscr{P}_{-1}$, $\mathcal{N} \in \mathscr{P}_0$. First, we prove that the sheaves $g_*\mathcal{M}$, $g_*\mathcal{N}$ have short locally free resolutions.

Recall that a morphism $f\colon X \to Y$ is *crepant* if $Lf^*(\omega_Y^{\cdot}) \simeq \omega_X^{\cdot}$.

LEMMA 3.13. *A proper morphism $f\colon X \to Y$ of Gorenstein varieties is crepant if and only if $f^!(\mathcal{O}_Y) \simeq \mathcal{O}_X$.*

Proof. As X and Y are Gorenstein, $\omega_X^{\cdot}$ and $\omega_Y^{\cdot}$ are, up to shift, line bundles. In particular, $\omega_Y^{\cdot}$ is a perfect complex, hence, by Lemma A.1, $f^!(\omega_Y^{\cdot}) \simeq f^!(\mathcal{O}_Y) \otimes Lf^*(\omega_Y^{\cdot})$. Isomorphism $\omega_X^{\cdot} \simeq f^!(\omega_Y^{\cdot}) \simeq f^!(\mathcal{O}_Y) \otimes Lf^*(\omega_Y)^{\cdot}$ implies $f^!(\mathcal{O}_Y) \simeq \omega_X^{\cdot} \otimes (Lf^*(\omega_Y^{\cdot}))^{-1}$. Hence, $Lf^*(\omega_Y^{\cdot}) \simeq \omega_X^{\cdot}$ if and only $f^!(\mathcal{O}_Y) \simeq \mathcal{O}_X$. $\square$

LEMMA 3.14. *Let X, Y be Gorenstein and $f\colon X \to Y$ a crepant morphism satisfying assumption (d). Assume Y is affine and $i\colon Y \to \mathcal{Y}$ an embedding of Y as a Cartier divisor into a smooth $\mathcal{Y}$. Denote $g = i \circ f\colon X \to \mathcal{Y}$. Then $g_*\mathcal{M}$ and $g_*\mathcal{N}$, for $\mathcal{M} \in \mathscr{P}_{-1}$, $\mathcal{N} \in \mathscr{P}_0$, admit locally free resolutions of length two.*

Proof. Let $\mathcal{M}$ be an object in $\mathscr{P}_{-1}$. As $\mathcal{Y}$ is affine, the length of the locally free resolution for $g_*\mathcal{M}$ is $l + 1$, for the maximal l such that $\mathcal{E}xt^l_{\mathcal{Y}}(g_*\mathcal{M}, \mathcal{O}_{\mathcal{Y}})$ is non-zero. Thus, the question is local on $\mathcal{Y}$ and, by restricting to a smaller affine open subset, we can assume that $\omega_{\mathcal{Y}}^{\cdot} \simeq \mathcal{O}_{\mathcal{Y}}[n+1]$ and $\omega_Y^{\cdot} \simeq \mathcal{O}_Y[n]$. Then $\omega_X^{\cdot} \simeq \mathcal{O}_X[n]$, because f is crepant. Let $\mathcal{N} = R\mathcal{H}om_X(\mathcal{M}, \mathcal{O}_X) = \mathcal{H}om_X(\mathcal{M}, \mathcal{O}_X)$. It is an object in $\mathscr{P}_0$, see [VdB04, Proposition 3.2.6]. As $\mathcal{N}$ is an object in ${}^0\mathrm{Per}(X/Y)$, we have $Rg_*\mathcal{N} \simeq g_*\mathcal{N}$. Then we have, by Grothendieck duality (A.1),

$$R\mathcal{H}om_{\mathcal{Y}}(g_*\mathcal{M}, \mathcal{O}_{\mathcal{Y}}) \simeq R\mathcal{H}om_{\mathcal{Y}}(Rg_*\mathcal{M}, \omega_{\mathcal{Y}})[-1] \simeq Rg_*R\mathcal{H}om_X(\mathcal{M}, \omega_X)[-1] \simeq g_*\mathcal{N}[-1].$$

Hence, $g_*\mathcal{M}$ is of projective dimension one. Analogously, for $\mathcal{N} \in \mathscr{P}_0$, its dual $\mathcal{M} = R\mathcal{H}om_X(\mathcal{N}, \mathcal{O}_X)$ lies in $\mathscr{P}_{-1}$ and $R\mathcal{H}om(g_*\mathcal{N}, \mathcal{O}_{\mathcal{Y}}) \simeq g_*\mathcal{M}[-1]$. $\square$

PROPOSITION 3.15. *Let $f\colon X \to Y$ be as in Lemma 3.14. Let further $p \in \{-1, 0\}$ and $\mathcal{R} \in \mathscr{P}_p$ be a projective object in ${}^p\mathrm{Per}(X/Y)$. Then the object $Lg^*Rg_*\mathcal{R}$ has cohomology*

$$\mathcal{H}^i_X(Lg^*Rg_*\mathcal{R}) = \begin{cases} \mathcal{R}, & \text{if } i = -1, \\ f^*f_*\mathcal{R}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The same cohomology is for the t-structure with heart ${}^p\mathrm{Per}(X/Y)$.

Proof. As $R^1g_*\mathcal{R} = 0$, the zero cohomology of $Lg^*Rg_*\mathcal{R}$ is the sheaf $g^*g_*\mathcal{R}$. The adjunction counit for $i\colon Y \to \mathcal{Y}$ gives an isomorphism $g^*g_* \xrightarrow{\simeq} f^*f_*$, hence $\mathcal{H}^0_X(Lg^*Rg_*\mathcal{R}) = f^*f_*\mathcal{R}$.

As $g_*\mathcal{R}$ admits a locally free resolution (see Lemma 3.14):

$$0 \to \mathcal{E}_{-1} \to \mathcal{E}_0 \to g_*\mathcal{R} \to 0,$$

the sheaves $\mathcal{H}^i_X(Lg^*Rg_*\mathcal{R})$ are zero, for $i \leqslant -2$. Moreover, $L^1g^*g_*\mathcal{R}$ is reflexive, as the kernel of a morphism of locally free sheaves [Har80, Proposition 1.1].

Applying Lf^* to exact triangle $f_*\mathcal{R}[1] \to Li^*g_*\mathcal{R} \to f_*\mathcal{R} \to f_*\mathcal{R}[2]$ (given by Theorem 3.1 for $Y \subset \mathcal{Y}$) implies the exact sequence

$$L^2f^*f_*\mathcal{R} \to f^*f_*\mathcal{R} \to L^1g^*g_*\mathcal{R} \to L^1f^*f_*\mathcal{R}.$$

As f is an isomorphism in codimension one, the sheaves $L^2f^*f_*\mathcal{R}$ and $L^1f^*f_*\mathcal{R}$ have the support of codimension at least two. As the universal map $f^*f_*\mathcal{R} \to (f^*f_*\mathcal{R})^{\vee\vee}$ is also an isomorphism in codimension one, the induced map $(f^*f_*\mathcal{R})^{\vee\vee} \to L^1g^*g_*\mathcal{R}$ is an isomorphism in

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codimension one. Hence, it is an isomorphism, see Lemma 3.10. It follows that $L^1 g^* g_* \mathcal{R} \simeq (f^* f_* \mathcal{R})^{\vee\vee}$. As $\mathcal{R}$ is locally free [VdB04, Proposition 3.2.6], $(f^* f_* \mathcal{R})^{\vee\vee} \simeq \mathcal{R}$ by Lemma 3.11.

By Lemma 3.8, $f^* f_* \mathcal{R}$ is an object of $^p\mathrm{Per}(X/Y)$. As the same holds for $\mathcal{R} \in \mathscr{P}_p \subset {}^p\mathrm{Per}(X/Y)$, the exact triangle $\mathcal{R}[1] \to Lg^* g_* \mathcal{R} \to f^* f_* \mathcal{R}$ is also the decomposition of $Lg^* g_* \mathcal{R}$ in the t-structure with heart $^p\mathrm{Per}(X/Y)$. $\qquad\square$

Proposition 3.15 implies that triangle

$$\mathcal{M}[1] \to Lg^* g_* \mathcal{M} \to f^* f_* \mathcal{M} \to \mathcal{M}[2] \tag{36}$$

is exact. By Lemma 3.14, object $Lg^* g_* \mathcal{M}$ is quasi-isomorphic to a complex $f^* \mathcal{F}_{-1} \to f^* \mathcal{F}_0$, for some locally free sheaves $\mathcal{F}_{-1}$, $\mathcal{F}_0$ on Y. Hence, sequence

$$0 \to \mathcal{M} \to f^* \mathcal{F}_{-1} \to f^* \mathcal{F}_0 \to f^* f_* \mathcal{M} \to 0 \tag{37}$$

is a projective resolution for $f^* f_* \mathcal{M}$ in $^{-1}\mathrm{Per}(X/Y)$, see Lemma 3.8.

Note that short exact sequence (15) and Lemma 3.8 imply that

$$0 \to f^* f_* \mathcal{M} \to \mathcal{M} \to \mathcal{P}[1] \to 0 \tag{38}$$

is a short exact sequence in $^{-1}\mathrm{Per}(X/Y)$.

3.5 2-periodicity

PROPOSITION 3.16. *Let $f\colon X \to Y$ satisfy assumption (d) and assume that $Y \subset \mathcal{Y}$ is a principal divisor in a smooth affine $\mathcal{Y}$. Then there exists a morphism $Lf^* \to Lf^*[2]$ of functors $\mathcal{D}_{qc}(Y) \to \mathcal{D}_{qc}(X)$, which, for any E in $\mathcal{D}^b(Y)$, is an isomorphism on almost all cohomology sheaves of $Lf^*(E)$.*

Proof. By applying Lf^* to functorial exact triangle (27), we obtain a morphism $Lf^* E \to Lf^* E[2]$ with cone $Lf^* Li^* i_* E[1]$. As $\mathcal{Y}$ is smooth, $\mathcal{D}^b(\mathcal{Y})$ coincides with the category of perfect complexes on $\mathcal{Y}$. It follows that the functor $Lf^* Li^* i_*$ takes $\mathcal{D}^b(Y)$ to $\mathcal{D}^b(X)$ (even to $\mathrm{Perf}(X)$), hence, for any $E \in \mathcal{D}^b(Y)$, there exists $l \in \mathbb{Z}$ such that $\mathcal{H}^i(Lf^* Li^* i_* E) = 0$, for $i < l$. Then $\mathcal{H}^i Lf^* E \simeq \mathcal{H}^{i-2} Lf^* E$, for any $i < l$. $\qquad\square$

Now let $f\colon X \to Y$ be a crepant morphism of Gorenstein varieties satisfying assumption (p) and assume that $Y \subset \mathcal{Y}$ is a principal divisor in a smooth $\mathcal{Y}$. Then $\mathcal{I}_Y \simeq \mathcal{O}_\mathcal{Y}$, hence composing functorial exact triangle (27) with Lf^* and precomposing with Rf_* gives a functorial exact triangle

$$Lf^* Rf_*[1] \to Lg^* Rg_* \to Lf^* Rf_* \to Lf^* Rf_*[2]. \tag{39}$$

Object $Lg^* g_* \mathcal{M}$ has only two non-zero cohomology sheaves described by Proposition 3.15, for any $\mathcal{M} \in \mathscr{P}_{-1}$. Then the long exact sequence of cohomology sheaves for triangle (39) applied to $\mathcal{M}$ implies short exact sequence

$$0 \to L^2 f^* f_* \mathcal{M} \to f^* f_* \mathcal{M} \to \mathcal{M} \to 0$$

and isomorphisms $L^l f^* f_* \mathcal{M} \simeq L^{l+2} f^* f_* \mathcal{M}$, for $l \geqslant 1$. Hence, we have 2-periodicity

$$L^l f^* f_* \mathcal{M} \simeq \begin{cases} \mathcal{P}, & \text{for even } l \geqslant 2, \\ 0, & \text{for odd } l. \end{cases} \tag{40}$$

Remark 3.17. From Propositions 3.16 and 3.15 it follows that

$$L^l f^* f_* \mathcal{N} \simeq \begin{cases} \mathcal{Q}, & \text{for odd } l \geqslant 1, \\ 0, & \text{for even } l \geqslant 2. \end{cases}$$

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Remark 3.18. Let $f\colon X \to Y$ be a crepant morphism of Gorenstein varieties satisfying assumption (d) such that $Y \subset \mathcal{Y}$ is a principal divisor in a smooth affine $\mathcal{Y}$. Using Proposition 3.15 we construct a projective resolution

$$0 \to \mathcal{N} \to f^*\mathcal{E}_{-1} \to f^*\mathcal{E}_0 \to f^*f_*\mathcal{N} \to 0 \tag{41}$$

of $f^*f_*\mathcal{N}$ in ${}^0\mathrm{Per}(X/Y)$. Here, $\mathcal{E}_{-1}$ and $\mathcal{E}_0$ are locally free sheaves on Y such that $Li^*g_*\mathcal{N}$ is quasi-isomorphic to $\mathcal{E}_{-1} \to \mathcal{E}_0$ (see Lemma 3.14).

PROPOSITION 3.19 (Cf. [DW16, Proposition 5.6]). *Let* $f\colon X \to Y$ *satisfy assumption (a),* $\mathcal{M}$ *be in* $\mathscr{P}_{-1}$ *and* $\mathcal{N}$ *in* $\mathscr{P}_0$. *Then there exist locally free sheaves* $\mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{E}_{-1}, \mathcal{E}_0$ *on* Y *such that*

$$0 \to \mathcal{M} \to f^*\mathcal{F}_{-1} \to f^*\mathcal{F}_0 \to \mathcal{M} \to \mathcal{P}_{\mathcal{M}}[1] \to 0$$
$$0 \to \mathcal{N} \to f^*\mathcal{E}_{-1} \to f^*\mathcal{E}_0 \to \mathcal{N} \to \mathcal{P}_{\mathcal{N}} \to 0$$

are projective resolutions of $\mathcal{P}_{\mathcal{M}}[1] = \mathcal{H}_X^{-1}\iota_f^*\mathcal{M}[1]$ *and* $\mathcal{P}_{\mathcal{N}} = \mathcal{H}_X^0\iota_f^*\mathcal{N}$ *respectively in* ${}^{-1}\mathrm{Per}(X/Y)$ *and* ${}^0\mathrm{Per}(X/Y)$.

Proof. Composing resolution (37) with sequence (38) gives the projective resolution for $\mathcal{P}_{\mathcal{M}}[1]$ Similarly, resolution (41) and sequence (32) give projective resolution for $\mathcal{P}_{\mathcal{N}}$. $\qquad\square$

4. The flop functor and Van den Bergh's functor

Now let X, X^+ and Y be quasi-projective Gorenstein varieties of dimension n such that X and X^+ are related by a flop over Y. We assume that Y has canonical hypersurface singularities of multiplicity two. Note that this condition is satisfied if Y has dimension three and terminal Gorenstein singularities. We assume that fibers of f have dimension bounded by one and the exceptional locus of f has codimension greater than one in X. Then morphism f^+ satisfies the same conditions.

Van den Bergh in [VdB04] proved an equivalence of $\mathcal{D}^b(X)$ with $\mathcal{D}^b(X^+)$ under these assumptions.

We consider a diagram

$$\begin{array}{ccc}
& X \times_Y X^+ & \\
{}^p\swarrow & & \searrow{}^{p^+} \\
X & & X^+ \\
{}_f\searrow & & \swarrow{}_{f^+} \\
& Y &
\end{array} \tag{42}$$

and the flop functor

$$F = Rp_*^+ Lp^*\colon \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(X^+). \tag{43}$$

In this section we give an alternative description of the flop functor and the functor Σ considered in [VdB04] under an extra assumption that Y is affine. We show that these are inverse to each other. Keeping the assumption that Y is affine we also provide an alternative description of the flop–flop functor

$$F^+F = Rp_* Lp^{+*} Rp_*^+ Lp^*\colon \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(X). \tag{44}$$

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We construct a functorial exact triangle relating Σ, F and the derived push-forward to a smooth scheme $\mathcal{Y}$. We conclude that the flop functor induces an equivalence $\mathcal{D}^b(X) \xrightarrow{\simeq} \mathcal{D}^b(X^+)$ and, following [Che02], show that F is an equivalence also in the case when Y is not affine.

LEMMA 4.1. *Let $g\colon X \to Z$ be a proper surjective morphism with connected fibers. If Z is normal, then $g_*\mathcal{O}_X \simeq \mathcal{O}_Z$.*

Proof. For Stein decomposition $g\colon X \to \operatorname{Spec}_Z g_*\mathcal{O}_X \xrightarrow{\varphi} Z$, morphism φ is finite and has connected fibers, hence it is birational. As a finite birational morphism onto a normal variety is an isomorphism, φ is an isomorphism. It follows that $g_*\mathcal{O}_X \simeq \mathcal{O}_Z$. $\qquad\square$

Remark 4.2. We assume that X and X^+ are normal varieties. As morphism p is proper, surjective and with connected fibers, Lemma 4.1 implies that $p_*\mathcal{O}_{X\times_Y X^+} \simeq \mathcal{O}_X$. Moreover, by Proposition 2.11, the sheaf $R^1 p_*\mathcal{O}_{X\times_Y X^+}$ vanishes. It follows that p satisfies assumption (d).

4.1 An alternative description of Van den Bergh's functor

LEMMA 4.3. *Let $f\colon X \to Y$ satisfy assumption (d), Y affine and $\mathcal{M}_1$, $\mathcal{M}_2$ objects in $\mathscr{P}_p$, for $p = -1$ or 0. Then $\operatorname{Ext}_X^i(\mathcal{M}_1, \mathcal{M}_2) = 0$, for $i \geqslant 1$.*

Proof. By [VdB04, Proposition 3.2.6], objects in $\mathscr{P}_p$ are vector bundles on X. Hence, $\mathcal{E}xt^i(\mathcal{M}_1, \mathcal{M}_2) \simeq 0$, for $i > 0$. As morphism f has fibers of dimension bounded by one, the local-to-global spectral sequence implies that $\operatorname{Ext}^i(\mathcal{M}_1, \mathcal{M}_2) \simeq H^0(Y, R^i f_* \mathcal{H}om(\mathcal{M}_1, \mathcal{M}_2))$ vanishes, for $i > 1$. Group $\operatorname{Ext}_X^1(\mathcal{M}_1, \mathcal{M}_2)$ is isomorphic to $\operatorname{Ext}^1_{p\operatorname{Per}(X/Y)}(\mathcal{M}_1, \mathcal{M}_2)$, hence also zero, as $\mathcal{M}_1$ is an object of $\mathscr{P}_p$. $\qquad\square$

Let $\mathscr{P} \subset \mathcal{B}$ be a full subcategory of an abelian category closed under extensions. Denote by $\operatorname{Hot}_{\mathcal{B}}^{-,b}(\mathscr{P})$ the homotopy category of bounded above complexes of objects in $\mathscr{P}$ with bounded cohomology in $\mathcal{B}$. We use the notation $\operatorname{Hot}^{-,b}(\mathscr{P})$ when the category $\mathcal{B}$ is clear from the context. Denote by $\operatorname{Hot}^{-,b}(\mathscr{P})^{\leqslant l}$ the full subcategory of complexes with non-zero cohomology in degree up to l. Denote by $\operatorname{Hot}^-(\mathscr{P})$ the category of bounded above complexes of objects in $\mathscr{P}$ without any constraint on cohomology. Finally, denote by $\operatorname{Hot}^b(\mathscr{P})$ the full subcategory of bounded complexes in $\operatorname{Hot}^{-,b}(\mathscr{P})$. Categories $\operatorname{Hot}^{-,b}(\mathscr{P})$, $\operatorname{Hot}^-(\mathscr{P})$ and $\operatorname{Hot}^b(\mathscr{P})$ are triangulated by [Nee90]. If $\mathscr{P}$ is the category of projective objects in $^p\operatorname{Per}(X/Y)$, for $p = -1, 0$, then we use the above notation for the embedding $\mathscr{P}_p \subset \operatorname{Coh}(X)$.

Assume that morphism $f\colon X \to Y$ satisfies assumption (a). As $^p\operatorname{Per}(X/Y)$ has enough projective objects, for $p = -1$ or 0, category $\mathcal{D}^b(^p\operatorname{Per}(X/Y))$ is equivalent to $\operatorname{Hot}^{-,b}(\mathscr{P}_p)$, see [GM03, Theorem III.5.21].

PROPOSITION 4.4. *Let $f\colon X \to Y$ satisfy assumption (d) and Y affine. Then categories $\mathcal{D}^b(X)$ and $\mathcal{D}^b(^p\operatorname{Per}(X/Y))$ are equivalent, for $p = -1$ or 0.*

Proof. As objects in $\mathscr{P}_p$ are coherent sheaves on X, we have a functor $\Theta\colon \operatorname{Hot}^{-,b}(\mathscr{P}_p) \to \mathcal{D}^b(X)$. There exists N_0 such that $\operatorname{Ext}_X^q(P, C) = 0$, for any $P \in \mathscr{P}_p$ and $C \in {}^p\operatorname{Per}(X/Y)$ and $q \geqslant N_0$. Indeed, C has non-zero cohomology sheaves in degrees -1 and 0 only, and objects in $\mathscr{P}_p$ are locally free sheaves on X (see [VdB04, Proposition 3.2.6]). Then Proposition E.4 implies that cohomology groups of complex $\prod_{j-i=p} \operatorname{Hom}_X(A^i, B^j)$ are isomorphic to $\operatorname{Hom}_X(A^{\cdot}, B^{\cdot})$, for any $A^{\cdot}, B^{\cdot} \in \operatorname{Hot}^{-,b}(\mathscr{P}_p)$ (see Lemma 4.3). Hence, $\operatorname{Hom}_X^{\cdot}(A^{\cdot}, B^{\cdot}) \simeq \operatorname{Hom}_{\operatorname{Hot}^{-,b}(\mathscr{P}_p)}(A^{\cdot}, B^{\cdot})$, i.e. Θ is fully faithful. It is essentially surjective, because $^p\operatorname{Per}(X/Y)$ is the heart of a bounded t-structure. $\qquad\square$

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By Proposition 3.12, functor

$$\Sigma_o := (f^{+*}f_*(-))^{\vee\vee} \colon \mathscr{P}_{-1} \to \mathscr{P}_0^+ \tag{45}$$

is fully faithful. Denote by

$$\Sigma \colon \mathrm{Hot}^{-,b}(\mathscr{P}_{-1}) \to \mathrm{Hot}^-(\mathscr{P}_0^+) \tag{46}$$

the functor obtained by applying Σ_o term-wise to complexes of objects of $\mathscr{P}_{-1}$.

THEOREM 4.5. *Let* $f \colon X \to Y$ *satisfy assumption* (a) *and* $f^+ \colon X^+ \to Y$ *be the flop of* f. *Then functor* Σ *induces an equivalence between* $\mathcal{D}^b(X)$ *and* $\mathcal{D}^b(X^+)$.

Proof. Lemma 3.11 implies that functor

$$T_o := (f^* f_*^+(-))^{\vee\vee} \colon \mathscr{P}_0^+ \to \mathscr{P}_{-1}$$

is the inverse of Σ_o. Let us extend it term-wise to a functor

$$T \colon \mathrm{Hot}^{-,b}(\mathscr{P}_0^+) \to \mathrm{Hot}^-(\mathscr{P}_{-1}).$$

Proposition 3.12 implies that Σ and T are inverse equivalences between $\mathscr{P}_{-1}$ and $\mathscr{P}_0^+$. Hence, they are also inverse equivalences between $\mathrm{Hot}^-(\mathscr{P}_{-1})$ and $\mathrm{Hot}^-(\mathscr{P}_0)$. By Proposition 4.4, we have $\mathcal{D}^b(X) \simeq \mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$ and $\mathcal{D}^b(X^+) \simeq \mathrm{Hot}^{-,b}(\mathscr{P}_0^+)$. Thus, it suffices to show that Σ takes $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$ to $\mathrm{Hot}^{-,b}(\mathscr{P}_0^+)$ and T takes $\mathrm{Hot}^{-,b}(\mathscr{P}_0^+)$ to $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$.

Let E be in $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$ and assume that $\Sigma(E)$ has unbounded cohomology. If cohomology of $\Sigma(E)$ is unbounded with respect to the standard t-structure it is also unbounded with respect to the t-structure with heart ${}^0\mathrm{Per}(X^+/Y)$, because these two t-structures are related by a tilt. Let $\mathcal{N}^+$ be the projective generator of ${}^0\mathrm{Per}(X^+/Y)$. Then $\dim_k \mathrm{Hom}_{X^+}^\cdot(\mathcal{N}^+, \Sigma(E))$ is infinite. By adjunction of inverse functors, we have that $\dim_k \mathrm{Hom}_X^\cdot(T(\mathcal{N}^+), E)$ is infinite. As $T_o(\mathcal{N}^+)$ is in $\mathscr{P}_{-1}$, it contradicts the boundedness of the cohomology of E with respect to the t-structure with heart ${}^{-1}\mathrm{Per}(X/Y)$. Analogously, T takes $\mathrm{Hot}^{-,b}(\mathscr{P}_0^+)$ to $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$. $\square$

Let us show that Theorem 4.5 may be considered as a write-up of Van den Bergh's theorem.

Let $f \colon X \to Y$ satisfy assumption (a). As f has fibers of dimension bounded by one, [VdB04, Lemma 3.2.2] implies that for an f-ample line bundle $\mathcal{L}$ both $\mathcal{L} \oplus \mathcal{O}_X$ and $\mathcal{O}_X \oplus \mathcal{L}^{-1}$ are compact generators of $\mathcal{D}_{\mathrm{qc}}(X)$. It follows that both the projective generator $\mathcal{M} = \bigoplus \mathcal{M}_i$ of ${}^{-1}\mathrm{Per}(X/Y)$ (defined as in Remark 2.7) and $\mathcal{N} = \mathcal{H}\mathrm{om}(\mathcal{M}, \mathcal{O}_X)$ of ${}^0\mathrm{Per}(X/Y)$ (see Remark 2.6) are compact generators of the category $\mathcal{D}_{\mathrm{qc}}(X)$. Analogous result holds for projective generators for ${}^{-1}\mathrm{Per}(X^+/Y)$ and ${}^0\mathrm{Per}(X^+/Y)$.

As $(f^{+*}f_*(-))^{\vee\vee}$ restricts to an equivalence $\mathscr{P}_{-1} \to \mathscr{P}_0^+$, Proposition 3.12, $\mathcal{N}^+$ (as in (35)) is a projective generator for ${}^0\mathrm{Per}(X^+/Y)$ and the endomorphisms algebras of $\mathcal{M}$ and $\mathcal{N}^+$ are isomorphic.

Moreover, by Lemma 4.3, both $\mathcal{M}$ and $\mathcal{N}^+$ have no higher self Ext groups. Hence, by the result of Keller [Kel06] both $\mathcal{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}_{\mathrm{qc}}(X^+)$ are equivalent to the derived category of DG modules over the algebra $A = \mathrm{Hom}_X(\mathcal{M}, \mathcal{M})$, cf. [BVdB03]. Denote by

$$\Sigma_{\mathrm{qc}} \colon \mathcal{D}_{\mathrm{qc}}(X) \xrightarrow{\simeq} \mathcal{D}_{\mathrm{qc}}(X^+)$$

the resulting equivalence.

PROPOSITION 4.6. *Let* $f \colon X \to Y$ *satisfy assumption* (a). *Functor* $\Sigma_{\mathrm{qc}}|_{\mathcal{D}^b(X)} \colon \mathcal{D}^b(X) \to \mathcal{D}^b(X^+)$ *is an equivalence that takes* ${}^{-1}\mathrm{Per}(X/Y)$ *to* ${}^0\mathrm{Per}(X^+/Y)$, *and it coincides with Van den Bergh's equivalence in [VdB04, Theorem 4.4.2].*

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Proof. Category $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$ is a subcategory of the homotopy category of complexes of projective A-modules; it consists of bounded above complexes. As Σ_o as in (45) maps projective generator for $^{-1}\mathrm{Per}(X/Y)$ to the projective generator for $^{0}\mathrm{Per}(X^+/Y)$, we have $\Sigma_{\mathrm{qc}}|_{\mathscr{P}_{-1}} = \Sigma_o$. We conclude that $\Sigma = \Sigma_{\mathrm{qc}}|_{\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})}$ is the equivalence defined by Van den Bergh in [VdB04, Theorem 4.4.2]. $\qquad\square$

Categories $\mathscr{P}_0$ and $\mathscr{P}_{-1}^+$ are also equivalent. We can thus analogously construct an equivalence $\Sigma_{\mathrm{qc}}^{-1} \colon \mathcal{D}^b(X) \to \mathcal{D}^b(X^+)$ that takes $^{0}\mathrm{Per}(X/Y)$ to $^{-1}\mathrm{Per}(X^+/Y)$.

4.2 An alternative description of the flop functor

Consider diagram (42). The counit of $f^* \dashv f_*$ adjunction

$$f^*f_* \to \mathrm{Id}$$

gives a morphism

$$p^*f^*f_* \to p^*.$$

Isomorphism $p^*f^* \simeq p^{+*}f^{+*}$ and $p^{+*} \dashv p_*^+$ adjunction lead to a base-change morphism

$$\epsilon \colon f^{+*}f_* \to p_*^+ p^*. \tag{47}$$

Note that the morphism ϵ is, by definition, the composition

$$\epsilon \colon f^{+*}f_* \to p_*^+ p^{+*} f^{+*} f_* \to p_*^+ p^*. \tag{48}$$

Analogous argument gives a derived base change

$$\widetilde{\epsilon} \colon Lf^{+*}Rf_* \to Rp_*^+ Lp^*. \tag{49}$$

Lemma 4.7. *Let f satisfy assumption (a) and $\mathcal{M}$ be an object in $\mathscr{P}_{-1}$. Then $\epsilon_{\mathcal{M}} \colon f^{+*}f_*\mathcal{M} \to p_*^+ p^*\mathcal{M}$ is an isomorphism.*

Proof. As $\mathcal{M}$ is locally free [VdB04, Propostion 3.2.6], applying p^* to sequence (15) we obtain an exact sequence

$$0 \to p^*\mathcal{P} \to p^*f^*f_*\mathcal{M} \to p^*\mathcal{M} \to 0$$

on $X \times_Y X^+$. Sheaf $\mathcal{P}$ is in $\mathscr{A}_f$, hence Proposition 2.11 implies an isomorphism

$$p_*^+ p^* f^* f_* \mathcal{M} \xrightarrow{\simeq} p_*^+ p^* \mathcal{M}.$$

As diagram (42) is commutative, $p_*^+ p^* f^* f_* \mathcal{M} \simeq p_*^+ p^{+*} f^{+*} f_* \mathcal{M}$. Isomorphism $f^{+*}f_*\mathcal{M} \simeq f^{+*}f_*^+\mathcal{N}^+$ (see Proposition 3.12) together with Lemma 3.5 implies that $f^{+*}f_*\mathcal{M}$ is torsion-free. Moreover, by Remark 4.2, $Rp_*^+\mathcal{O}_{X\times_Y X^+} = \mathcal{O}_{X^+}$. Thus, conditions of Lemma 2.8 are satisfied and we have an isomorphism $f^{+*}f_*\mathcal{M} \xrightarrow{\simeq} p_*^+ p^{+*} f^{+*} f_* \mathcal{M}$. Hence, $\epsilon_{\mathcal{M}}$ is a composite of two isomorphisms in (48). $\qquad\square$

Proposition 4.8. *Let f satisfy assumption (a). Then the flop functor $Rp_*^+ Lp^*$ on $\mathscr{P}_{-1}$ is isomorphic to the non-derived flop functor $p_*^+ p^*$.*

Proof. Every object in $\mathscr{P}_{-1}$ locally free, [VdB04, Propostion 3.2.6], hence $Lp^*|_{\mathscr{P}_{-1}} \simeq p^*|_{\mathscr{P}_{-1}}$. Moreover, because $R^1f_*\mathscr{P}_{-1} \simeq 0$, [VdB04, Lemma 3.1.2] Proposition 2.11 implies that $R^1p_*^+ p^*\mathscr{P}_{-1} \simeq 0$. $\qquad\square$

We define functor

$$f^{+*}f_* \colon \operatorname{Hot}^{-,b}(\mathscr{P}_{-1}) \to \operatorname{Hot}^-(f^{+*}f_*\mathscr{P}_{-1})$$

term-wise. We use the same notation for the composite of the above functor with the canonical functor $\operatorname{Hot}^-(f^{+*}f_*\mathscr{P}_{-1}) \to \mathcal{D}^-(X^+)$. Also term-wise, we define a functor

$$p_*^+ p^* \colon \operatorname{Hot}^{-,b}(\mathscr{P}_{-1}) \to \mathcal{D}^-(X^+).$$

LEMMA 4.9. *Let f satisfy assumption (a). Then the functor $f^{+*}f_*$ maps $\operatorname{Hot}^{-,b}(\mathscr{P}_{-1})$ to $\mathcal{D}^b(X^+)$. The functor $f^* f_*$ maps $\operatorname{Hot}^{-,b}(\mathscr{P}_{-1})$ to $\mathcal{D}^b(X)$.*

Proof. The $i^* \dashv i_*$ adjunction counit induces an isomorphism $g^{+*}g_* \xrightarrow{\cong} f^{+*}f_*$. Hence, it is enough to show that the term-wise extension of $g^{+*}g_*$ maps $\operatorname{Hot}^{-,b}(\mathscr{P}_{-1})$ to $\mathcal{D}^b(X^+)$.

Functor Rg_* preserves the bounded derived category of coherent sheaves, because g is proper, and so does Lg^{+*}, because $\mathcal{Y}$ is smooth. Hence, for $P^{\cdot}$ in $\operatorname{Hot}^{-,b}(\mathscr{P}_{-1}) \simeq \mathcal{D}^b(X)$ (see Proposition 4.4), $Lg^{+*}Rg_*(P^{\cdot})$ is an object in $\mathcal{D}^b(X^+)$. For $P \in \mathscr{P}_{-1}$, we denote by $P^+ = (f^{+*}f_*P)^{\vee\vee}$ the corresponding object in $\mathscr{P}_0^+$ (see Proposition 3.12). As $f_*^+ P^+ = f_* P$ and $g_*^+ P^+ = g_* P$, Proposition 3.15 for g^+, f^+ and P^+ instead of g, f and $\mathcal{R}$, implies that the first sheet of the spectral sequence $E_1^{p,q} = \mathcal{H}_{X^+}^q(Lg^{+*}g_*P^p) \Rightarrow \mathcal{H}_{X^+}^{p+q}(Lg^{+*}Rg_*P^{\cdot})$ has two non-zero rows:

$$E_1^{\bullet,0} = f^{+*}f_*^+ P^{+\bullet} = g^{+*}g_*^+ P^{+\bullet} = g^{+*}g_* P^{\bullet}, \quad E_1^{\bullet,-1} = P^{+\bullet}.$$

By Theorem 4.5, the complex $P^{+\bullet}$ is in $\operatorname{Hot}^{-,b}(\mathscr{P}_0^+)$, hence cohomology of the -1st row is bounded: $E_2^{N,-1} = 0$, for $N \leqslant N_0$. The spectral sequence implies that, for sufficiently negative N, $\mathcal{H}_{X^+}^N(g^{+*}g_* P^{\bullet}) = E_2^{N,0} = E_\infty^{N,0} = \mathcal{H}_{X^+}^N(Lg^{+*}g_* P^{\bullet}) = 0$, i.e. $g^{+*}g_*(P^{\bullet}) \in \mathcal{D}^b(X^+)$.

The proof of the boundedness for $f^* f_*(P^{\bullet}) = g^* g_*(P^{\bullet})$ is similar with the use of Proposition 3.15 for $\mathcal{M}$ instead of $\mathcal{N}^+$. $\qquad\square$

PROPOSITION 4.10. *Let f satisfy assumption (a). Then functors $f^{+*}f_*$ and $p_*^+ p^*$ are isomorphic on the category $\operatorname{Hot}^{-,b}(\mathscr{P}_{-1})$. They are also isomorphic to the flop functor restricted to $\mathcal{D}^b(X)$ under the equivalence $\mathcal{D}^b(X) \simeq \operatorname{Hot}^{-,b}(\mathscr{P}_{-1})$.*

Proof. Proposition 4.8 implies that the canonical morphism $p_*^+ p^*|_{\mathscr{P}_{-1}} \to Rp_*^+ p^*|_{\mathscr{P}_{-1}}$ is an isomorphism. All objects in $\mathscr{P}_{-1}$ are locally free [VdB04, Propostion 3.2.6], hence canonical morphism $Rp_*^+ Lp^*|_{\mathscr{P}_{-1}} \to Rp_*^+ p^*|_{\mathscr{P}_{-1}}$ is also an isomorphism. This implies isomorphism of $p_*^+ p^*|_{\mathscr{P}_{-1}}$ and $Rp_*^+ Lp^*|_{\mathscr{P}_{-1}}$.

Lemma 4.7 assures that the base change $\epsilon_{\mathcal{M}} \colon f^{+*}f_* \mathcal{M} \to p_*^+ p^* \mathcal{M}$ is an isomorphism, for all $\mathcal{M} \in \mathscr{P}_{-1}$.

Thus, all three functors are isomorphic on $\mathscr{P}_{-1}$. Induction on triangles shows that the isomorphism extends to $\operatorname{Hot}^b(\mathscr{P}_{-1})$. By Lemmas 4.9 and 4.11, they are also isomorphic on $\operatorname{Hot}^{-,b}(\mathscr{P}_{-1})$. $\qquad\square$

In the proof of Proposition 4.10 we use the fact that in order to check that a natural transformation gives an isomorphism of two functors defined on $\operatorname{Hot}^{-,b}(\mathscr{P}_{-1})$, it suffices to check that it gives an isomorphism of the functors restricted to $\operatorname{Hot}^b(\mathscr{P}_{-1})$, which we prove now.

LEMMA 4.11. *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories, $\mathscr{P} \subset \mathcal{B}$ a full subcategory closed under extensions and $F, G \colon \operatorname{Hot}_{\mathcal{B}}^{-,b}(\mathscr{P}) \to \mathcal{D}^-(\mathcal{A})$ exact functors. Assume there exists n_0 such that both F and G map $\operatorname{Hot}_{\mathcal{B}}^{-,b}(\mathscr{P})^{\leqslant 0}$ to $\mathcal{D}^-(\mathcal{A})^{\leqslant n_0}$ and that the image of either F or G is contained in $\mathcal{D}^b(\mathcal{A})$. Let $\kappa \colon F|_{\operatorname{Hot}_{\mathcal{B}}^b(\mathscr{P})} \to G|_{\operatorname{Hot}_{\mathcal{B}}^b(\mathscr{P})}$ be an isomorphism of functors. Then κ admits an extension to a functorial isomorphism $\widetilde{\kappa} \colon F \to G$.*

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Proof. Let E be an object in $\mathrm{Hot}^{-,b}(\mathscr{P})$. We denote by $\sigma_{\geqslant k}E$ the 'stupid' truncation:

$$\sigma_{\geqslant k}(E)^i = \begin{cases} E^i, & \text{if } i \geqslant k, \\ 0, & \text{otherwise.} \end{cases}$$

Morphism $\sigma_{\geqslant l}E \to E$ induces isomorphisms $\tau_{\geqslant m}F(\sigma_{\geqslant l}E) \xrightarrow{\simeq} \tau_{\geqslant m}F(E)$, $\tau_{\geqslant m}G(\sigma_{\geqslant l}E) \xrightarrow{\simeq} \tau_{\geqslant m}G(E)$, for any $m \in \mathbb{Z}$, and $l \leqslant m - n_0$. We have

$$\tau_{\geqslant m}F(E) \simeq \tau_{\geqslant m}F(\sigma_{\geqslant l}E) \simeq \tau_{\geqslant m}G(\sigma_{\geqslant l}E) \simeq \tau_{\geqslant m}G(E), \tag{50}$$

for any $l \leqslant m - n_0$. Let us assume that $F(E)$ is an object in $\mathcal{D}^b(\mathcal{A})$. There exists N such that morphism $F(E) \to \tau_{\geqslant m}F(E)$ is an isomorphism, for any $m \leqslant N$. Thus, isomorphisms (50) imply that $G(E)$ is also an object in $\mathcal{D}^b(\mathcal{A})$. We define $\widetilde{\kappa}$ as the composite of isomorphisms

$$F(E) \to \tau_{\geqslant m}F(E) \to \tau_{\geqslant m}F(\sigma_{\geqslant l}E) \xrightarrow{\tau_{\geqslant m}\kappa_{\sigma_{\geqslant l}E}} \tau_{\geqslant m}G(\sigma_{\geqslant l}E) \to \tau_{\geqslant m}G(E) \to G(E),$$

for any $m \leqslant N$ and $l \leqslant m - n_0$. As κ is a natural transformation, $\tau_{\geqslant m}\kappa_{\sigma_{\geqslant l}E} \simeq \tau_{\geqslant m}\kappa_{\sigma_{\geqslant l'}E}$, for any $l, l' \leqslant m - n_0$. Hence, $\widetilde{\kappa}$ does not depend on the choice of $m \leqslant N$ and $l \leqslant m - n_0$. $\square$

4.3 A functorial exact triangle of functors $\mathcal{D}_{\mathbf{qc}}(X) \to \mathcal{D}_{\mathbf{qc}}(X^+)$

Assume that $f \colon X \to Y$ satisfies assumption (a), i.e. there exists a closed embedding $i \colon Y \to \mathcal{Y}$, for a smooth $\mathcal{Y}$ of dimension $n + 1$. Then, $X \times_Y X^+ \simeq X \times_{\mathcal{Y}} X^+$, i.e. diagram

$$\tag{51}$$

is fibered.

Recall, that we denote by $F = Rp_*^+ Lp^*$ the flop functor and by $\Sigma \colon \mathrm{Hot}^{-,b}(\mathscr{P}_{-1}) \to \mathcal{D}^b(X^+)$ the term-wise extension of $\Sigma_o = (f^{+*}f_*(-))^{\vee\vee} \colon \mathscr{P}_{-1} \to \mathscr{P}_0^+$.

LEMMA 4.12. *Let* $f \colon X \to Y$ *satisfy assumption* (a). *Functors* $Rf_*^+\Sigma, Rf_* \colon \mathrm{Hot}^{-,b}(\mathscr{P}_{-1}) \to \mathcal{D}^b(Y)$ *are isomorphic.*

Proof. Consider morphism $\alpha \colon f_*|_{\mathscr{P}_{-1}} \to f_*^+\Sigma_o|_{\mathscr{P}_{-1}}$ defined as the composite

$$f_* \xrightarrow{\eta f_*} f_*^+ f^{+*} f_* \xrightarrow{f_*^+ \beta f^{+*} f_*} f_*^+(f^{+*}f_*(-)^{\vee\vee}) = f_*^+\Sigma_o,$$

for the unit η of $f^{+*} \dashv f_*^+$ adjunction and the reflexification $\beta \colon (-) \to (-)^{\vee\vee}$. Lemma 3.11 implies that α is an isomorphism of functors (see Lemma 4.11).

Morphism α yields morphism $\alpha^- \colon f_*^- \to (f_*^+\Sigma)$ of the term-wise extension of functors $f_*|_{\mathscr{P}_{-1}}$ and $f_*^+\Sigma_o|_{\mathscr{P}_{-1}}$ to the category $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$. As α is an isomorphism, the same is true about α^-.

Now let E be an object in $\mathcal{D}^b(X)$. It is isomorphic to a complex $\mathcal{P}.$ in $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$. The first layer of spectral sequence

$$E_{p,q}^1 = R^p f_* \mathcal{P}_q \Rightarrow R^{p+q} f_* E$$

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has only one non-zero row. Hence, complex $f_*\mathcal{P}.$ is isomorphic to Rf_*E. Analogously, because $\Sigma_o(\mathcal{P}_{-1}) \simeq \mathscr{P}_0^+$, complex $f_*^+\Sigma(\mathcal{P}.)$ is isomorphic to $Rf_*^+\Sigma E$. Thus, α^- induces an isomorphism of Rf_* and $Rf_*^+\Sigma$. $\qquad\square$

PROPOSITION 4.13. *Let $f\colon X \to Y$ satisfy assumption (a) and let Σ_{qc} be as in Proposition 4.6. The derived base change $\widetilde{\epsilon}$ induces a functorial exact triangle of functors $\mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(X^+)$:*

$$\Sigma_{\mathrm{qc}}[1] \to Lg^{+*}Rg_* \xrightarrow{\widetilde{\epsilon}} F \to \Sigma_{\mathrm{qc}}[2].$$

Proof. We have already seen in §4.1 that there exists a choice of compact generators $\mathcal{M}$ and $\mathcal{N}^+$ of $\mathcal{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}_{\mathrm{qc}}(X^+)$, respectively, such that $\mathcal{D}_{\mathrm{qc}}(X) \simeq \mathcal{D}(\mathrm{Mod}\text{–}A) \simeq \mathcal{D}_{\mathrm{qc}}(X^+)$, for an algebra $A = \mathrm{Hom}_X(\mathcal{M}, \mathcal{M})$ (see Lemma 4.3). Under these equivalences, functor Σ_{qc} is given by algebra A considered as an $A^{\mathrm{op}} \otimes A$ bimodule.

Appendix C, allows us to lift the base change $Lg^{+*}Rg_* \to Rp_*^+Lp^*$ to a 1-morphism in **Bimod**. It induces a functorial exact triangle

$$\Sigma' \to Lg^{+*}Rg_* \to Rp_*^+Lp^* \to \Sigma'[1]. \tag{52}$$

Let us show that functors $\Sigma'|_{\mathscr{P}_{-1}}$ and $\Sigma_o[1]|_{\mathscr{P}_{-1}}$ are isomorphic. To this end, we consider an object $\mathcal{M}$ in $\mathscr{P}_{-1}$. By Proposition 3.12, object $\mathcal{N}^+ := (f^{+*}f_*(\mathcal{M}))^{\vee\vee}$ lies in $\mathscr{P}_0^+$. Thus, Proposition 3.15 yields an exact triangle

$$\mathcal{N}^+[1] \to Lg^{+*}Rg_*^+\mathcal{N}^+ \to f^{+*}f_*^+\mathcal{N}^+ \to \mathcal{N}^+[2]. \tag{53}$$

By Lemma 3.11, we have $f_*^+\mathcal{N}^+ \simeq f_*\mathcal{M}$, hence $g_*^+\mathcal{N}^+ \simeq g_*\mathcal{M}$. Proposition 4.10 assures that $f^{+*}f_*\mathcal{M} \simeq F(\mathcal{M})$, hence triangle (53) reads

$$\Sigma_o(\mathcal{M})[1] \to Lg^{+*}Rg_*\mathcal{M} \to F(\mathcal{M}) \to \Sigma_o(\mathcal{M})[2]. \tag{54}$$

There exists a morphism from triangle (54) to triangle (52) applied to $\mathcal{M}$ which is equal to the identity morphism on $Lg^{+*}g_*\mathcal{M}$ and on $F(\mathcal{M}) = Rp_*^+Lp^*\mathcal{M}$. Indeed, the map $Lg^{+*}Rg_*\mathcal{M} \to F(\mathcal{M})$ in (54) is the composite

$$Lg^{+*}Rg_*\mathcal{M} \xrightarrow{\tau_{\geqslant 0}} f^{+*}f_*\mathcal{M} \xrightarrow{\simeq} p_*^+p^*\mathcal{M} \xrightarrow{\simeq} Rp_*^+Lp^*\mathcal{M}$$

of the truncation $\tau_{\geqslant 0}$, which in this case is taking the zeroth cohomology, followed by the base-change isomorphism (see Proposition 4.10) and then by the isomorphism $p_*^+p^*\mathcal{M} \simeq Rp_*^+Lp^*\mathcal{M}$ (see Proposition 4.8). Note that, under the isomorphism $f^{+*}f_* \simeq g^{+*}g_*$ of functors $\mathrm{Coh}(X) \to \mathrm{Coh}(X^+)$, induced by the $i^* \dashv i_*$ adjunction counit, the base change $f^{+*}f_* \to p_*^+p^*$ corresponds to the base change $g^{+*}g_* \to p_*^+p^*$. On the other hand, the map $Lg^{+*}Rg_* \to Rp_*^+Lp^*$ in (52) is the (derived) base change. For the degree reason (as $Rp_*^+Lp^*\mathcal{M}$ is a sheaf by Proposition 4.8), the derived base change for $\mathcal{M}$ factors via $H^0(Lg^{+*}Rg_*\mathcal{M})$. The existence of the desired morphism of triangles follows from the fact that the restriction of the derived base change to zeroth cohomology is the non-derived base change. Thus, for any $\mathcal{M}$ in $\mathscr{P}_{-1}$, there exists an isomorphism $\alpha_\mathcal{M}\colon \Sigma_o(\mathcal{M})[1] \to \Sigma'(\mathcal{M})$. It is unique, as $\mathrm{Hom}_{X^+}(\Sigma_o(\mathcal{M})[1], F(\mathcal{M})[-1]) = \mathrm{Ext}_{X^+}^{-2}(\mathcal{N}^+, f^{+*}f_*^+\mathcal{N}^+) = 0$ (for degree reasons). Morphism $\alpha_\mathcal{M}$ when composed with $\Sigma'(\mathcal{M}) \to Lg^{+*}Rg_*(\mathcal{M})$ is equal to $\beta_\mathcal{M}$, for a morphism $\beta = \eta\Sigma_o\colon \Sigma_o[1] \to Lg^{+*}Rg_*$. Here, $\eta\colon \mathrm{Id}[1] \to g^{+!}Rg_*^+[1] \simeq Lg^{+*}Rg_*^+$ is the adjunction unit (we use $Rg_*^+\Sigma_o \simeq Rg_*$ which follows from Lemma 4.12).

For any $\mathcal{M}, \mathcal{M}_1 \in \mathscr{P}_{-1}$ and $\varphi\colon \mathcal{M} \mapsto \mathcal{M}_1$, both compositions

$$\Sigma_o(\mathcal{M})[1] \xrightarrow{\Sigma_o(\varphi)[1]} \Sigma_o(\mathcal{M}_1)[1] \xrightarrow{\alpha_{\mathcal{M}_1}} \Sigma'(\mathcal{M}_1), \quad \Sigma_o(\mathcal{M})[1] \xrightarrow{\alpha_\mathcal{M}} \Sigma'(\mathcal{M}) \xrightarrow{\Sigma'(\varphi)} \Sigma'(\mathcal{M}_1)$$

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fit into the following commuting diagram.

$$\begin{array}{ccccc}
F(\mathcal{M}_1)[-1] & \longrightarrow & \Sigma'(\mathcal{M}_1) & \longrightarrow & Lg^{+*}Rg_*(\mathcal{M}_1) \\
{\scriptstyle F(\varphi)[-1]}\big\uparrow & & \big\uparrow & & \big\uparrow{\scriptstyle Lg^{+*}Rg_*(\varphi)} \\
F(\mathcal{M})[-1] & \longrightarrow & \Sigma_o(\mathcal{M})[1] & \longrightarrow & Lg^{+*}Rg_*(\mathcal{M})
\end{array}$$

As both $\mathrm{Hom}_{X^+}(\Sigma_o(\mathcal{M})[1], F(\mathcal{M}_1))$ and $\mathrm{Hom}_{X^+}(\Sigma_o(\mathcal{M})[1], F(\mathcal{M}_1)[-1])$ vanish (also for degree reasons), we have $\Sigma'(\varphi) \circ \alpha_\mathcal{M} = \alpha_{\mathcal{M}_1} \circ \Sigma_o(\varphi)[1]$, i.e. α extends to a morphism of functors $\alpha\colon \Sigma_o[1]|_{\mathscr{P}_{-1}} \to \Sigma'|_{\mathscr{P}_{-1}}$. Moreover, any ψ in $\mathrm{Hom}_{X^+}(\Sigma_o(\mathcal{M})[1], \Sigma_o(\mathcal{M}_1)[1]) \simeq \mathrm{Hom}_X(\mathcal{M}, \mathcal{M}_1)$ uniquely determines $\widetilde{\psi}\colon \Sigma'(\mathcal{M}) \to \Sigma'(\mathcal{M}_1)$. Similarly, as $\mathrm{Hom}_{X^+}(\Sigma'(\mathcal{M}), F(\mathcal{M}_1)) = 0 = \mathrm{Hom}_{X^+}(\Sigma'(\mathcal{M}), F(\mathcal{M}_1)[-1])$, any $\tau \in \mathrm{Hom}_{X^+}(\Sigma'(\mathcal{M}), \Sigma'(\mathcal{M}_1))$ determines $\widetilde{\tau}\colon \Sigma_o(\mathcal{M})[1] \to \Sigma_o(\mathcal{M}_1)[1]$. Thus, morphism $\alpha\colon \mathrm{Hom}_{X^+}(\Sigma_o(\mathcal{M})[1], \Sigma_o(\mathcal{M}_1)[1]) \to \mathrm{Hom}_{X^+}(\Sigma'(\mathcal{M}), \Sigma'(\mathcal{M}_1))$ is a bijection. It follows that α is an isomorphism of functors $\alpha\colon \Sigma_o[1]|_{\mathscr{P}_{-1}} \xrightarrow{\simeq} \Sigma'|_{\mathscr{P}_{-1}}$.

Lemma 4.3 implies that $\mathcal{D}_{qc}(X) \simeq \mathcal{D}(\mathrm{Mod}\text{–}A)$, for an algebra A. Furthermore, by definition, functor Σ_{qc} is isomorphic to the bimodule functor Φ_A, for A considered as $A^{\mathrm{op}} \otimes A$ bimodule. By construction (via an exact triangle), Σ' is also a bimodule functor. Thus, assumptions of Lemma C.12 are satisfied and we conclude that $\Sigma'[-1] \simeq \Sigma_{qc}$. Hence, functorial exact triangle (52) reads

$$\Sigma_{qc}[1] \to Lg^{+*}Rg_* \to F \to \Sigma_{qc}[2]. \qquad \square$$

COROLLARY 4.14. *Let $f\colon X \to Y$ satisfy assumption (p). The flop functor F takes $\mathcal{D}^b(X)$ to $\mathcal{D}^b(X^+)$.*

Proof. The statement is local in Y, therefore we can assume that morphism f satisfies assumption (a). As both $Lg^{+*}Rg_*$ and Σ_{qc} take $\mathcal{D}^b(X)$ to $\mathcal{D}^b(X^+)$ (see Theorem 4.5), the functorial exact triangle of Proposition 4.13 implies that the flop functor F takes $\mathcal{D}^b(X)$ to $\mathcal{D}^b(X^+)$. $\square$

4.4 An alternative description of the flop–flop functor

We show that F^+F on category $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$ is isomorphic to f^*f_*. First, we show vanishing of higher inverse images of projective objects in $\mathscr{A}_f$ and then extend Proposition 4.8 to the sheaf $f^*f_*\mathcal{N}$.

LEMMA 4.15. *Let $f\colon X \to Y$ satisfy assumption (a) and $\mathcal{P}$ be projective in $\mathscr{A}_f$. Then $L^j p^*\mathcal{P} = 0$, for $j \geqslant 2$.*

Proof. The statement is local in X, in particular in Y. Therefore, we can assume that f satisfies assumption (c) and $\mathcal{P}$ is a direct sum of copies of $\mathcal{P}_i$ as in (19).

Consider the counit $Lg^*g_*\mathcal{M}_i \to \mathcal{M}_i$ of the $Lg^* \dashv Rg_*$ adjunction. It gives an exact triangle

$$Lg^*g_*\mathcal{M}_i \to \mathcal{M}_i \to S_{\mathcal{M}_i} \to Lg^*g_*\mathcal{M}_i[1]. \qquad (55)$$

Applying functor Lp^* to it yields an exact triangle on $X \times_Y X^+$:

$$L(p^*g^*)g_*\mathcal{M}_i \to p^*\mathcal{M}_i \to Lp^*S_{\mathcal{M}_i} \to L(p^*g^*)g_*\mathcal{M}_i[1].$$

Lemma 3.14 implies that $L^j(p^*g^*)g_*\mathcal{M}_i = 0$, for $j \geqslant 2$. Thus, $L^j p^*S_{\mathcal{M}_i} = 0$, for $j > 2$.

Proposition 3.15, the long exact sequence of cohomology associated to triangle (55) and sequence (15) imply that $S_{\mathcal{M}_i}$ has two non-zero cohomology sheaves:

$$\mathcal{M}_i[2] \to S_{\mathcal{M}_i} \to \mathcal{P}_i[1] \to \mathcal{M}_i[3].$$

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By applying Lp^* and looking at the cohomology sheaves of the obtained triangle, we obtain an isomorphism $L^j p^* \mathcal{P}_i \simeq L^{j+1} p^* S_{\mathcal{M}_i} \simeq 0$, for $j \geqslant 3$, and an exact sequence

$$0 \to L^2 p^* \mathcal{P}_i \to p^* \mathcal{M}_i \to L^2 p^* S_{\mathcal{M}_i} \to L^1 p^* \mathcal{P}_i \to 0.$$

Sheaf $L^2 p^* \mathcal{P}_i$ is supported on the exceptional divisor of p, hence it is torsion. As $p^* \mathcal{M}_i$ is locally free, $L^2 p^* \mathcal{P}_i = 0$. $\square$

LEMMA 4.16. *Let* $f \colon X \to Y$ *satisfy assumption (a). Then, for any* $\mathcal{N} \in \mathscr{P}_0$, *the pullbacks of* $\mathcal{N}$ *and* $\mathcal{M}^+ = (f^{+*} f_* \mathcal{N})^{\vee\vee}$ *fit into the short exact sequence*

$$0 \to p^{+*} \mathcal{M}^+ \to p^* \mathcal{N} \to L^1 p^{+*} (f^{+*} f_*^+ \mathcal{M}^+) \to 0 \tag{56}$$

on $X \times_Y X^+$.

Proof. By Proposition 3.15, we have an exact triangle

$$\mathcal{N}[1] \to Lg^* g_* \mathcal{N} \to f^* f_* \mathcal{N} \to \mathcal{N}[2]. \tag{57}$$

By applying Lp^* to (57), we obtain an exact triangle

$$Lp^* \mathcal{N}[1] \to L(p^* g^*) g_* \mathcal{N} \to Lp^* (f^* f_* \mathcal{N}) \to Lp^* \mathcal{N}[2]. \tag{58}$$

As $g_* \mathcal{N}$ has a locally free resolution of length two (see Lemma 3.14), $L^2 (p^* g^*) g_* \mathcal{N} = 0$. The long exact sequence of cohomology for triangle (58) implies the exact sequence

$$0 \to L^2 p^* (f^* f_* \mathcal{N}) \to p^* \mathcal{N} \to L^1 (p^* g^*) g_* \mathcal{N} \to L^1 p^* (f^* f_* \mathcal{N}) \to 0. \tag{59}$$

As $\mathcal{N}$ is locally free, $L^{>0} p^* \mathcal{N} = 0$. Hence, by applying Lp^* to sequence (32), we obtain isomorphisms $L^3 p^* \mathcal{Q} \simeq L^2 p^* (f^* f_* \mathcal{N})$ and $L^2 p^* \mathcal{Q} \simeq L^1 p^* (f^* f_* \mathcal{N})$, where $\mathcal{Q} = \mathcal{H}_X^0 (\iota_{f*} \iota_f^* \mathcal{N})$ is a projective object in $\mathscr{A}_f$. In view of Lemma 4.15, sequence (59) is reduced to an isomorphism $p^* \mathcal{N} \simeq L^1 (p^* g^*) g_* \mathcal{N}$.

Let $\mathcal{M}^+ = (f^{+*} f_* \mathcal{N})^{\vee\vee}$ be an object in $\mathscr{P}_{-1}^+$. Proposition 3.12 implies that $f_*^+ \mathcal{M}^+ \simeq f_* \mathcal{N}$, hence $g_*^+ \mathcal{M}^+ \simeq g_* \mathcal{N}$. As diagram (51) commutes, we have

$$p^* \mathcal{N} \simeq L^1 (p^* g^*) g_* \mathcal{N} \simeq L^1 (p^{+*} g^{+*}) g_*^+ \mathcal{M}^+. \tag{60}$$

By applying Lp^{+*} to sequence (36), we obtain an exact sequence

$$0 \to L^2 p^{+*} (f^{+*} f_*^+ \mathcal{M}^+) \to p^{+*} \mathcal{M}^+ \to L^1 (p^{+*} g^{+*}) g_*^+ \mathcal{M}^+ \to L^1 p^{+*} (f^{+*} f_*^+ \mathcal{M}^+) \to 0. \tag{61}$$

Now consider sequence (15) with f^+, $\mathcal{M}^+$ and $\mathcal{P}^+$ instead of f, $\mathcal{M}$ and $\mathcal{P}$. Apply functor Lp^{+*} to it. Then Lemma 4.15 for f^+, p^+ and $\mathcal{P}^+$ instead of f,p and $\mathcal{P}$, implies that $L^2 p^{+*} (f^{+*} f_*^+ \mathcal{M}^+) \simeq 0$. Hence, sequence (61) can be rewritten as

$$0 \to p^{+*} \mathcal{M}^+ \to L^1 (p^{+*} g^{+*}) g_*^+ \mathcal{M}^+ \to L^1 p^{+*} (f^{+*} f_*^+ \mathcal{M}^+) \to 0. \tag{62}$$

Sequence (62) together with isomorphism (60) imply that (56) is exact. $\square$

PROPOSITION 4.17. *Let* f *satisfy assumption (a). Then flop functor* $Rp_*^+ Lp^*$ *on category* $f^* f_* \mathscr{P}_0$ *is isomorphic to the non-derived flop functor* $p_*^+ p^*$.

Proof. In view of Lemma 4.15, $L^j p^* \mathcal{Q} \simeq 0$, for any $\mathcal{Q}$ projective in $\mathscr{A}_f$ and any $j \geqslant 2$. As any object $\mathcal{N}$ in $\mathscr{P}_0$ is locally free [VdB04, Proposition 3.2.6], by applying Lp^* to sequence (32), we obtain an isomorphism $Lp^* f^* f_* \mathcal{N} \simeq p^* f^* f_* \mathcal{N}$. As diagram (42) commutes, the latter sheaf is isomorphic to $p^{+*} f^{+*} f_* \mathcal{N}$. Lemma 2.8 implies that $R^1 p_*^+ p^{+*}(E) \simeq 0$, for any sheaf E on X^+. Thus, $Rp_*^+ p^{+*} f^{+*} f_* \mathcal{N} \simeq p_*^+ p^{+*} f^{+*} f_* \mathcal{N} \simeq p_*^+ p^* f^* f_* \mathcal{N}$. $\square$

Base change ϵ given by (47) and an analogous $\epsilon^+ \colon f^* f_*^+ \to p_* p^{+*}$ together with the adjunction unit $\eta^{f^+} \colon \mathrm{Id} \to f_*^+ f^{+*}$ give $\widetilde{\epsilon} \colon f^* f_* \to p_* p^{+*} p_*^+ p^*$ which is the composite

$$\widetilde{\epsilon} \colon f^* f_* \xrightarrow{f^* \eta^{f^+} f_*} f^* f_*^+ f^{+*} f_* \xrightarrow{\epsilon^+ (f^{+*} f_*)} p_* p^{+*} f^{+*} f_* \xrightarrow{(p_* p^{+*}) \epsilon} p_* p^{+*} p_*^+ p^*. \tag{63}$$

PROPOSITION 4.18. *Let f satisfy assumption (a). Morphism $\widetilde{\epsilon}$ is an isomorphism on the category* $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$.

Proof. We check that, for any $\mathcal{M} \in \mathscr{P}_{-1}$, the map $\widetilde{\epsilon}_{\mathcal{M}}$ is a composition of three isomorphisms.

Take $\mathcal{N}^+ := (f^{+*} f_* \mathcal{M})^{\vee\vee}$ as in (35). In view of Lemma 3.11, $f_* \mathcal{M} \simeq f_*^+ \mathcal{N}^+$. Then Lemma 2.9 implies that morphism $\eta_{f_* \mathcal{M}}^{f^+} \colon f_* \mathcal{M} \to f_*^+ f^{+*} f_* \mathcal{M}$ is an isomorphism, hence so is $f^* \eta^{f^+} f_*$ on $\mathcal{M}$.

Natural transformation ϵ^+ is the composite of the adjunction unit η^p and the counit ε^{f^+} as in (48). We check that both maps

$$f^* f_*^+ f^{+*} f_* \mathcal{M} \xrightarrow{\eta_{f^* f_*^+ f^{+*} f_* \mathcal{M}}^p} p_* p^* f^* f_*^+ f^{+*} f_* \mathcal{M},$$

$$p_* p^{+*} f^{+*} f_*^+ f^{+*} f_* \mathcal{M} \xrightarrow{p_* p^{+*} \varepsilon_{f^{+*} f_* \mathcal{M}}^{f^+}} p_* p^{+*} f^{+*} f_* \mathcal{M}$$

are isomorphisms. As $f^* f_*^+ f^{+*} f_* \mathcal{M} \simeq f^* f_* \mathcal{M}$, morphism $\eta_{f^* f_*^+ f^{+*} f_* \mathcal{M}}^p$ is an isomorphism if and only if $\eta_{f^* f_* \mathcal{M}}^p$ is also an isomorphism. Object $\mathcal{N}^+$ satisfies $R^1 f_*^+ \mathcal{N}^+ = 0$, hence, by Proposition 2.11, $R^1 p_* p^{+*} \mathcal{N}^+ = 0$. As, by Lemma 2.8, $R^i p_* p^* \mathcal{M} = 0$, for $i \geqslant 1$, applying Rp_* to sequence (56) with the roles of f and f^+ exchanged gives $R^1 p_* L^1 p^* (f^* f_* \mathcal{M}) \simeq R^1 p_* p^{+*} \mathcal{N}^+ = 0$. Then, short exact sequence (20), for $E = f^* f_* \mathcal{M}$ and p instead of f, implies that the adjunction unit $\eta_{f^* f_* \mathcal{M}}^p \colon f^* f_* \mathcal{M} \xrightarrow{\simeq} p_* p^* (f^* f_* \mathcal{M})$ is an isomorphism.

By the definition of the adjunction, morphism $\varepsilon_{f^{+*} f_* \mathcal{M}}^{f^+}$ fits into the following commutative triangle.

$$
\begin{array}{ccc}
 & f^{+*} f_*^+ f^{+*} f_* \mathcal{M} & \\
{\scriptstyle f^{+*} \eta_{f_* \mathcal{M}}^{f^+}} \nearrow & & \searrow {\scriptstyle \varepsilon_{f^{+*} f_* \mathcal{M}}^{f^+}} \\
f^{+*} f_* \mathcal{M} & \xrightarrow{\quad \mathrm{Id}_{f^{+*} f_* \mathcal{M}} \quad} & f^{+*} f_* \mathcal{M}
\end{array}
$$

We have already seen that $\eta_{f_* \mathcal{M}}^{f^+}$ is an isomorphism, hence the same is true about $f^{+*} \eta_{f_* \mathcal{M}}^{f^+}$. As, clearly, $\mathrm{Id}_{f^{+*} f_* \mathcal{M}}$ is an isomorphism, so is the third morphism in the triangle $\varepsilon_{f^{+*} f_* \mathcal{M}}^{f^+}$.

Finally, by Lemma 4.7, $\epsilon_{\mathcal{M}} \colon f^+ f_* \mathcal{M} \to p_*^+ p^* \mathcal{M}$ is an isomorphism, hence so is $p_* p^{+*} \epsilon_{\mathcal{M}}$.

Thus, for any $\mathcal{M} \in \mathscr{P}_{-1}$, we have an isomorphism $f^* f_* \mathcal{M} \to p_* p^{+*} p_*^+ p^* \mathcal{M}$. By induction on triangles, the isomorphism extends to $\mathrm{Hot}^b(\mathscr{P}_{-1})$. Lemmas 4.9 and 4.11 imply that base change $\widetilde{\epsilon}$ is an isomorphism on $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$. $\qquad\square$

PROPOSITION 4.19. *Let $f \colon X \to Y$ satisfy assumption (a). The flop–flop functor $F^+ F$ restricted to $\mathcal{D}^b(X)$ is isomorphic to term-wise functor $f^* f_*$ on the category $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$ under the equivalence $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1}) \simeq \mathcal{D}^b(X)$.*

Proof. In view of Proposition 4.18, it suffices to check that functors $p_* p^{+*} p_*^+ p^*$ and $Rp_* Lp^{+*} Rp_*^+ Lp^*$ are isomorphic on $\mathrm{Hot}^{-,b}(\mathscr{P}_{-1})$. By Proposition 4.8, we have an isomorphism $Rp_*^+ Lp^* \mathcal{M} \simeq p_*^+ p^* \mathcal{M}$, for any $\mathcal{M} \in \mathscr{P}_{-1}$. Proposition 4.10 implies that $p_*^+ p^* \mathcal{M} \simeq f^{+*} f_* \mathcal{M}$, hence $Rp_* Lp^{+*} p_*^+ p^* \mathcal{M} \simeq p_* p^{+*} p_*^+ p^* \mathcal{M}$ by Proposition 4.8. By induction on triangles, we have an isomorphism on $\mathrm{Hot}^b(\mathscr{P}_{-1})$. We conclude by Lemma 4.11. $\qquad\square$

4.5 The flop functor as the inverse of Van den Bergh's functor

We show that the flop functor $F^+\colon \mathcal{D}^b(X^+) \to \mathcal{D}^b(X)$ and functor Σ are inverse to each other. In view of Theorem 4.5, this statement is equivalent to the flop functor F^+ being isomorphic to the term-wise functor $T = (f^*f^+_*(-))^{\vee\vee}$.

PROPOSITION 4.20. *Let $f\colon X \to Y$ satisfy assumption (a). The flop functor is isomorphic to $(f^{+*}f_*(-))^{\vee\vee}$ as a functor $\mathscr{P}_0 \to \mathscr{P}^+_{-1}$.*

Proof. Let $\mathcal{N}$ be an object in $\mathscr{P}_0$ and $\mathcal{M}^+ = (f^{+*}f_*\mathcal{N})^{\vee\vee}$ its image under the equivalences of Lemma 3.11. Lemma 2.9 and Proposition 4.18 give isomorphisms

$$p^+_* p^{+*} f^{+*} f^+_* \mathcal{M}^+ \simeq p^+_* p^{+*} p^+_* p^* p_* p^{+*} \mathcal{M}^+ \simeq p^+_* p^* p_* p^{+*} \mathcal{M}^+ \simeq f^{+*} f^+_* \mathcal{M}^+.$$

Thus, sequence (20) implies that $Rp^+_*(L^1 p^{+*} f^{+*} f^+_* \mathcal{M}^+) \simeq 0$. Moreover, we have $Rp^+_* p^{+*} \mathcal{M}^+ \simeq \mathcal{M}^+$. Hence, by applying Rp^+_* to sequence (56), we obtain

$$Rp^+_* p^* \mathcal{N} \simeq Rp^+_* p^{+*} \mathcal{M}^+ \simeq \mathcal{M}^+.$$

Hence, the flop functor takes $\mathcal{N}$ to $\mathcal{M}^+$.

Let $\epsilon_{\mathcal{N}}\colon f^{+*} f_* \mathcal{N} \to p^+_* p^* \mathcal{N} \simeq \mathcal{M}^+$ be the base change. As $f_* \mathcal{N} \simeq f^+_* \mathcal{M}^+$, we have $f^{+*} f_* \mathcal{N} \simeq f^{+*} f^+_* \mathcal{M}^+$. The kernel of reflexification $f^{+*} f^+_* \mathcal{M}^+ \to (f^{+*} f^+_* \mathcal{M}^+)^{\vee\vee} \simeq \mathcal{M}^+$ is torsion and $p^{+*} p_* \mathcal{N} \simeq \mathcal{M}^+$ is torsion-free, hence, by applying functor $\operatorname{Hom}_{X^+}(-, p^{+*} p_* \mathcal{N})$ to sequence (15) with f and $\mathcal{M}$ replaced by f^+ and $\mathcal{M}^+$, we obtain an isomorphism $\operatorname{Hom}_{X^+}(\mathcal{M}^+, p^{+*} p_* \mathcal{N}) \simeq \operatorname{Hom}_{X^+}(f^{+*} f^+_* \mathcal{M}^+, p^{+*} p_* \mathcal{N})$. It follows that $\epsilon_{\mathcal{N}}$ factors uniquely via a morphism $\widetilde{\epsilon}_{\mathcal{N}}\colon \mathcal{M}^+ \simeq (f^{+*} f_* \mathcal{N})^{\vee\vee} \to p^+_* p^* \mathcal{N}$ which is easily checked to be functorial. Thus, there exists a morphism $\widetilde{\epsilon}\colon (f^{+*} f_*(-))^{\vee\vee} \to p^+_* p^*(-)$ of functors.

Objects of $\mathscr{P}_0$ and $\mathscr{P}^+_{-1}$ are reflexive on X and X^+, hence any morphism $\mathcal{N}_1 \to \mathcal{N}_2$ (and $\mathcal{M}^+_1 \to \mathcal{M}^+_2$) is determined by its restrictions to any open set with the complement of codimension greater than one. In particular, by its restriction to the complement $U \subset X$ of the exceptional set of f (and f^+). Flop functor is $Rp^+_* Lp^*$ and the other functor is $(f^{+*} f_*(-))^{\vee\vee}$. Then, taking into account that maps f, f^+, p and p^+ are isomorphisms outside the exceptional sets and $(f^+)^{-1} \circ f = p^+ \circ p^{-1}$ on U, the morphism $\operatorname{Hom}_X(\mathcal{N}_1, \mathcal{N}_2) \to \operatorname{Hom}_{X^+}(\mathcal{M}^+_1, \mathcal{M}^+_2)$ induced by the flop functor F coincides with the morphism given by $(f^{+*} f_*(-))^{\vee\vee}$. It follows that $\widetilde{\epsilon}$ is an isomorphism of functors. $\square$

THEOREM 4.21. *Let $f\colon X \to Y$ satisfy assumption (a). The flop functor F takes ${}^0\mathrm{Per}(X/Y)$ to ${}^{-1}\mathrm{Per}(X^+/Y)$. It is isomorphic to $\Sigma^{-1}_{\mathrm{qc}}$ in the category of bimodule functors $\mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(X^+)$.*

Proof. The same argument as in the proof of Proposition 4.6 shows that $\Sigma^{-1}_{\mathrm{qc}}$ is the term-wise extension of $(f^{+*} f_*(-))^{\vee\vee}$ to the category $\mathrm{Hot}^{-,b}(\mathscr{P}_0)$. Proposition 4.20 implies that $F|_{\mathscr{P}_0} \simeq \Sigma^{-1}_{\mathrm{qc}}|_{\mathscr{P}_0}$.

Categories $\mathcal{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}_{\mathrm{qc}}(X^+)$ have compact generators $\mathcal{N} \in \mathscr{P}_0$ and $\mathcal{M}^+ = (f^{+*} f_* \mathcal{N})^{\vee\vee} \in \mathscr{P}^+_{-1}$. For algebras $A_X = \operatorname{Hom}_X(\mathcal{N}, \mathcal{N})$ and $A_{X^+} = \operatorname{Hom}_{X^+}(\mathcal{M}^+, \mathcal{M}^+)$, we have equivalences $\mathcal{D}_{\mathrm{qc}}(X) \simeq \mathcal{D}(\mathrm{Mod}\text{–}A_X)$, $\mathcal{D}_{\mathrm{qc}}(X^+) \simeq \mathcal{D}(\mathrm{Mod}\text{–}A_{X^+})$. By construction, functor $\Sigma^{-1}_{\mathrm{qc}}$ is isomorphic to a bimodule functor Φ_{M_1}, for some $A^{\mathrm{op}}_X \otimes A_{X^+}$ DG bimodule M_1, see Appendix C. Similarly, the flop functor F is isomorphic to Φ_{M_2}, for some DG bimodule M_2. Moreover, from the construction of $\Sigma^{-1}_{\mathrm{qc}}$, it follows that $\mathcal{H}^i(M_1) = 0$, for $i \neq 0$ (see the discussion before Proposition 4.6). By Lemma C.12, isomorphism $F|_{\mathscr{P}_0} \simeq \Sigma^{-1}_{\mathrm{qc}}|_{\mathscr{P}_0}$ implies an isomorphism $F \simeq \Sigma^{-1}_{\mathrm{qc}}$. $\square$

As an immediate consequence of Theorems 4.5 and 4.21 we obtain the following corollary.

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COROLLARY 4.22. *Let $f\colon X \to Y$ satisfy assumption (a). Then the flop functor F induces an equivalence of $\mathcal{D}^b(X)$ with $\mathcal{D}^b(X^+)$.*

In view of Proposition 4.13 and Theorem 4.21 we have a functorial exact triangle of functors $\mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(X^+)$

$$(F^+)^{-1}[1] \to Lg^{+*}Rg_* \to F \to (F^+)^{-1}[2]. \tag{64}$$

Moreover, it restricts to a functorial exact triangle of functors $\mathcal{D}^b(X) \to \mathcal{D}^b(X^+)$.

COROLLARY 4.23. *Let $f\colon X \to Y$ satisfy assumption (a). The flop functor $F\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X^+)$ is t-exact when $\mathcal{D}^b(X)$ is endowed with the t-structure with heart ${}^p\mathrm{Per}(X/Y)$ and $\mathcal{D}^b(X^+)$ with the t-structure with heart ${}^{p-1}\mathrm{Per}(X^+/Y)$. Moreover, F induces an equivalence of $\mathscr{A}_f$ with $\mathscr{A}_{f+}[1]$.*

Proof. As $Rf^+_* \circ F \simeq Rf_*$ and, by Theorem 4.21, functor F takes ${}^0\mathrm{Per}(X/Y)$ to ${}^{-1}\mathrm{Per}(X^+/Y)$, the flop F induces an equivalence of $\mathscr{A}_f \subset {}^0\mathrm{Per}(X/Y)$ with $\mathscr{A}_{f+}[1] \subset {}^{-1}\mathrm{Per}(X^+/Y)$.

Category ${}^p\mathrm{Per}(X/Y)$ is defined by two conditions; we require that Rf_*E is a pure sheaf, for any $E \in {}^p\mathrm{Per}(X/Y)$, and that $\mathscr{A}_f[-p] \subset {}^p\mathrm{Per}(X/Y)$. It follows from the properties of F that $F({}^p\mathrm{Per}(X/Y)) \subset {}^{p-1}\mathrm{Per}(X^+/Y)$. As F is an equivalence, both categories are hearts of bounded t-structures on $\mathcal{D}^b(X^+)$. It follows that $F({}^p\mathrm{Per}(X/Y)) = {}^{p-1}\mathrm{Per}(X^+/Y)$. □

COROLLARY 4.24. *Let morphism $f\colon X \to Y$ satisfy assumption (a). Then categories $\mathcal{D}^b(X)$ and $\mathcal{D}^b({}^p\mathrm{Per}(X/Y))$ are equivalent, for $p \in \mathbb{Z}$.*

Proof. Corollary 4.23 implies that, for any $p \in \mathbb{Z}$, category ${}^p\mathrm{Per}(X/Y)$ is equivalent either to ${}^0\mathrm{Per}(X/Y)$ or to ${}^{-1}\mathrm{Per}(X/Y)$. Hence, $\mathcal{D}^b({}^p\mathrm{Per}(X/Y)) \simeq \mathcal{D}^b({}^q\mathrm{Per}(X/Y))$, for $q \in \{-1, 0\}$, $q \cong p \mod 2$. By Proposition 4.4, the latter category is equivalent to $\mathcal{D}^b(X)$, which finishes the proof. □

Let $f\colon X \to Y$ satisfy assumption (a). Proposition 2.11 implies that $Rp_*\mathcal{O}_{X \times_Y X^+} = \mathcal{O}_X$. Thus, commutativity of diagram (42) yields isomorphisms $F^+ \circ Lf^{+*} \simeq Rp_*Lp^{+*}Lf^{+*} \simeq Rp_*Lp^*Lf^* \simeq Lf^*$. Hence, also $F^+ \circ Lg^{+*} \simeq Lg^*$. Composing triangle (64) with F^+ leads to a functorial exact triangle of functors $\mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(X)$:

$$\mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(X)}[1] \to Lg^*Rg_* \to F^+F \to \mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(X)}[2]. \tag{65}$$

4.6 Reduction to the affine case

We have shown with Corollary 4.22 that, if a morphism $f\colon X \to Y$ satisfies assumption (a), then the flop functor F is an equivalence of $\mathcal{D}^b(X)$ with $\mathcal{D}^b(X^+)$. Now, following [Che02], we show that the flop functor is also an equivalence, provided morphism f satisfies assumption (p).

Note that the condition on Y to have hypersurface singularities means that completions of the local rings at all closed points of Y are defined by one equation in regular complete local rings.

LEMMA 4.25. *Let Y be an irreducible variety of dimension n with hypersurface singularities. Then it has a finite open covering by affine subvarieties Y_i such that every Y_i admits a closed embedding as a principal divisor into a smooth affine $\mathcal{Y}_i$.*

For the reader's convenience, we give the proof of this statement.

Proof. Let $y \in Y$ be a closed point. As Zariski topology is quasi-compact, it is enough to show the existence of an affine open neighbourhood of y which admits a closed embedding into a smooth affine $\mathcal{Y}$ of dimension $n + 1$. As the statement is local, we can assume that Y is affine. Then Y is realised as a closed subscheme in $\mathbb{A}^m$. We can assume that $m > n + 1$, because otherwise the

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proof is obvious. Let $\mathbb{A}^m \subset \mathbb{P}^m$ be the compactification to a projective space and $\bar{Y} \subset \mathbb{P}^m$ the closure of Y in $\mathbb{P}^m$.

As Y has hypersurface singularities, the Zariski tangent space at point $y \in Y$ has dimension $\leqslant n+1$. It follows that we can choose a point z in $\mathbb{P}^m \setminus \bar{Y}$ such that the line $(zy) \subset \mathbb{P}^m$ does not contain any point of $\bar{Y}$ except for y and is not tangent to Y at y.

Denote by Y_1 the image of $\bar{Y}$ under the projection $p : \mathbb{P}^m \to \mathbb{P}^{m-1}$ with centre in z. By the choice of z, the morphism $\bar{Y} \to Y_1$ is an isomorphism for some Zariski neighbourhoods of y and $p(y)$. We can keep projecting in the same manner, if necessary, by replacing $\bar{Y} \subset \mathbb{P}^m$ with $Y_1 \subset \mathbb{P}^{m-1}$, until we come to the projective space of dimension $n+1$ and a closed irreducible subvariety Y' in it of dimension n together with a morphism $\pi : \bar{Y} \to Y'$ which is an isomorphism in neighbourhoods of y and $\pi(y)$.

We can find an affine open neighbourhood $\mathcal{Y} \subset \mathbb{P}^{n+1}$ of $\pi(y)$ such that π gives an isomorphism of $Y' \cap \mathcal{Y}$ with a neighbourhood of $y \in Y$. By choosing a smaller affine $\mathcal{Y}$, if necessary, we can assume that $Y' \cap \mathcal{Y} \subset \mathcal{Y}$ is cut out by a single equation. Then $Y' \cap \mathcal{Y}$ is affine and embedded as a principal divisor in $\mathcal{Y}$, so we are done. □

We consider $f : X \to Y$ which satisfies assumption (p). Lemma 4.25 implies the existence of a finite affine covering $Y = \bigcup Y_i$ such that, for every i, space Y_i admits an embedding as a principal Cartier divisor to a smooth affine $\mathcal{Y}_i$.

For every i, we pull back the whole diagram to Y_i as follows.

Note that, for any i, morphism f_i satisfies assumption (a).

PROPOSITION 4.26. *Let $f : X \to Y$ and $f_i : X_i \to Y_i$ be as above. Assume that flop functors $F_i : \mathcal{D}^b(X_i) \to \mathcal{D}^b(X_i^+)$ are equivalences, for every i. Then the flop functor $F : \mathcal{D}^b(X) \to \mathcal{D}^b(X^+)$ is an equivalence.*

Proof. As X is Gorenstein, category $\mathcal{D}^b(X)$ has a spanning class $\Omega = \{\mathcal{O}_x\}$, where x runs over all closed points in X (see [HLdS07, Lemma 1.26]). With such a choice of a spanning class the same argument as in [Che02, Proposition 3.2] proves that F is an equivalence if all F_i are. □

5. Spherical pairs

With the notation of (42), we consider the category $\mathcal{D}^b(X \times_Y X^+)$ and its quotient by the intersection $\mathcal{K}^b$ of kernels of Rp_* and Rp_*^+. We prove that $\mathcal{D}^b(\mathscr{A}_f)$ is equivalent to the full subcategory of $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ defined as the kernel of Rp_*^+. We show that $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ admits SODs $\langle \mathcal{D}^b(\mathscr{A}_f), \mathcal{D}^b(X^+) \rangle = \langle \mathcal{D}^b(X), \mathcal{D}^b(\mathscr{A}_f) \rangle$ and similarly for $\mathscr{A}_{f+}$. We conclude that $(\mathcal{D}^b(X), \mathcal{D}^b(X^+))$ and $(\mathcal{D}^b(\mathscr{A}_f), \mathcal{D}^b(\mathscr{A}_{f+}))$ are spherical pairs, which yields a geometric incarnation of the schober [KS14] related to the flop.

5.1 Category $\mathcal{D}(\mathscr{A}_f)$ as a full subcategory of $\mathcal{D}_{qc}(X \times_Y X^+)$

LEMMA 5.1. *Let $f : X \to Y$ satisfy assumption (a) and let $\mathcal{P}$ be a projective object in $\mathscr{A}_f$. Then $p_* p^* \mathcal{P} \simeq \mathcal{P}$, $Rp_* L^1 p^* \mathcal{P} = 0$ and $Rp_* p^* \mathcal{P} \simeq \mathcal{P}$.*

Proof. By Corollary 4.23, the flop functor F^+ induces an equivalence of $\mathscr{A}_{f^+}$ and $\mathscr{A}_f[1]$. Hence, there exists projective $\mathcal{P}^+ \in \mathscr{A}_{f^+}$ such that $\mathcal{P}[1] \simeq Rp_* Lp^{+*}\mathcal{P}^+$. As $L^i p^{+*}\mathcal{P}^+ = 0$, for $i > 1$, by Lemma 4.15, and $Rp_* p^{+*}\mathcal{P}^+ = 0$, by Corollary 2.12, we have $\mathcal{P} \simeq p_* L^1 p^{+*}\mathcal{P}^+$. Then, according to Lemma 2.9, we have $p_* p^*\mathcal{P} \simeq \mathcal{P}$ and $Rp_* L^1 p^*\mathcal{P} = 0$. As $Rp_* Lp^*\mathcal{P} \simeq \mathcal{P}$, it follows that $Rp_* p^*\mathcal{P} \simeq \mathcal{P}$. $\qquad\square$

LEMMA 5.2. *Let* $f \colon X \to Y$ *satisfy assumption* (a) *and let* $\mathcal{M}$ *be a projective object in* $^{-1}\mathrm{Per}(X/Y)$. *There exists an exact triangle*

$$p^*\mathcal{P} \to Lp^{+*}(f^{+*}f_*\mathcal{M}) \to Lp^*\mathcal{M} \to p^*\mathcal{P}[1], \qquad (66)$$

for $\mathcal{P} := \mathcal{H}_X^{-1}(\iota_f^*\mathcal{M})$ *projective in* $\mathscr{A}_f$.

Proof. By Lemma 4.15, we have $L^j p^*\mathcal{P} = 0$, for $j > 1$ and any projective object $\mathcal{P}$ in $\mathscr{A}_f$. Thus, applying Lp^* to sequence (15) and using the fact that $\mathcal{M}$ is locally free, we get an isomorphism $L^j p^*\mathcal{P} \simeq L^j p^*(f^* f_*\mathcal{M})$, for $j > 0$. Further, as $Lp^*\mathcal{M} \simeq p^*\mathcal{M}$, i.e. $L^i p^*\mathcal{M} = 0$, for $i \geqslant 1$, the sequence

$$0 \to p^*\mathcal{P} \to p^* f^* f_*\mathcal{M} \to p^*\mathcal{M} \to 0 \qquad (67)$$

obtained by applying p^* to sequence (15) is short exact. We also have

$$p^* f^* f_*\mathcal{M} \simeq Lp^{+*}(f^{+*}f_*^+ \mathcal{N}^+) \simeq Lp^{+*}(f^{+*}f_*\mathcal{M}), \qquad (68)$$

for $\mathcal{N}^+ \in \mathscr{P}_0^+$ given by (35). Indeed, commutativity of diagram (42) implies that $p^* f^* f_*\mathcal{M} \simeq p^{+*} f^{+*} f_*\mathcal{M}$. Further, Lemma 3.11 yields $f_*\mathcal{M} \simeq f_*^+ \mathcal{N}^+$. Finally, $\mathcal{P}^+ = \mathcal{H}_X^0(\iota_{f^+}^* \mathcal{N}^+)$ is a projective object in $\mathscr{A}_{f^+}$, according to Lemma 3.5, hence $L^j p^{+*}\mathcal{P}^+ = 0$, for $j > 1$ (by Lemma 4.15). Thus, applying p^{+*} to sequence (32) on X^+ yields $L^j p^{+*}(f^{+*} f_*^+ \mathcal{N}^+) = 0$, for $j > 0$.

It follows that the triangle given by short exact sequence (67) is isomorphic to a triangle of the form (66). $\qquad\square$

PROPOSITION 5.3. *Let* $f \colon X \to Y$ *satisfy assumption* (a) *and let* $\mathcal{M}$ *be a projective object in* $^{-1}\mathrm{Per}(X/Y)$. *Then* $p^*\mathcal{P} \in \mathrm{Perf}(X \times_Y X^+)$, $\mathrm{Hom}_{X \times_Y X^+}(p^*\mathcal{P}, p^*\mathcal{P}) \simeq \mathrm{Hom}_X(\mathcal{P}, \mathcal{P})$ *and* $\mathrm{Ext}_{X \times_Y X^+}^i(p^*\mathcal{P}, p^*\mathcal{P}) = 0$, *for* $i > 0$, *where* $\mathcal{P} = \mathcal{H}_X^{-1}(\iota_f^*\mathcal{M})$ *is projective in* $\mathscr{A}_f$.

Proof. Sheaf $\mathcal{M}$ is locally free on X, hence $Lp^*\mathcal{M} \in \mathrm{Perf}(X \times_Y X^+)$. Sequence (41) implies that $f^{+*} f_*^+ \mathcal{N}^+ \in \mathrm{Perf}(X^+)$, hence $Lp^{+*} f^{+*} f_*\mathcal{N}$ is also a perfect complex on $X \times_Y X^+$. Triangle (66) implies that $p^*\mathcal{P} \in \mathrm{Perf}(X \times_Y X^+)$.

By adjunction and Lemma 5.1, we have $\mathrm{Hom}_{X \times_Y X^+}(p^*\mathcal{P}, p^*\mathcal{P}) \simeq \mathrm{Hom}_X(\mathcal{P}, \mathcal{P})$. Thus, it suffices to check that $\mathrm{Ext}_{X \times_Y X^+}^i(p^*\mathcal{P}, p^*\mathcal{P}) = 0$, for $i > 0$. Applying $\mathrm{Hom}(-, p^*\mathcal{P})$ to triangle (66) yields, for any $i \in \mathbb{Z}$, an isomorphism

$$\mathrm{Ext}_{X \times_Y X^+}^{i+1}(Lp^*\mathcal{M}, p^*\mathcal{P}) \simeq \mathrm{Ext}_{X \times_Y X^+}^i(p^*\mathcal{P}, p^*\mathcal{P}),$$

because $\mathrm{Ext}_{X \times_Y X^+}^j(Lp^{+*}(f^{+*}f_*\mathcal{M}), p^*\mathcal{P}) \simeq \mathrm{Ext}_{X^+}^j(f^{+*}f_*\mathcal{M}, Rp_*^+ p^*\mathcal{P}) = 0$ (see Corollary 2.12). Sheaf $\mathcal{M}$ is projective in $^{-1}\mathrm{Per}(X/Y)$ and $\mathcal{P}[1] \in {}^{-1}\mathrm{Per}(X/Y)$. Thus, by Lemma 5.1 again, we conclude that

$$\mathrm{Ext}_{X \times_Y X^+}^{i+1}(Lp^*\mathcal{M}, p^*\mathcal{P}) \simeq \mathrm{Ext}_X^{i+1}(\mathcal{M}, Rp_* p^*\mathcal{P}) \simeq \mathrm{Ext}_X^i(\mathcal{M}, \mathcal{P}[1]) = 0,$$

for $i \neq 0$, which finishes the proof. $\qquad\square$

Now let $\mathcal{M}$ be a projective generator of $^{-1}\mathrm{Per}(X/Y)$ and $\mathcal{P} = \mathcal{H}_X^{-1}(\iota_f^*\mathcal{M})$. Denote by

$$A_P := \mathrm{Hom}_{X \times_Y X^+}(p^*\mathcal{P}, p^*\mathcal{P}).$$

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the endomorphism algebra of $p^*\mathcal{P}$. Functor $\mathrm{Hom}_{X\times_Y X^+}(p^*\mathcal{P}, -)\colon \mathrm{QCoh}(X\times_Y X^+)\to \mathrm{Mod}\text{–}A_P$ has left adjoint $(-)\otimes_{A_P} p^*P\colon \mathrm{Mod}\text{–}A_P\to \mathrm{QCoh}(X\times_Y X^+)$, [Pop73, Theorem 3.6.3]. We denote by $\bar{p}^*\colon \mathcal{D}(\mathscr{A}_f)\to \mathcal{D}_{\mathrm{qc}}(X\times_Y X^+)$ its derived functor.

As $\mathcal{P}$ is a projective generator of $\mathscr{A}_f$, categories $\mathcal{D}(\mathscr{A}_f)$ and $\mathcal{D}(\mathrm{Mod}\text{–}A_P)$ are equivalent. We also have $\mathcal{D}(\mathrm{Mod}\text{–}A_P)\simeq [SF\text{–}A_P]$, for the category $SF\text{–}A_P\subset \mathrm{DGMod}\text{–}A_P$ of semi-free A_P DG modules, i.e. DG modules that admit a filtration with direct sums of shifts of representable DG modules as the graded factors of the filtration [Dri04].

PROPOSITION 5.4. *Functor* $\bar{p}^*\colon \mathcal{D}(\mathscr{A}_f)\to \mathcal{D}_{\mathrm{qc}}(X\times_Y X^+)$ *is fully faithful.*

Proof. By [Kel94, Lemma 4.2], in order to prove that functor $\mathcal{D}(\mathrm{Mod}\text{–}A_P)\to \mathcal{D}_{qc}(X\times_Y X^+)$ is fully faithful, it suffices to show that it induces a bijection $\mathrm{Hom}_{\mathcal{D}(A_f)}(\mathcal{P}, \mathcal{P}[n])\to \mathrm{Hom}_{X\times_Y X^+}(p^*\mathcal{P}, p^*\mathcal{P}[n])$, for any $n\in\mathbb{Z}$ and that $p^*\mathcal{P}\in \mathcal{D}_{\mathrm{qc}}(X\times_Y X^+)$ is compact. Both statements follow from Proposition 5.3. $\square$

Functor $\bar{p}^*$ restricts to a fully faithful functor $\mathcal{D}^b(\mathscr{A}_f)\to \mathcal{D}^-(X\times_Y X^+)$. We show that $\mathcal{D}^b(\mathscr{A}_f)$ is, in fact, a subquotient of $\mathcal{D}^b(X\times_Y X^+)$. Following an idea of Kapranov, we show that the flop–flop functor is a spherical cotwist arising from a spherical pair.

5.2 Spherical pairs $(\mathcal{D}^b(X), \mathcal{D}^b(X^+))$, $(\mathcal{D}^b(\mathscr{A}_f), \mathcal{D}^b(\mathscr{A}_{f^+}))$
For a morphism $f\colon X\to Y$ satisfying (a) and its flop $f^+\colon X^+\to Y$, we consider

$$\mathcal{K} = \{E\in \mathcal{D}^-(X\times_Y X^+)\,|\, Rp_*(E) = 0,\ Rp_*^+(E) = 0\} \qquad (69)$$

and quotient category $\mathcal{D}^-(X\times_Y X^+)/\mathcal{K}$. In the above, as in diagram (42), morphisms p and p^+ denote the projections for $X\times_Y X^+$. As both p and p^+ have fibers of dimension bounded by one, category $\mathcal{K}$ inherits the standard t-structure from $\mathcal{D}^-(X\times_Y X^+)$ (Lemma 2.2). In particular, $K\in\mathcal{K}$ if and only if $\mathcal{H}^i(K)\in\mathcal{K}$, for all $i\in\mathbb{Z}$.

Let $\mathrm{Mor}_\mathcal{K}\subset \mathcal{D}^-(X\times_Y X^+)$ denote the category with objects as in $\mathcal{D}^-(X\times_Y X^+)$ and morphisms $f\colon E\to F$ such that the cone of f lies in $\mathcal{K}$ (see [Ver77, Nee01]). The quotient category $\mathcal{D}^-(X\times_Y X^+)/\mathcal{K}$ has the same objects as $\mathcal{D}^-(X\times_Y X^+)$. For any pair of objects E, F, morphisms $E\to F$ in $\mathcal{D}^-(X\times_Y X^+)/\mathcal{K}$ are equivalence classes of triples (f, Z, g), where $Z\xrightarrow{f} E$ is a morphism in $\mathrm{Mor}_\mathcal{K}$ and $Z\xrightarrow{g} F$ is a morphism in $\mathcal{D}^-(X\times_Y X^+)$. We have $(f, Z, g)\sim (f', Z', g')$ if there exists a triple (f'', Z'', g'') and $Z''\xrightarrow{u} Z$, $Z''\xrightarrow{v} Z'$ for which diagram

$$
\begin{array}{ccc}
 & Z & \\
{\scriptstyle f}\nearrow & \uparrow{\scriptstyle u} & \searrow{\scriptstyle g} \\
E \xleftarrow{f''} & Z'' & \xrightarrow{g''} F \\
{\scriptstyle f'}\searrow & \downarrow{\scriptstyle v} & \nearrow{\scriptstyle g'} \\
 & Z' &
\end{array}
\qquad (70)
$$

commutes, cf. [Nee01].

We denote by $\mathcal{K}^b$ the intersection $\mathcal{K}\cap \mathcal{D}^b(X\times_Y X^+)$. The embedding $\mathcal{D}^b(X\times_Y X^+)\to \mathcal{D}^-(X\times_Y X^+)$ induces a functor

$$\chi\colon \mathcal{D}^b(X\times_Y X^+)/\mathcal{K}^b\to \mathcal{D}^-(X\times_Y X^+)/\mathcal{K}. \qquad (71)$$

LEMMA 5.5. *Functor χ is fully faithful.*

Proof. It is clear from the above definitions that χ is well-defined. Let us check that χ is faithful. To this end, we assume that (f, Z, g), (f', Z', g') are morphisms $E_1 \to E_2$ in $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ which are equivalent as morphisms in $\mathcal{D}^-(X \times_Y X^+)/\mathcal{K}$. Let (f'', Z'', g'') with $Z'' \in \mathcal{D}^-(X \times_Y X^+)$ be the triple defining the equivalence as in (70). Choose $l \in \mathbb{Z}$ such that E_1, E_2, Z and Z' belong to $\mathcal{D}^b(X \times_Y X^+)^{\geqslant l}$. Then

$$
\mathcal{H}^i(\mathrm{Cone}(\tau_{\geqslant l} f'')) = \begin{cases} \mathcal{H}^i(\mathrm{Cone}(f'')), & \text{for } i \geqslant l - 1, \\ 0, & \text{for } i < l - 1, \end{cases} \tag{72}
$$

implies that all cohomology sheaves of $\mathrm{Cone}(\tau_{\geqslant l} f'')$ belong to $\mathcal{K}$, i.e. $\mathrm{Cone}(\tau_{\geqslant l} f'')$ is an object in $\mathcal{K}^b$. The truncation $\tau_{\geqslant l}$ of diagram (70) is isomorphic to the original diagram on the boundary diamond, whereas the middle line $(\tau_{\geqslant l} f'', \tau_{\geqslant l} Z'', \tau_{\geqslant l} g'')$ defines a morphism in $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$. Hence, $(f, Z, g) \sim (f', Z', g')$ holds in $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$.

Finally, let (f, Z, g) be a morphism $E_1 \to E_2$ in $\mathcal{D}^-(X \times_Y X^+)/\mathcal{K}$ and assume there exists $l \in \mathbb{Z}$ such that both $\mathcal{H}^i(E_1)$ and $\mathcal{H}^i(E_2)$ vanish, for $i \leqslant l$. Then (72) implies that $\tau_{\geqslant l} f \in \mathrm{Mor}_{\mathcal{K}}$. Hence, $(\tau_{\geqslant l} f, \tau_{\geqslant l} Z, \tau_{\geqslant l} g)$ is a morphism $E_1 \to E_2$ in $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ which functor χ takes to a morphism equivalent to (f, Z, g). Thus, functor χ is also full. $\qquad\square$

In view of Lemma 5.5, we can regard $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ as a full subcategory of $\mathcal{D}^-(X \times_Y X^+)/\mathcal{K}$.

PROPOSITION 5.6. *The composite of $Lp^* \colon \mathcal{D}^b(X) \to \mathcal{D}^-(X \times_Y X^+)$ with the quotient functor $Q \colon \mathcal{D}^-(X \times_Y X^+) \to \mathcal{D}^-(X \times_Y X^+)/\mathcal{K}$ takes $\mathcal{D}^b(X)$ to the essential image of χ.*

Proof. Let $E \in \mathcal{D}^b(X)^{\geqslant l}$. As p has fibers of dimension bounded by one and $Rp_* Lp^*(E) \simeq E$, we have $Rp_* \mathcal{H}^j(Lp^* E) = 0$, for $j < l - 1$.

As $^{-1}\mathrm{Per}(X/Y)$ is obtained from $\mathrm{Coh}(X)$ by means of the tilt in a torsion pair, sequence (34) implies that $\mathcal{H}^j_{{}^{-1}\mathrm{Per}(X/Y)}(E) = 0$, for $j < l$. By Corollary 4.23, object $Rp_*^+ Lp^*(E)$ lies in $\mathcal{D}^b(X^+)^{\geqslant l}_{{}^0\mathrm{Per}(X^+/Y)}$. Category $\mathrm{Coh}(X^+)$ is obtained from $^0\mathrm{Per}(X^+/Y)$ by the tilt in a torsion pair, hence $Rp_*^+ Lp^* E \in \mathcal{D}^b(X^+)^{\geqslant l-1}_{X^+}$. Finally, the dimension of fibers of p^+ being bounded by one implies that $Rp_*^+ \mathcal{H}^j(Lp^* E) = 0$, for $j < l - 2$. Thus, $\tau_{\leqslant l-3} Lp^* E$ lies in $\mathcal{K}$, i.e. morphism $Lp^* E \to \tau_{\geqslant l-2} Lp^* E$ is an isomorphism in $\mathcal{D}^-(X \times_Y X^+)/\mathcal{K}$. $\qquad\square$

We denote by

$$
\widetilde{Lp^*} \colon \mathcal{D}^b(X) \to \mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b
$$

the functor invoked by Lemma 5.5 and Proposition 5.6.

In order to show existence of a right adjoint to $\widetilde{Lp^*}$, we first prove the following general statement.

LEMMA 5.7. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{K}$ a thick triangulated subcategory. Let further $E \in \mathcal{D}$ be such that $\mathrm{Hom}_{\mathcal{D}}(E, K) = 0$, for any $K \in \mathcal{K}$. Then, for any $F \in \mathcal{D}$, we have an isomorphism*

$$
\mathrm{Hom}_{\mathcal{D}}(E, F) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{D}/\mathcal{K}}(E, F).
$$

Proof. We have the map

$$
\alpha \colon \mathrm{Hom}_{\mathcal{D}}(E, F) \to \mathrm{Hom}_{\mathcal{D}/\mathcal{K}}(E, F), \quad \alpha \colon h \mapsto (\mathrm{id}, E, h).
$$

Now let (f, Z, g) be any element in $\mathrm{Hom}_{\mathcal{D}/\mathcal{K}}(E, F)$. By definition, f is a morphism in $\mathrm{Mor}_{\mathcal{K}}$. In other words we have an exact triangle

$$
Z \xrightarrow{f} E \to K \to Z[1],
$$

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with $K \in \mathcal{K}$. As both $\mathrm{Hom}_{\mathcal{D}}(E, K)$ and $\mathrm{Hom}_{\mathcal{D}}(E, K[-1])$ vanish, object Z is uniquely decomposed as $Z \simeq K[-1] \oplus E$ and $f = (0, \mathrm{id})$ is the projection to E. Accordingly, we have a decomposition $g = (g_K, g_E)$. Diagram

$$
\begin{array}{ccc}
& K[-1] \oplus E & \\
\swarrow (0,\mathrm{id}) \quad \uparrow (0,\mathrm{id}) \quad (g_K, g_E) \searrow & \\
E \xleftarrow{\quad \mathrm{id} \quad} E \xrightarrow{\quad g_E \quad} F
\end{array}
$$

commutes, i.e. $(f, Z, g) \sim (\mathrm{id}, E, g_E)$. Thus, map α is surjective.

Now let $(\mathrm{id}, E, g) \sim (\mathrm{id}, E, g')$ in $\mathcal{D}/\mathcal{K}$. For the triple (f'', Z'', g'') giving the equivalence (see diagram (70)), we have, as above, the decomposition $Z'' = K''[-1] \oplus E$. Commutativity of

$$
\begin{array}{ccccc}
& & E & & \\
& \nearrow^{\mathrm{id}} & \uparrow{(0,\mathrm{id})} & \searrow^{g} & \\
E & \xleftarrow{\;(0,\mathrm{id})\;} & K''[-1] \oplus E & \xrightarrow{(g''_K, g''_E)} & F \\
& \searrow_{\mathrm{id}} & \downarrow{(0,\mathrm{id})} & \nearrow_{g'} & \\
& & E & &
\end{array}
$$

yields $g = g''_E = g'$, i.e. α is also a monomorphism. $\qquad \square$

PROPOSITION 5.8. *Functor $\widetilde{Lp^*}$ is fully faithful and has the right adjoint $Rp_* \colon \mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b \to \mathcal{D}^b(X)$.*

Proof. First, note that functor $Rp_* \colon \mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b \to \mathcal{D}^b(X)$ is well-defined, because $\mathcal{K}^b \subset \mathrm{Ker} Rp_*$.

For any $E \in \mathcal{D}^b(X)$ and $F \in \mathcal{D}^-(X \times_Y X^+)$, we have an isomorphism $\mathrm{Hom}_{X \times_Y X^+}(Lp^*E, F) \simeq \mathrm{Hom}_X(E, Rp_*F)$. As functor χ is fully faithful (Lemma 5.5), in order to prove that Rp_* is right adjoint to $\widetilde{Lp^*}$, it suffices to show that, for any $F \in \mathcal{D}^-(X \times_Y X^+)$, we have

$$
\mathrm{Hom}_{X \times_Y X^+}(Lp^*E, F) \simeq \mathrm{Hom}_{\mathcal{D}^-(X \times_Y X^+)/\mathcal{K}}(Lp^*E, F). \tag{73}
$$

By adjunction, $\mathrm{Hom}_{X \times_Y X^+}(Lp^*E, K) = 0$, for any $K \in \mathcal{K}$, thus (73) follows from Lemma 5.7.

As functor $Lp^* \colon \mathcal{D}^b(X) \to \mathcal{D}^-(X \times_Y X^+)$ is fully faithful, isomorphism (73), for $F = Lp^*(E')$, implies that functor $\widetilde{Lp^*}$ is fully faithful too. $\qquad \square$

As $Rp_* \widetilde{Lp^*} \simeq \mathrm{Id}_{\mathcal{D}^b(X)}$, we have a SOD [Bon89, Lemma 3.1]

$$
\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b = \langle \mathcal{C}, \widetilde{Lp^*} \mathcal{D}^b(X) \rangle, \tag{74}
$$

with $\mathcal{C} = \{ E \in \mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b \,|\, Rp_*E = 0 \}$. We aim at showing that $\mathcal{C} \simeq \mathcal{D}^b(\mathscr{A}_{f^+})$.

By Proposition 5.3, we know that, for a projective object $\mathcal{M} \in {}^{-1}\mathrm{Per}(X/Y)$ and $\mathcal{P} = \mathcal{H}_X^{-1} \iota_f^* \mathcal{M}$, the endomorphisms algebras of $p^*\mathcal{P}$ and $\mathcal{P}$ are isomorphic. Next, we show that they remain the same when we pass to the quotient category $\mathcal{D}^-(X \times_Y X^+)/\mathcal{K}$.

LEMMA 5.9. *Let $\mathcal{P}$ be as above. Then, for any $E \in \mathcal{D}^-(X \times_Y X^+)$,*

$$
\mathrm{Hom}_{X \times_Y X^+}(p^*\mathcal{P}, E) \simeq \mathrm{Hom}_{\mathcal{D}^-(X \times_Y X^+)/\mathcal{K}}(p^*\mathcal{P}, E).
$$

Proof. By Lemma 5.7, it suffices to check that $\mathrm{Hom}_{X \times_Y X^+}(p^*\mathcal{P}, K) = 0$, for any $K \in \mathcal{K}$. It immediately follows from applying $\mathrm{Hom}(-, K)$ to triangle (66). $\qquad \square$

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Consider the composite functor

$$\widetilde{p}^*\colon \mathcal{D}^b(\mathscr{A}_f) \xrightarrow{\overline{p}^*|_{\mathcal{D}^b(\mathscr{A}_f)}} \mathcal{D}^-(X \times_Y X^+) \xrightarrow{Q} \mathcal{D}^-(X \times_Y X^+)/\mathcal{K},$$

where Q denotes the quotient functor and $\overline{p}^*$ is as in Proposition 5.4. Recall that we assume $Y = \operatorname{Spec} R$ to be affine and Noetherian. Endomorphism algebra A_P of $\mathcal{P}$ is finite over R, hence Noetherian. Thus, the category mod–A_P of finitely generated right A_P modules is abelian. As $\mathcal{P}$ is a projective generator of $\mathscr{A}_f$, $\mathcal{D}^b(\mathscr{A}_f)$ is equivalent to $\mathcal{D}^b(\text{mod–}A_P)$. Using this equivalence and taking into account Proposition 5.3, we can define $\widetilde{p}_*$, right adjoint to $\widetilde{p}^*$, as

$$\widetilde{p}_*\colon \mathcal{D}^b(X \times_Y X^+) \to \mathcal{D}^b(\text{mod–}A_P), \quad \widetilde{p}_*\colon E \mapsto R\operatorname{Hom}_{X \times_Y X^+}(p^*\mathcal{P}, E). \tag{75}$$

Finally, let $\mathscr{P}$ be the category of projective objects in $\mathscr{A}_f$ of the form $\mathcal{H}_X^{-1}(\iota_f^*\mathcal{M})$, for $\mathcal{M} \in \mathscr{P}_{-1}$. As there is no non-trivial extension of objects in $\mathscr{P}$, it is an exact subcategory of $\mathscr{A}_f$. Moreover, for any $E \in \mathscr{A}_f$, there exists $\mathcal{M} \in \mathscr{P}_{-1}$ and $\varphi\colon \mathcal{M} \to E[1]$, surjective in $^{-1}\operatorname{Per}(X/Y)$. As functor $\iota_{f*}\colon \mathscr{A}_f[1] \to {}^{-1}\operatorname{Per}(X/Y)$ is t-exact, its left adjoint ι_f^* is right exact, i.e. morphism $\mathcal{H}_X^{-1}(\iota_f^*\mathcal{M}) \to \mathcal{H}_X^{-1}(\iota_f^*E[1]) \simeq E$ is surjective in $\mathscr{A}_f$. This implies that

$$\mathcal{D}^b(\mathscr{A}_f) \simeq \operatorname{Hot}^{-,b}(\mathscr{P}).$$

The last equivalence allows us to view $\widetilde{p}^*$ as a functor

$$\widetilde{p}^*\colon \operatorname{Hot}^{-,b}(\mathscr{P}) \to \mathcal{D}^-(X \times_Y X^+)/\mathcal{K}. \tag{76}$$

It is the term-wise extension of the functor $p^*\colon \mathscr{P} \to \mathcal{D}^-(X \times_Y X^+)/\mathcal{K}$.

Lemma 5.1 implies that the adjunction unit $\operatorname{Id} \to \widetilde{p}_*\widetilde{p}^*$ is an isomorphism on $\mathscr{P}$. By Lemma 4.11 we have an isomorphism

$$\operatorname{Id}_{\mathcal{D}^b(\mathscr{A}_f)} \to \widetilde{p}_*\widetilde{p}^* \tag{77}$$

of functors $\mathcal{D}^b(\mathscr{A}_f) \to \mathcal{D}^b(\mathscr{A}_f)$.

The embedding $\mathscr{A}_f \to {}^0\operatorname{Per}(X/Y)$ is an exact functor of abelian categories. Hence, it induces the functor

$$\Psi\colon \mathcal{D}(\mathscr{A}_f) \to \mathcal{D}_{\operatorname{qc}}(X).$$

LEMMA 5.10. *There exists an isomorphism $Rp_*\widetilde{p}^* \xrightarrow{\simeq} \Psi|_{\mathcal{D}^b(\mathscr{A}_f)}$ of functors $\operatorname{Hot}^{-,b}(\mathscr{P}) \to \mathcal{D}^b(X)$.*

Proof. Lemma 5.1 implies that the adjunction unit gives an isomorphism $\Psi|_{\mathscr{P}} \xrightarrow{\simeq} p_*p^*|_{\mathscr{P}}$ of functors $\mathscr{P} \to \operatorname{Coh}(X)$. As functor $\Psi|_{\mathcal{D}^b(\mathscr{A}_f)}$ takes $\operatorname{Hot}^{-,b}(\mathscr{P})$ to $\mathcal{D}^b(X)$, Lemma 4.11 implies that the term-wise extension of p_*p^* to the category $\operatorname{Hot}^{-,b}(\mathscr{P})$ is isomorphic to $\Psi|_{\mathcal{D}^b(\mathscr{A}_f)}$.

Now let $\mathcal{P}_.$ be a complex in $\operatorname{Hot}^{-,b}(\mathscr{P})$ and $F = \widetilde{p}^*\mathcal{P}_.$. Lemma 5.1 implies that the first layer of spectral sequence

$$E_1^{q,r} = R^q p_* p^* \mathcal{P}_r \Rightarrow R^{q+r} p_* F.$$

has one non-zero row only. Hence, Rp_*F is quasi-isomorphic to $p_*p^*\mathcal{P}_.$, which finishes the proof. $\square$

PROPOSITION 5.11. *There exists a fully faithful functor*

$$\widetilde{p}^*\colon \mathcal{D}^b(\mathscr{A}_f) \to \mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b.$$

Proof. Proposition 5.4 implies that $\widetilde{p}^*\colon \mathcal{D}^b(\mathscr{A}_f) \to \mathcal{D}^-(X \times_Y X^+)$ is fully faithful. In order to show that the composite $\mathcal{D}^b(\mathscr{A}_f) \xrightarrow{\widetilde{p}^*} \mathcal{D}^-(X \times_Y X^+) \xrightarrow{Q} \mathcal{D}^-(X \times_Y X^+)/\mathcal{K}$ is fully faithful,

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we check that, for any $E \in \mathcal{D}^b(\mathscr{A}_f)$ and $F \in \mathcal{D}^-(X \times_Y X^+)$, we have

$$\mathrm{Hom}_{X \times_Y X^+}(\widetilde{p}^* E, F) \simeq \mathrm{Hom}_{\mathcal{D}^-(X \times_Y X^+)/\mathcal{K}}(\widetilde{p}^* E, F).$$

By Lemma 5.7, it suffices to check that $\mathrm{Hom}(\widetilde{p}^* E, K) = 0$, for any $E \in \mathcal{D}^b(\mathscr{A}_f)$ and any $K \in \mathcal{K}$. By induction on triangles, Lemma 5.9 implies that, for any $i \in \mathbb{Z}$, $\mathrm{Hom}(\sigma_{\geqslant i}\widetilde{p}^* E, K) = 0$. As, for any $P \in \mathscr{P}$, the space $\mathrm{Hom}(p^* P, K)$ vanishes, we have $\mathrm{Ext}^j((\widetilde{p}^* E)^i, K) = 0$, for any i and j. It then follows from Lemma E.2 that $\mathrm{Hom}(\widetilde{p}^* E, K) = 0$.

Thus, it suffices to show that functor $\widetilde{p}^*$ takes $\mathcal{D}^b(\mathscr{A}_f)$ to the image of $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ in $\mathcal{D}^-(X \times_Y X^+)/\mathcal{K}$, see (71). Consider $E \in \mathcal{D}^b(\mathcal{A}_f)$. We shall show that $\widetilde{p}^* E$ is quasi-isomorphic in $\mathcal{D}^-(X \times_Y X^+)/\mathcal{K}$ to a complex with bounded cohomology. To this end, we shall find N such that $\tau_{<N}\widetilde{p}^* E$ is an object in $\mathcal{K}$.

First, we note that $Rp_*^+(p^* \mathcal{P}_i) = 0$, for any $i \in \mathbb{Z}$ (Corollary 2.12). Thus, the term-wise definition of $\widetilde{p}^*$ (76) implies that $Rp_*^+(\widetilde{p}^* E) = 0$ and, as p^+ has fibers of dimension bounded by one, $Rp_*^+ \mathcal{H}^i(\widetilde{p}^* E) = 0$, for any $i \in \mathbb{Z}$ (see Lemma 2.2).

By Lemma 5.10, we have $Rp_* \widetilde{p}^* E \simeq \Psi(E) \in \mathcal{D}^b(X)^{\geqslant k}$, for some $k \in \mathbb{Z}$. As p has fibers of dimension bounded by one, it follows that $Rp_* \mathcal{H}^i(\widetilde{p}^* E) = 0$, for $i < k-1$. Thus, $\tau_{<k-1}\widetilde{p}^* E$ lies in $\mathcal{K}$. $\square$

LEMMA 5.12. *Let $\mathcal{M}$ be a projective generator of $^{-1}\mathrm{Per}(X/Y)$ and put $\mathcal{P} = \mathcal{H}_X^{-1}(\iota_f^* \mathcal{M})$. Further, let $C \in \mathcal{D}^-(X \times_Y X^+)$ be such that $Rp_*^+ C = 0$ and $R\mathrm{Hom}_{X \times_Y X^+}(p^* P, C) = 0$. Then $Rp_* C = 0$.*

Proof. Applying $R\mathrm{Hom}_{X \times_Y X^+}(-, C)$ to triangle (66) yields $R\mathrm{Hom}_{X \times_Y X^+}(Lp^* \mathcal{M}, C) \simeq R\mathrm{Hom}_X(\mathcal{M}, Rp_* C) = 0$. As $\mathcal{M}$ is a compact generator of $\mathcal{D}_{\mathrm{qc}}(X)$, cf. [VdB04, Lemma 3.2.2] and the proof of Proposition 4.6, it follows that $Rp_* C = 0$. $\square$

PROPOSITION 5.13. *Category $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ admits a SOD*

$$\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b = \langle \widetilde{p}^* \mathcal{D}^b(\mathscr{A}_f), \widetilde{L}p^{+*}\mathcal{D}^b(X^+) \rangle.$$

Proof. As $\widetilde{p}^*$ is defined term-wise, Corollary 2.12 implies that $Rp_*^+ \circ \widetilde{p}^* = 0$. In view of SOD (74) and Proposition 5.11, it suffices to show that any $E \in \mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ such that $Rp_*^+ E = 0$ is isomorphic to $\widetilde{p}^* F$, for some $F \in \mathcal{D}^b(\mathscr{A}_f)$.

Consider triangle

$$\widetilde{p}^* \widetilde{p}_*(E) \to E \to C_E \to \widetilde{p}^* \widetilde{p}_*(E)[1] \tag{78}$$

in $\mathcal{D}^-(X \times_Y X^+)$. As $Rp_*^+ E = 0$ and $Rp_*^+ \widetilde{p}^*(F) = 0$, for any $F \in \mathcal{D}^b(\mathscr{A}_f)$, by applying Rp_*^+ to triangle (78) we obtain $Rp_*^+ C_E = 0$. In view of isomorphism (77), by applying $\widetilde{p}_*$ to triangle (78), we obtain $\widetilde{p}_* C_E = 0$. By formula (75), $R\mathrm{Hom}(p^* \mathcal{P}, C_E) = 0$. By Lemma 5.12, $Rp_* C_E = 0$, i.e. $C_E \in \mathcal{K}$. Hence, $\widetilde{p}^* \widetilde{p}_* E \to E$ is an isomorphism in $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$. $\square$

The following corollary gives a geometric description for the category $\mathcal{D}^b(\mathscr{A}_f)$.

COROLLARY 5.14. *We have an equivalence of categories*

$$\mathcal{D}^b(\mathscr{A}_f) = \{E \in \mathcal{D}^b(X \times_Y X^+) \mid Rp_*^+(E) = 0\}/\mathcal{K}^b.$$

Proposition 5.13 implies the existence of $\widetilde{p}_!: \mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b \to \mathcal{D}^b(\mathscr{A}_f)$, left adjoint to $\widetilde{p}^*$, which is defined by the functorial exact triangle

$$\widetilde{L}p^{+*} Rp_*^+ \to \mathrm{Id} \to \widetilde{p}^* \widetilde{p}_! \to \widetilde{L}p^{+*} Rp_*^+[1] \tag{79}$$

associated to a SOD. By exchanging the role of X and X^+ we also obtain $\widetilde{p}_!^+: \mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b \to \mathcal{D}^b(\mathscr{A}_{f+})$.

LEMMA 5.15. *The composite* $\widetilde{p}_!\widetilde{p}^{+*}\widetilde{p}_!^+\widetilde{p}^* \colon \mathcal{D}^b(\mathscr{A}_f) \to \mathcal{D}^b(\mathscr{A}_f)$ *is isomorphic to* $\mathrm{Id}_{\mathcal{D}^b(\mathscr{A}_f)}[4]$.

Proof. First, we check that $\widetilde{p}_!^+\widetilde{p}^*$ restricts to a functor $\mathscr{P} \to \mathscr{P}^+$. By Lemma 5.10, $Rp_*\widetilde{p}^* \simeq \Psi$. Hence, for $P \in \mathscr{A}_f$ projective, triangle (79), with p and p^+ interchanged, applied to $p^*P = \widetilde{p}^*P$ yields an exact triangle

$$\widetilde{L}p^*P \to p^*P \to \widetilde{p}^{+*}\widetilde{p}_!^+\widetilde{p}^*P \to \widetilde{L}p^*P[1].$$

By Lemma 4.15 object Lp^*P has two non-zero cohomology sheaves, p^*P and L^1p^*P. It follows that $\widetilde{p}^{+*}\widetilde{p}_!^+\widetilde{p}^*P \simeq L^1p^*P[2]$. Hence, in view of Lemma 5.10, $\Psi\widetilde{p}_!^+\widetilde{p}^*P \simeq Rp_*^+\widetilde{p}^{+*}\widetilde{p}_!^+\widetilde{p}^*P \simeq Rp_*^+L^1p^*P[2]$. As $Rp_*^+Lp^*$ restricts to an equivalence $\mathscr{A}_f \xrightarrow{\simeq} \mathscr{A}_{f+}[1]$, Corollary 4.23, object $Rp_*^+L^1p^*P = P^+$ is projective in $\mathscr{A}_{f+}$. Hence $\widetilde{p}_!^+p^*P[-2] = P^+$ is projective in $\mathscr{A}_{f+}$.

As $\widetilde{p}_!^+\widetilde{p}^*[-2]$ maps objects of $\mathscr{P}$ to objects of $\mathscr{A}_{f+}$, it maps acyclic complexes in $\mathscr{P}$ to acyclic complexes. Hence, the class $\mathscr{P}$ is adapted to $\widetilde{p}_!^+\widetilde{p}^*[-2]$, i.e. the functor can be calculated term-wise using the equivalence $\mathcal{D}^b(\mathscr{A}_f) \simeq \mathrm{Hot}^{-,b}(\mathscr{P})$. The same is true for the functor $\widetilde{p}_!\widetilde{p}^{+*}[-2]$ in the opposite direction. Moreover, because $\widetilde{p}_!^+\widetilde{p}^*[-2]$ maps complexes of objects of $\mathscr{P}$ to complexes of objects of $\mathscr{P}^+$ also the composite $\widetilde{p}_!\widetilde{p}^{+*}\widetilde{p}_!^+\widetilde{p}^*[-4]$ can be calculated term-wise. Therefore, it suffices to check that $\widetilde{p}_!\widetilde{p}^{+*}\widetilde{p}_!^+\widetilde{p}^*[-4]|_{\mathscr{P}} \simeq \mathrm{Id}_{\mathscr{P}}$.

Precomposing triangle (79), with p and p^+ interchanged, with $\widetilde{p}^*$, composing with $\widetilde{p}_!$ and using the isomorphism $\widetilde{p}_!\widetilde{p}^* \simeq \mathrm{Id}$, we arrive at an exact triangle

$$\widetilde{p}_!\widetilde{L}p^*Rp_*p^*P \to P \to \widetilde{p}_!\widetilde{p}^{+*}\widetilde{p}_!^+\widetilde{p}^*P \to \widetilde{p}_!\widetilde{L}p^*Rp_*p^*P[1]. \tag{80}$$

First, we calculate the image of $\widetilde{p}_!\widetilde{L}p^*Rp_*p^*P \simeq \widetilde{p}_!\widetilde{L}p^*\Psi P \simeq \widetilde{p}_!\widetilde{L}p^*P$ under $\Psi = Rp_*\widetilde{p}^*$ using triangle (79) composed with Rp_* and applied to $\widetilde{L}p^*P$:

$$Rp_*Lp^{+*}Rp_*^+\widetilde{L}p^*P \to Rp_*\widetilde{L}p^*P \to \Psi\widetilde{p}_!\widetilde{L}p^*P \to Rp_*Lp^{+*}Rp_*^+\widetilde{L}p^*P[1].$$

By Lemma 5.1 we have $Rp_*\widetilde{L}p^*P \simeq P$. As $Rg_*P = 0$, triangle (65) implies that $Rp_*Lp^{+*}Rp_*^+\widetilde{L}p^*P \simeq P[2]$. It follows that $\Psi\widetilde{p}_!\widetilde{L}p^*P$ has non-zero cohomology in degrees 0 and -3 only, both isomorphic to P. The map $\widetilde{p}_!\widetilde{L}p^*Rp_*p^*P \to P$ in (80) is the composition of two adjunction counits $\widetilde{L}p^*Rp_* \to \mathrm{Id}$ and $\widetilde{p}_!\widetilde{p}^* \to \mathrm{Id}$. The first when applied to p^*P is the truncation at the zero cohomology, the second is an isomorphism. Hence, $\widetilde{p}_!\widetilde{p}^{+*}\widetilde{p}_!^+\widetilde{p}^*P \simeq P[4]$. $\qquad\square$

PROPOSITION 5.16. *Category* $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ *admits a SOD*

$$\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b \simeq \langle \widetilde{L}p^*\mathcal{D}^b(X), \widetilde{p}^*\mathcal{D}^b(\mathscr{A}_f)\rangle. \tag{81}$$

Proof. Functor $\widetilde{p}_*(-) \simeq R\mathrm{Hom}_{\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b}(p^*\mathcal{P}, -)$ is right adjoint to the fully faithful functor $\widetilde{p}^*$ of Proposition 5.11.

First, we check that $\widetilde{p}_*\widetilde{L}p^*(E) = 0$, for any $E \in \mathcal{D}^b(X)$. It follows from Lemma 5.9 that the above statement is equivalent to

$$\mathrm{Hom}_{X \times_Y X^+}(p^*\mathcal{P}, Lp^*E) = 0,$$

for any $E \in \mathcal{D}^b(X)$.

By applying $\mathrm{Hom}_{X \times_Y X^+}(-, Lp^*E)$ to triangle (66), we learn that $\mathrm{Hom}_{X \times_Y X^+}(p^*\mathcal{P}, Lp^*E)$ is a cone of a morphism

$$\theta_E \colon \mathrm{Hom}_{X \times_Y X^+}(Lp^*\mathcal{M}, Lp^*E) \to \mathrm{Hom}_{X \times_Y X^+}(Lp^{+*}(f^{+*}f_*\mathcal{M}), Lp^*E).$$

As $Rp_*\mathcal{O}_{X \times_Y X^+} \simeq \mathcal{O}_X$, we have

$$\mathrm{Hom}_{X \times_Y X^+}(Lp^*\mathcal{M}, Lp^*E) \simeq \mathrm{Hom}_X(\mathcal{M}, E). \tag{82}$$

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Moreover, in view of Proposition 4.10, we have

$$\operatorname{Hom}_{X \times_Y X^+}(Lp^{+*}f^{+*}f_*\mathcal{M}, Lp^*E) \simeq \operatorname{Hom}_{X^+}(F(\mathcal{M}), F(E)). \tag{83}$$

We check that under isomorphisms (82) and (83) morphism θ_E coincides with the morphism ν_E induced by functor F:

$$\nu_E \colon \operatorname{Hom}_X(\mathcal{M}, E) \to \operatorname{Hom}_{X^+}(F(\mathcal{M}), F(E)).$$

By (67) and (68), morphism $\varphi \colon Lp^{+*}f^{+*}f_*\mathcal{M} \to Lp^*\mathcal{M}$ in triangle (66) is the composite of the isomorphism $\kappa \colon Lp^{+*}f^{+*}f_*\mathcal{M} \xrightarrow{\simeq} p^{+*}f^{+*}f_*\mathcal{M} \xrightarrow{\simeq} p^*f^*f_*\mathcal{M}$ with the (non-derived) pull-back p^* of the $f^* \dashv f_*$ adjunction counit $\varepsilon_\mathcal{M} \colon f^*f_*\mathcal{M} \to \mathcal{M}$. We use isomorphism (82) to identify $\alpha \in \operatorname{Hom}_X(\mathcal{M}, E)$ with $Lp^*(\alpha) \in \operatorname{Hom}(Lp^*(\mathcal{M}), Lp^*(E))$. Under isomorphism (83), the composition $Lp^*(\alpha) \circ \varphi$ reads as $Rp_*^+ Lp^*(\alpha) \circ Rp_*^+(p^*(\varepsilon_\mathcal{M}) \circ \kappa) \circ \eta_\mathcal{M}$, for the $Lp^{+*} \dashv Rp_*^+$ adjunction unit $\eta_\mathcal{M} \colon f^{+*}f_*\mathcal{M} \to Rp_*^+ Lp^{+*}f^{+*}f_*\mathcal{M}$. The composite $f^{+*}f_*\mathcal{M} \xrightarrow{\eta_\mathcal{M}} Rp_*^+ Lp^{+*}f^{+*}f_*\mathcal{M} \xrightarrow{Rp_*^+(\kappa)} Rp_*^+ p^*f^*f_*\mathcal{M} \xrightarrow{Rp_*^+(\varepsilon_\mathcal{M})} Rp_*^+ p^*\mathcal{M}$ is the base change (49). It follows that $Rp_*^+ Lp^*(\alpha) \circ Rp_*^+(p^*(\varepsilon_\mathcal{M}) \circ \kappa) \circ \eta_\mathcal{M}$ is the image of $F(\alpha) = Rp_*^+ Lp^*(\alpha) \in \operatorname{Hom}(F(\mathcal{M}), F(E))$ under the isomorphism $\operatorname{Hom}(F(\mathcal{M}), F(E)) \simeq \operatorname{Hom}(f^{+*}f_*\mathcal{M}, F(E))$ given by Proposition 4.10.

As the flop functor is an equivalence of categories (see Corollary 4.22), ν_E is an isomorphism, i.e. $\operatorname{Hom}_{X \times_Y X^+}(p^*\mathcal{P}, Lp^*E) \simeq \operatorname{Cone}(\theta_E) \simeq 0$.

In order to prove the SOD (81) it suffices to show that, for any $E \in \mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ such that $\widetilde{p}_*(E) \simeq 0$, we have $E \simeq \widetilde{L}p^*(E')$, for some $E' \in \mathcal{D}^b(X)$.

Now let $E \in \mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ satisfy $\widetilde{p}_* E \simeq 0$. Consider an exact triangle

$$\widetilde{L}p^* Rp_* E \to E \to \widetilde{C}_E \to \widetilde{L}p^* Rp_* E[1]. \tag{84}$$

We have $Rp_* \widetilde{C}_E = 0$. Moreover, because $\widetilde{p}_* \widetilde{L}p^* Rp_* E = 0$, it follows that $\widetilde{p}_* C_E = 0$. Now, we proceed as in the proof of Lemma 5.12 to show that $Rp_*^+ \widetilde{C}_E \simeq 0$.

Applying $R\operatorname{Hom}_{X \times_Y X^+}(-, \widetilde{C}_E)$ to triangle (66) yields

$$R\operatorname{Hom}_{X \times_Y X^+}(Lp^{+*}(f^{+*}f_*\mathcal{M}), \widetilde{C}_E) \simeq R\operatorname{Hom}_{X^+}(f^{+*}f_*\mathcal{M}, Rp_*^+ \widetilde{C}_E) = 0.$$

By Proposition 4.10, the sheaf $f^{+*}f_*\mathcal{M}$ is isomorphic to the flop $F(\mathcal{M})$. As the flop functor F is an equivalence (see Corollary 4.22), $F(\mathcal{M})$ is a compact generator of $\mathcal{D}^b(X^+)$. Thus, vanishing of $R\operatorname{Hom}_{X^+}(F(\mathcal{M}), Rp_*^+ \widetilde{C}_E)$ implies that $Rp_*^+ \widetilde{C}_E \simeq 0$.

Hence, in triangle (84) sheaf $\widetilde{C}_E$ is an object of $\mathcal{K}^b$. Therefore, $E \simeq \widetilde{L}p^* Rp_* E$ in $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$. $\qquad\square$

THEOREM 5.17. *Category $\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b$ admits 4-periodical SODs*

$$\mathcal{D}^b(X \times_Y X^+)/\mathcal{K}^b = \langle \widetilde{p}^*\mathcal{D}^b(\mathscr{A}_f), \widetilde{L}p^{+*}\mathcal{D}^b(X^+)\rangle = \langle \widetilde{L}p^{+*}\mathcal{D}^b(X^+), \widetilde{p}^{+*}\mathcal{D}^b(\mathscr{A}_{f+})\rangle$$

$$= \langle \widetilde{p}^{+*}\mathcal{D}^b(\mathscr{A}_{f+}), \widetilde{L}p^*\mathcal{D}^b(X)\rangle = \langle \widetilde{L}p^*\mathcal{D}^b(X), \widetilde{p}^*\mathcal{D}^b(\mathscr{A}_f)\rangle.$$

In particular, $(\widetilde{p}^*\mathcal{D}^b(\mathscr{A}_f), \widetilde{p}^{+*}\mathcal{D}^b(\mathscr{A}_{f+}))$ and $(\widetilde{L}p^*\mathcal{D}^b(X), \widetilde{L}p^*\mathcal{D}^b(X^+))$ are spherical pairs.

Proof. Propositions 5.13 and 5.16 for X and X^+ imply the 4-periodical SODs. Proposition B.3 claims that $(\widetilde{p}^*\mathcal{D}^b(\mathscr{A}_f), \widetilde{p}^{+*}\mathcal{D}^b(\mathscr{A}_{f+}))$ and $(\widetilde{L}p^*\mathcal{D}^b(X), \widetilde{L}p^*\mathcal{D}^b(X^+))$ are spherical pairs. $\qquad\square$

COROLLARY 5.18. *Functor $\Psi \colon \mathcal{D}^b(\mathscr{A}_f) \to \mathcal{D}^b(X)$ is spherical. The unit and counit for the $\Psi^* \dashv \Psi$ adjunction fit into functorial exact triangles:*

$$\Psi^*\Psi \to \operatorname{Id}_{\mathcal{D}^b(\mathscr{A}_f)} \to \operatorname{Id}_{\mathcal{D}^b(\mathscr{A}_f)}[4] \to \Psi^*\Psi[1], \quad F^+ F \to \operatorname{Id}_{\mathcal{D}^b(X)} \to \Psi\Psi^* \to F^+ F[1]. \tag{85}$$

Proof. By Proposition B.3 and Theorem 5.17 $(\widetilde{p}^*\mathcal{D}^b(\mathscr{A}_f), \widetilde{p}^{+*}\mathcal{D}^b(\mathscr{A}_{f+}))$ is a spherical pair. By Proposition B.2 the corresponding spherical functor is $Rp_*\widetilde{p}^*$. By Lemma 5.10 the latter is isomorphic to Ψ.

By Proposition B.2 the cone of the counit $\Psi^*\Psi \to \mathrm{Id}_{\mathcal{D}^b(\mathscr{A}_f)}$ is isomorphic to $\widetilde{p}_!\widetilde{p}^{+*}\widetilde{p}_!^+\widetilde{p}^*$ and the cone of the unit $\mathrm{Id}_{\mathcal{D}^b(X)} \to \Psi\Psi^*$ is $Rp_*\widetilde{L}p^{+*}Rp_*^+\widetilde{L}p^* \simeq Rp_*Lp^{+*}Rp_*^+Lp^* = F^+F$. We conclude by the isomorphism $\widetilde{p}_!\widetilde{p}^{+*}\widetilde{p}_!^+\widetilde{p}^* \simeq \mathrm{Id}_{\mathcal{D}^b(\mathscr{A}_f)}[4]$, Lemma 5.15. $\square$

6. Contraction algebra as the endomorphism algebra of a projective generator

In [DW16] Donovan and Wemyss considered a morphism $f \colon X \to Y$ between Gorenstein varieties of dimension three that contracts a rational irreducible curve C to a point. If Y is a spectrum of a complete local Noetherian ring, the category $^{-1}\mathrm{Per}(X/Y)$ has projective generator. In [DW16] a *contraction algebra* A_{con} is defined as the quotient of the endomorphism algebra of the projective generator for $^{-1}\mathrm{Per}(X/Y)$ by morphisms which factor through $\mathcal{O}_X$. Algebra A_{con} is shown to govern non-commutative deformations of $\mathcal{O}_C(-1)$. If $N_{X/C} \simeq \mathcal{O}_C \oplus \mathcal{O}_C(-2)$, algebra A_{con} is commutative and was first considered in terms of deformation theory by Toda in [Tod07].

We prove in our, more general, set-up that A_{con} is isomorphic to the endomorphism algebra of a projective generator for $\mathscr{A}_f$.

Category $^0\mathrm{Per}(X/Y)$ has a projective generator $\mathcal{N}$ and $Q = \mathcal{H}^0_X(\iota_f^*\mathcal{N})$ is a projective generator for $\mathscr{A}_f$, Lemma 3.5. Endomorphisms algebras $A_Q = \mathrm{End}_X(\mathcal{Q})$ and $A_X = \mathrm{End}_X(\mathcal{N})$ are Noetherian, and we have equivalences $\mathcal{D}(\mathrm{Mod}\text{-}A_Q) \simeq \mathcal{D}(\mathscr{A}_f)$, $\mathcal{D}(\mathrm{Mod}\text{-}A_X) \simeq \mathcal{D}_{\mathrm{qc}}(X)$.

Projective generator $\mathcal{Q}$ for $\mathscr{A}_f$ is an object in $^0\mathrm{Per}(X/Y)$. It follows that $\mathcal{Q}' := R\,\mathrm{Hom}_X(\mathcal{N}, \mathcal{Q})$ is an $A_Q^{\mathrm{op}} \otimes_k A_X$ module and, under the above equivalences, functor Ψ is isomorphic to

$$\Psi \simeq \Phi_{Q'} \colon \mathcal{D}(\mathrm{Mod}\text{-}A_Q) \to \mathcal{D}(\mathrm{Mod}\text{-}A_X).$$

Proposition 3.19 yields a finite projective resolution of $\mathcal{Q}$ in $^0\mathrm{Per}(X/Y)$. Hence, $\mathcal{Q}'$ is a perfect A_X DG module and by Proposition C.6 the right adjoint $\Psi^!$ exists and is isomorphic to a bimodule functor.

PROPOSITION 6.1. *Let f satisfy assumption (a), let $\mathcal{M}_i$ be as in (17) and let $\mathcal{P}_i$ be as in (19). Then $\Psi\Psi^!(\mathcal{M}_i) \simeq \Psi\mathcal{P}_i[-2]$.*

Proof. Considering the right adjoint to the functorial exact triangles of Corollary 5.18 yields an exact triangle:

$$\Psi\Psi^!(\mathcal{M}_i) \to \mathcal{M}_i \to (F^+F)^{-1}(\mathcal{M}_i) \to \Psi\Psi^!(\mathcal{M}_i)[1]. \tag{86}$$

Put $\mathcal{K}_i = (F^+F)^{-1}(\mathcal{M}_i)$. As $Rf_*F^+F = Rf_*$, we have $Rg_*\mathcal{K}_i \simeq Rg_*\mathcal{M}_i$. Hence, by Proposition 3.15, $\mathcal{H}^0_X(Lg^*Rg_*\mathcal{K}_i) = f^*f_*\mathcal{M}_i$, $\mathcal{H}^{-1}_X(Lg^*Rg_*\mathcal{K}_i) = \mathcal{M}_i$ and the remaining cohomology sheaves vanish. Considering long exact sequence of cohomology sheaves of the exact triangle obtained by applying functorial exact triangle (65) to $\mathcal{K}_i$:

$$\mathcal{K}_i[1] \to Lg^*Rg_*\mathcal{K}_i \to \mathcal{M}_i \to \mathcal{K}_i[2],$$

we get that $\mathcal{H}^0(\mathcal{K}_i) \simeq \mathcal{M}_i$ and $\mathcal{H}^1(\mathcal{K}_i) \simeq \mathcal{P}_i$ (as $\mathcal{P}_i$ is the kernel of the surjective morphism $f^*f_*\mathcal{M}_i \to \mathcal{M}_i$). The long exact sequence of cohomology sheaves of (86) yields an isomorphism $\Psi\Psi^!(\mathcal{M}_i) \simeq \mathcal{P}_i[-2] \simeq \Psi\mathcal{P}_i[-2]$. $\square$

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THEOREM 6.2. *Let f satisfy assumption (c), let $\mathcal{M}_i$ be as in (17) and let $\mathcal{P}_i$ be as in (19). The endomorphism algebra*

$$A_P = \operatorname{Hom}_{\mathscr{A}_f}\left(\bigoplus_{i=1}^n \mathcal{P}_i, \bigoplus_{i=1}^n \mathcal{P}_i\right)$$

is isomorphic to the quotient of $\operatorname{Hom}_X(\bigoplus_{i=1}^n \mathcal{M}_i, \bigoplus_{i=1}^n \mathcal{M}_i)$ by the subspace of morphisms that factor via direct sums of copies of the structure sheaf.

Proof. By Proposition 6.1 $\Psi\Psi^!(\mathcal{M}_i) \simeq \Psi\mathcal{P}_i[-2]$. As $\Psi|_{\mathscr{A}_f}$ is fully faithful, we have $\Psi^!(\mathcal{M}_i) \simeq \mathcal{P}_i[-2]$. Thus, functor $\Psi^!$ gives a homomorphism of algebras

$$\varphi \colon \operatorname{Hom}_X(\mathcal{M}, \mathcal{M}) \to A_P = \operatorname{Hom}_{\mathcal{D}^b(\mathscr{A}_f)}(\Psi^!\mathcal{M}, \Psi^!\mathcal{M}) \simeq \operatorname{Hom}_X(\Psi\Psi^!\mathcal{M}, \mathcal{M}),$$

where $\mathcal{M} = \bigoplus_i \mathcal{M}_i$. The triangle of functors right adjoint to the second triangle of (85) implies that the morphism φ fits into an exact sequence

$$\operatorname{Hom}_X((F^+F)^{-1}\mathcal{M}, \mathcal{M}) \xrightarrow{\psi} \operatorname{Hom}_X(\mathcal{M}, \mathcal{M}) \xrightarrow{\varphi} \operatorname{Hom}_X(\Psi\Psi^!\mathcal{M}, \mathcal{M})$$
$$\to \operatorname{Hom}_X((F^+F)^{-1}\mathcal{M}[-1], \mathcal{M}).$$

By Corollary 4.22,

$$\operatorname{Hom}_X(F^{-1}F^{+-1}(\mathcal{M})[-1], \mathcal{M}) \simeq \operatorname{Ext}^1_{X^+}(F^{+-1}(\mathcal{M}), F(\mathcal{M})).$$

Theorem 4.21 implies that $F^{+-1}(\mathcal{M}) \simeq \mathcal{N}^+$ is a projective object in $^0\mathrm{Per}(X^+/Y)$. On the other hand, $F(\mathcal{M}) \simeq f^{+*}f_*\mathcal{M}$ by Proposition 4.19. By Lemma 3.11, $f^{+*}f_*\mathcal{M} \simeq f^{+*}f_*^+\mathcal{N}^+$. Then, Lemma 2.9 implies that $R^1f_*^+(f^{+*}f_*\mathcal{M}) = 0$, i.e. $f^{+*}f_*\mathcal{M}$ is an object in $^0\mathrm{Per}(X^+/Y)$ (see Remark 2.3 and formula (8) therein). Thus,

$$\operatorname{Hom}_X((F^+F)^{-1}\mathcal{M}[-1], \mathcal{M}) \simeq \operatorname{Ext}^1_{X^+}(\mathcal{N}^+, f^{+*}f_*\mathcal{M}) \simeq \operatorname{Ext}^1_{^0\mathrm{Per}(X/Y^+)}(\mathcal{N}^+, f^{+*}f_*\mathcal{M}) = 0,$$

i.e. morphism φ is surjective.

Let us now describe the image of ψ. Triangle (64) with roles of X and X^+ interchanged, when applied to $\mathcal{N}^+$, gives an exact triangle

$$\mathcal{M}[-2] \to (F^+F)^{-1}\mathcal{M} \to Lg^*Rg_*\mathcal{M}[-1] \to \mathcal{M}[-1],$$

in view of $(F^+F)^{-1}\mathcal{M} \simeq F^{-1}(\mathcal{N}^+)$, $Rg_*^+\mathcal{N}^+ \simeq Rg_*\mathcal{M}$, and $F^+(\mathcal{N}^+) \simeq \mathcal{M}$. As $\operatorname{Ext}^1_X(\mathcal{M}, \mathcal{M}) = 0 = \operatorname{Ext}^2_X(\mathcal{M}, \mathcal{M})$, we have isomorphisms $\operatorname{Hom}_X(\mathcal{M}, (F^+F)^{-1}\mathcal{M}) \simeq \operatorname{Hom}_X(\mathcal{M}, Lg^*Rg_*\mathcal{M}[-1])$ and $\operatorname{Hom}_X((F^+F)^{-1}\mathcal{M}, \mathcal{M}) \simeq \operatorname{Hom}_X(Lg^*Rg_*\mathcal{M}[-1], \mathcal{M})$. Thus, a morphism $\mathcal{M} \to \mathcal{M}$ is decomposed as $\mathcal{M} \to (F^+F)^{-1}\mathcal{M} \to \mathcal{M}$ if an only if it is decomposed as $\mathcal{M} \to Lg^*Rg_*\mathcal{M}[-1] \to \mathcal{M}$. By Lemma 3.14, object $Lg^*Rg_*\mathcal{M}$ is quasi-isomorphic to a complex $\{\mathcal{O}_X^S \to \mathcal{O}_X^S\}$, for some $S \in \mathbb{Z}_+$. Thus, 'stupid' truncation gives an exact triangle

$$\mathcal{O}_X^S[-2] \to \mathcal{O}_X^S \to Lg^*Rg_*\mathcal{M}[-1] \to \mathcal{O}_X^S[-1].$$

As $\operatorname{Ext}^1_X(\mathcal{O}_X^S, \mathcal{M}) = 0 = \operatorname{Ext}^2_X(\mathcal{O}_X^S, \mathcal{M})$, we have $\operatorname{Hom}_X(\mathcal{M}, Lg^*Rg_*\mathcal{M}[-1]) \simeq \operatorname{Hom}_X(\mathcal{M}, \mathcal{O}_X^S)$ and $\operatorname{Hom}_X(Lg^*Rg_*\mathcal{M}[-1], \mathcal{M}) \simeq \operatorname{Hom}_X(\mathcal{O}_X^S, \mathcal{M})$. Hence, a morphism $\mathcal{M} \to \mathcal{M}$ factors via $Lg^*Rg_*\mathcal{M}[-1]$ if and only if it factors via $\mathcal{O}_X^S$.

Therefore, the image of ψ is contained in the space of morphisms $\mathcal{M} \to \mathcal{M}$ that factor via $\mathcal{O}_X^S$. Conversely, for a projective generator $\mathcal{P}$ of $\mathscr{A}_f$,

$$R\operatorname{Hom}_{\mathcal{D}^b(\mathscr{A}_f)}(\Psi^!\mathcal{M}, \Psi^!\mathcal{O}_X) \simeq R\operatorname{Hom}_{\mathcal{D}^b(\mathscr{A}_f)}(\mathcal{P}, \Psi^!\mathcal{O}_X[2]) \simeq R\operatorname{Hom}_X(\Psi\mathcal{P}, \mathcal{O}_X[2])$$

$$\simeq R\operatorname{Hom}_X(\Psi\mathcal{P}, f^!\mathcal{O}_Y[2]) \simeq R\operatorname{Hom}_Y(Rf_*\Psi\mathcal{P}, \mathcal{O}_Y[2]) = 0,$$

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which implies that the space of morphisms $\mathcal{M} \to \mathcal{M}$ that factor via direct sums of copies of $\mathcal{O}_X$ is annihilated by $\Psi^!$, i.e. it is contained in the kernel of φ. $\qquad\square$

Appendix A. Grothendieck duality and the twisted inverse image functor

Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes. The direct image functor Rf_*, considered as a functor $\mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(Y)$, admits the right adjoint denoted by $f^!$. Functor $f^!$ maps $D^+_{\mathrm{qc}}(Y)$ to $D^+_{\mathrm{qc}}(X)$ and, for any $E^\cdot \in \mathcal{D}_{\mathrm{qc}}(X)$ and $F^\cdot \in \mathcal{D}^+_{\mathrm{qc}}(Y)$, the *sheafified Grothendieck duality*

$$Rf_* R\mathcal{H}\mathrm{om}_X(E^\cdot, f^!(F^\cdot)) = R\mathcal{H}\mathrm{om}_Y(Rf_*(E^\cdot), F^\cdot), \tag{A.1}$$

is satisfied, see [LH09, Corollary 4.2.2].

We construct a morphism $\alpha\colon Lf^*(-) \otimes f^!(\mathcal{O}_Y) \to f^!(-)$. In view of adjunction

$$\mathrm{Hom}_X(Lf^*(F^\cdot) \otimes f^!(\mathcal{O}_Y), f^!(F^\cdot)) \simeq \mathrm{Hom}_Y(Rf_*(Lf^*(F^\cdot) \otimes f^!(\mathcal{O}_Y)), F^\cdot),$$

constructing $\alpha_{F^\cdot}$ is equivalent to finding a morphism $Rf_*(Lf^*(F^\cdot) \otimes f^!(\mathcal{O}_Y)) \to F^\cdot$. By projection formula, we have

$$Rf_*(Lf^*(F^\cdot) \otimes f^!(\mathcal{O}_Y)) = F^\cdot \otimes Rf_* f^!(\mathcal{O}_Y).$$

Let $\varepsilon\colon Rf_* f^!(\mathcal{O}_Y) \to \mathcal{O}_Y$ be the counit of adjunction. Then morphism

$$\mathrm{id}_{F^\cdot} \otimes \varepsilon\colon F^\cdot \otimes Rf_* f^!(\mathcal{O}_Y) \to F^\cdot \otimes \mathcal{O}_Y \simeq F^\cdot$$

gives a morphism

$$\alpha_{F^\cdot}\colon Lf^*(F^\cdot) \otimes f^!(\mathcal{O}_Y) \to f^!(F^\cdot). \tag{A.2}$$

LEMMA A.1. *Let $f\colon X \to Y$ be a proper morphism. Then $\alpha_{F^\cdot}$ is an isomorphism for $F^\cdot \in$ $\mathrm{Perf}(Y)$.*

Proof. Morphism $\alpha_{\mathcal{O}_Y}$ is clearly an isomorphism. By Deligne's appendix to [Har66], functor $f^!$ is defined locally on Y, hence α is also an isomorphism for any complex $F^\cdot$ which is locally quasi-isomorphic to a complex of locally free sheaves, i.e. for $F^\cdot \in \mathrm{Perf}(Y)$. $\qquad\square$

By [LH09, §4.7], morphism $Lf^*(F^\cdot) \otimes f^!(\mathcal{O}_Y) \to f^!(F^\cdot)$ is an isomorphism for all $F^\cdot$ in $\mathcal{D}_{\mathrm{qc}}(Y)$ if and only if f is proper and of finite Tor dimension.

Appendix B. Functorial exact triangles, spherical functors and pairs

Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories. The category of exact functors $\mathcal{C} \to \mathcal{D}$ is not triangulated, thus we cannot speak about exact triangles of functors. Instead, we work with a weaker notion of a *functorial exact triangle* by which we mean a triple of exact functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them

$$F_1(-) \to F_2(-) \to F_3(-) \to F_1(-)[1] \tag{B.1}$$

such that when applied to any object $C \in \mathcal{C}$ they give an exact triangle in $\mathcal{D}$. Note that (B.1) is not an exact triangle in any triangulated category.

A basic example of a functorial exact triangle is given by an SOD

$$\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle.$$

Denote by $i_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{D}$ the embedding functor and by $i_{\mathcal{A}}^*$ its left adjoint. Similarly, let $i_{\mathcal{B}} \colon \mathcal{B} \to \mathcal{D}$ be the embedding and $i_{\mathcal{B}}^!$ its right adjoint. Then the $i_{\mathcal{A}}^* \dashv i_{\mathcal{A}}$ adjunction unit η and the $i_{\mathcal{B}}^! \dashv i_{\mathcal{B}}$ adjunction counit ε fit into a functorial exact triangle [Bon89]:

$$i_{\mathcal{B}} i_{\mathcal{B}}^! \xrightarrow{\varepsilon} \mathrm{Id}_{\mathcal{D}} \xrightarrow{\eta} i_{\mathcal{A}} i_{\mathcal{A}}^* \to i_{\mathcal{B}} i_{\mathcal{B}}^![1]. \tag{B.2}$$

Consider a general adjoint pair $F \dashv F^!$ of exact functors with $F \colon \mathcal{C} \to \mathcal{D}$. If the adjunction counit ε fits into a functorial exact triangle

$$FF^! \xrightarrow{\varepsilon} \mathrm{Id}_{\mathcal{D}} \to T \to FF^![1]$$

we refer to T as a *twist functor*. If the adjunction unit η fits into a functorial exact triangle

$$C \to \mathrm{Id}_{\mathcal{C}} \xrightarrow{\eta} F^! F \to C[1]$$

we refer to C as a *cotwist functor*.

Constructions and uniqueness of twist and cotwist functors are discussed in details in §C.4 in the framework of 2-categories. Let us briefly say that twist and cotwist are well-defined if F is a functor $\mathcal{D}(A) \to \mathcal{D}(B)$, for some DG algebras A and B, given by the tensor product with an $A^{\mathrm{op}} \otimes B$ DG bimodule and both of his adjoints are of this form too. If F satisfies this condition, we say that it is an *A and B perfect bimodule functor*. Indeed, in this case the bimodule defining F is perfect as an A and B module, cf. Proposition C.6. Once the bimodule corresponding to F is fixed, the twist T and the cotwist C are defined uniquely up to isomorphisms.

Consider an *A and B perfect bimodule functor* $F \colon \mathcal{D}(A) \to \mathcal{D}(B)$. Here F is *spherical* [AL17] if it admits both adjoints $F^* \dashv F \dashv F^!$ and the twist T and the cotwist C are both equivalences. If this is the case we refer to T as a *spherical twist* and to C as a *spherical cotwist*.

By [AL17], a functor F is spherical if and only if the dual twist T' and the dual cotwist C' are equivalences, where T' and C' are functors which fit into functorial exact triangles

$$F^* F \xrightarrow{\varepsilon'} \mathrm{Id}_{\mathcal{C}} \to C' \to F^* F[1], \quad T' \to \mathrm{Id}_{\mathcal{D}} \xrightarrow{\eta'} FF^* \to T'[1],$$

for the $F^* \dashv F$ adjunction unit η' and counit ε'.

Spherical functors can be constructed via spherical pairs. We recall the definition following [KS14]. Let $\mathcal{E}$ be a triangulated category and $\mathcal{E}_+, \mathcal{E}_- \subset \mathcal{E}$ a pair of admissible subcategories [Bon89]. Let $i_\pm \colon \mathcal{E}_\pm \to \mathcal{E}$ be the inclusions. They admit left $i_\pm^*$ and right $i_\pm^!$ adjoint functors $\mathcal{E} \to \mathcal{E}_\pm$. By $j_\pm \colon {}^\perp \mathcal{E}_\pm \to \mathcal{E}$ we denote the inclusions of the left orthogonal complements and by $j_\pm^! \colon \mathcal{E} \to {}^\perp \mathcal{E}_\pm$ their right adjoints. We assume that all these functors are induced by appropriate DG functors.

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Definition B.1 [KS14, Definition 3.4]. The pair $(\mathcal{E}_+, \mathcal{E}_-)$ of admissible subcategories is a *spherical pair* if

(i) the composites $i_+^* i_-\colon \mathcal{E}_- \to \mathcal{E}_+$, $i_-^* i_+\colon \mathcal{E}_+ \to \mathcal{E}_-$ are equivalences;
(ii) the composites $j_+^! j_-\colon {}^\perp\mathcal{E}_- \to {}^\perp\mathcal{E}_+$, $j_-^! j_+\colon {}^\perp\mathcal{E}_+ \to {}^\perp\mathcal{E}_-$ are equivalences.

Proposition B.2 [KS14, Propositions 3.6, 3.7]. *Let $(\mathcal{E}_+, \mathcal{E}_-) \subset \mathcal{E}$ be a spherical pair. Then functor $\Psi := j_-^! i_+\colon \mathcal{E}_+ \to {}^\perp\mathcal{E}_-$ is spherical. The unit and the counit for the $\Psi^* \dashv \Psi$ adjunction fit into functorial exact triangles:*

$$\Psi^*\Psi \xrightarrow{\varepsilon} \mathrm{Id}_{\mathcal{E}_+} \to i_+^* i_- i_-^* i_+ \to \Psi^*\Psi[1], \quad j_-^! j_+ j_+^! j_- \to \mathrm{Id}_{{}^\perp\mathcal{E}_-} \xrightarrow{\eta} \Psi\Psi^* \to j_-^! j_+ j_+^! j_-[1]. \tag{B.3}$$

The functorial exact triangles (B.3) are deduced from functorial exact triangles (B.2) for the SODs related to (admissible subcategories) $\mathcal{E}_+$ and $\mathcal{E}_-$. Therefore, in the presence of spherical pairs, one can define spherical functors purely in the realm of triangulated categories.

With the following proposition we describe spherical pairs associated to 4-periodical SODs

$$\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle = \langle \mathcal{B}, \mathcal{C} \rangle = \langle \mathcal{C}, \mathcal{D} \rangle = \langle \mathcal{D}, \mathcal{A} \rangle. \tag{B.4}$$

Proposition B.3. *Let $\mathcal{T}$ be a triangulated category with 4-periodical SODs (B.4). Then pairs $(\mathcal{A}, \mathcal{C})$, $(\mathcal{B}, \mathcal{D})$ of subcategories of $\mathcal{T}$ are spherical.*

Proof. We prove that $(\mathcal{A}, \mathcal{C})$ is a spherical pair. Let $i_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{T}$ denote the inclusion functor and $i_{\mathcal{A}}^*$ its left adjoint, similarly for $\mathcal{C} \subset \mathcal{T}$. The composite $i_{\mathcal{C}}^* i_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{C}$ is the right mutation over $\mathcal{B}$, hence it is an equivalence [BK89, Lemma 1.9]. Similarly, the right mutation $i_{\mathcal{A}}^* i_{\mathcal{C}}$ over $\mathcal{D}$ is an equivalence, (i), the first condition of Definition B.1 is satisfied.

Now let $i_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{T}$ denote the inclusion functor and $i_{\mathcal{B}}^!\colon \mathcal{T} \to \mathcal{B}$ its right adjoint, similarly for $\mathcal{D} \subset \mathcal{T}$. The composites $i_{\mathcal{B}}^! i_{\mathcal{D}}$, respectively $i_{\mathcal{D}}^! i_{\mathcal{B}}$, are the left mutations over $\mathcal{C}$, respectively $\mathcal{A}$, hence they are equivalences. It follows that $(\mathcal{A}, \mathcal{C})$ is a spherical pair in $\mathcal{T}$. The proof for the pair $(\mathcal{B}, \mathcal{D})$ is analogous. □

Remark B.4. In [HS16, Theorem 3.11], Halpern-Leistner and Shipman proved that any spherical functor $F\colon \mathcal{C} \to \mathcal{D}$ is given by 4-periodical SODs.

Appendix C. Lifting to the bicategory Bimod

Let k be a field. For a k-linear DG category $\mathcal{E}$, we denote by $[\mathcal{E}]$ the *homotopy category* of $\mathcal{E}$, i.e. the category with the same objects as in $\mathcal{E}$ and morphisms the zeroth cohomology of the DG complexes of morphisms in $\mathcal{E}$ (see [BK90]). It is an ordinary k-linear category.

By a *DG enhancement* of a triangulated category $\mathcal{C}$ we mean a choice of a pre-triangulated DG category $\mathcal{E}$ together with an equivalence $\mathcal{C} \simeq [\mathcal{E}]$ which is compatible with the induced triangulated structure on $[\mathcal{E}]$ (see [BK90]). A choice of suitable DG enhancements for our categories prompts a convenient replacement for the category of functors, as well as a lift of adjunction (co)units to morphisms in this replacement. This allows us to get an interpretation for 'cones' of morphisms of functors that we need in the main body of the text.

For the sake of simplicity, we take as DG enhancements of our categories only suitable categories of DG modules over DG algebras. The category of functors is replaced by the derived category of bimodules, and natural transformations by morphisms in the latter category. As a result we arrive at the bicategory **Bimod** whose objects are DG algebras and 1-morphisms are objects of the derived categories of DG bimodules. There is (after a suitable choice of universe) a bifunctor $\Phi\colon \mathbf{Bimod} \to \mathbf{Cat}$ to the 2-category of categories. We are interested in lifting of adjoint pairs of functors in **Cat** along the functor Φ.

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In the general framework of 2-categories (or bicategories), we have the uniqueness of the 2-categorical adjunction for a given 1-morphism. In addition, we show that 2-adjunctions can be transported over equivalences of objects in 2-categories, which implies an invariance of 2-categorical adjunctions in **Bimod** under the choice of DG enhancements.

By fixing a lift of a functor to a 1-morphism in **Bimod**, we get an essentially unique 2-categorical adjunction. This allows us to define a unique twist and cotwist of 2-categorical adjunctions as 1-morphisms in **Bimod**, thus interpreting constructions of Anno and Logvinenko [AL17]. In particular, we have the notion of spherical twists for spherical functors and, slightly more generally, for spherical couples.

Further we show how the 2-adjunction theory can be applied to functors between derived categories of (quasi-)coherent sheaves. Here another suitable 2-category is the category **FM** whose objects are schemes and 1-morphisms are objects of the derived category of the product of two schemes. By the results of Lunts and Schnürer [LS16] the 2-categorical adjunctions, twists and cotwists are readily transferred to this context. We discuss uniqueness of the functorial exact triangles associated to an FM functor and its adjoints. We also construct a 2-categorical lift to **Bimod** of the base-change morphism.

Finally, we provide a criterion for isomorphism of functors in terms of the restriction to one (generating) object.

C.1 Abstract 2-categorical adjunctions and spherical couples

Let $\mathcal{C}$ be a (strict) 2-category. A quadruple $(s, r, \eta, \varepsilon)$ of 1-morphisms $s \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $r \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ and 2-morphisms $\eta \colon \mathrm{Id}_A \to rs$, $\varepsilon \colon sr \to \mathrm{Id}_B$ is a *2-categorical* (A, B)-*adjunction* if

$$s \xrightarrow{s\eta} srs \xrightarrow{\varepsilon s} s, \quad r \xrightarrow{\eta r} rsr \xrightarrow{r\varepsilon} r \tag{C.1}$$

are equal to the identity morphisms of s and r, respectively. In this case, r is said to be a *right 2-categorical adjoint* of s, s a *left 2-categorical adjoint* of r and η and ε the *unit* and *counit* of the adjunction. We write $s \dashv r$ if there exist η and ε such that $(s, r, \eta, \varepsilon)$ is a 2-categorical adjunction. The choice of such η and ε is not unique.

The proof of the following fact is standard.

LEMMA C.1 (Cf. [Bén65, TV15]). *Let s be a 1-morphism. If s has a right 2-categorical adjoint, then the 2-categorical adjunction $(s, r, \eta, \varepsilon)$ is unique up to a canonical 2-isomorphism. More precisely, for any other adjunction $(s, r', \eta', \varepsilon')$, the composite $\alpha \colon r \xrightarrow{\eta' r} r'sr \xrightarrow{r\varepsilon} r'$ is a unique 2-isomorphism commuting with the units and counits. Conversely, any 2-isomorphism $\alpha \colon r \to r'$ yields a 2-categorical adjunction $(s, r', \alpha s \circ \eta, \varepsilon \circ s\alpha^{-1})$.*

An (A, B)-*equivalence* in $\mathcal{C}$ is a quadruple (f, g, ν, μ) of 1-morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ and 2-isomorphisms $\nu \colon \mathrm{Id}_A \xrightarrow{\simeq} gf$, $\mu \colon fg \xrightarrow{\simeq} \mathrm{Id}_B$. It is an *adjoint* (A, B)-*equivalence* if (f, g, ν, μ) is, in addition, a 2-categorical (A, B)-adjunction.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be 2-categories. A *pseudo-functor* $\theta \colon \mathcal{C}_1 \to \mathcal{C}_2$ consists of a map $\theta \colon \mathrm{Ob}(\mathcal{C}_1) \to \mathrm{Ob}(\mathcal{C}_2)$, functors $\theta_{C,C'} \colon \mathrm{Hom}_{\mathcal{C}_1}(C, C') \to \mathrm{Hom}_{\mathcal{C}_2}(\theta(C), \theta(C'))$, for any pair (C, C') of objects of $\mathcal{C}_1$, and 2-isomorphisms $u \colon \theta(\mathrm{Id}_C) \to \mathrm{Id}_{\theta(C)}$, for any object $C \in \mathcal{C}_1$, and $a \colon \theta(g \circ f) \to \theta(g) \circ \theta(f)$, for any pair (f, g) of composable 1-morphisms in $\mathcal{C}_1$. These need to satisfy coherence conditions: for any triple (h, g, f) of composable 1-morphisms in $\mathcal{C}_1$, the two possible 2-morphisms $\theta(h \circ g \circ f) \to \theta(h) \circ \theta(g) \circ \theta(f)$ induced by a are equal, and so are the 2-morphisms $\theta(f) \to \theta(f) \circ \theta(\mathrm{Id})$, $\theta(g) \to \theta(\mathrm{Id}) \circ \theta(g)$ induced by a and u^{-1}.

We say that a pseudo-functor $\theta\colon \mathcal{C}_1 \to \mathcal{C}_2$ is a *pseudo-equivalence* if θ_{C_1,C_2} is an equivalence for all $(C_1, C_2) \in \mathcal{C}_1$.

Let $\alpha\colon e \to f$, $\alpha'\colon e' \to f'$ be 2-morphisms in a 2-category $\mathcal{C}$. We say that α and α' are *isomorphic*, and depict it by $\alpha \simeq \alpha'$, if there exist isomorphisms $\tau_e\colon e \xrightarrow{\simeq} e'$, $\tau_f\colon f \xrightarrow{\simeq} f'$ such that $\tau_f \circ \alpha = \alpha' \circ \tau_e$.

LEMMA C.2. *Let $\theta\colon \mathcal{C}_1 \to \mathcal{C}_2$ be a pseudo-functor and $(s, r, \eta, \varepsilon)$ a 2-categorical (A, B)-adjunction in $\mathcal{C}_1$. Then $\theta(s)$ fits into a 2-categorical $(\theta(A), \theta(B))$-adjunction $(\theta(s), \theta(r), \eta', \varepsilon')$ such that $\theta_{A,A}(\eta) \simeq \eta'$, $\theta_{B,B}(\varepsilon) \simeq \varepsilon'$.*

Proof. A standard diagram chasing shows that for η' and ε' defined as the composites

$$\eta'\colon \mathrm{Id}_{\theta(A)} \xrightarrow{u^{-1}} \theta(\mathrm{Id}_A) \xrightarrow{\theta(\eta)} \theta(rs) \xrightarrow{a} \theta(r)\theta(s),$$

$$\varepsilon'\colon \theta(s)\theta(r) \xrightarrow{a^{-1}} \theta(sr) \xrightarrow{\theta(\varepsilon)} \theta(\mathrm{Id}_B) \xrightarrow{u} \mathrm{Id}_{\theta(B)}$$

equalities (C.1) hold, i.e. $(\theta(s), \theta(r), \eta', \varepsilon')$ is a 2-categorical $(\theta(A), \theta(B))$-adjunction.

By definition of η' and ε', diagrams

$$
\begin{array}{ccccccc}
\mathrm{Id}_{\theta(A)} & \xrightarrow{\eta'} & \theta(r)\theta(s) & \quad & \theta(s)\theta(r) & \xrightarrow{\varepsilon'} & \mathrm{Id}_{\theta(B)} \\
u \uparrow \simeq & & \simeq \uparrow a & & a \uparrow \simeq & & \simeq \uparrow u \\
\theta(\mathrm{Id}_A) & \xrightarrow{\theta_{A,A}(\eta)} & \theta(rs) & & \theta(sr) & \xrightarrow{\theta_{B,B}(\varepsilon)} & \theta(\mathrm{Id}_B)
\end{array}
$$

commute, which proves isomorphisms $\theta_{A,A}(\eta) \simeq \eta'$ and $\theta_{B,B}(\varepsilon) \simeq \varepsilon'$. $\qquad\square$

PROPOSITION C.3. *Let $\mathcal{C}$ be a 2-category, (f_A, g_A, ν_A, μ_A) an (A, A')-equivalence and (f_B, g_B, ν_B, μ_B) a (B, B')-equivalence in $\mathcal{C}$. Denote by $\mathcal{C}_{A,B}$, respectively $\mathcal{C}_{A',B'}$, the full 2-subcategories of $\mathcal{C}$ with two objects A and B, respectively A' and B'. The four functors*

$$\theta_{S,T}(-) := f_T \circ (-) \circ g_S\colon \mathrm{Hom}_{\mathcal{C}}(S, T) \to \mathrm{Hom}_{\mathcal{C}}(S', T'),$$

defined for pairs $(S, T) \in \{(A, A), (A, B), (B, A), (B, B)\}$ of objects of $\mathcal{C}_{A,B}$ and the corresponding pair (S', T') of objects of $\mathcal{C}_{A',B'}$, can be extended to a pseudo-equivalence $\theta\colon \mathcal{C}_{A,B} \to \mathcal{C}_{A',B'}$.

Proof. By changing the 2-isomorphism ν_A, μ_A, ν_B, and μ_B if necessary, one can assume that (f_A, g_A, ν_A, μ_A) and (f_B, g_B, ν_B, μ_B) are adjoint equivalences [Mac71].

Functor $\theta_{S,T}$ is an equivalence with quasi-inverse $g_T \circ (-) \circ f_S$. We need to define morphisms u and a from the definition of pseudo-functors. The 2-isomorphisms u are defined as μ_A and μ_B:

$$\theta_{A,A}(\mathrm{Id}_A) = f_A \circ g_A \xrightarrow{\mu_A} \mathrm{Id}_{A'}, \quad \theta_{B,B}(\mathrm{Id}_B) = f_B \circ g_B \xrightarrow{\mu_B} \mathrm{Id}_{B'}.$$

For a pair (f, g) of 1-morphisms such that B is the target of f and the source of g, the 2-isomorphism a:

$$f_{T_g} \circ g \circ f \circ g_{S_f} \xrightarrow{f_{T_g} \circ g \circ \nu_B \circ f \circ g_{S_f}} f_{T_g} \circ g \circ f_B \circ g_B \circ f \circ g_{S_f},$$

where $S_f, T_g \in \{A, B\}$ are the source of f and the target of g, respectively. Similarly, we use ν_A if A is the target of f and the source of g.

Equalities (C.1) for 2-adjunctions (f_A, g_A, ν_A, μ_A) and (f_B, g_B, ν_B, μ_B) imply that u and a defined in this way satisfy the required coherence conditions. $\qquad\square$

We say that a 2-category $\mathcal{C}$ is *1-triangulated* if, for any pair (B, C) of objects of $\mathcal{C}$, the category $\mathrm{Hom}_{\mathcal{C}}(B, C)$ is triangulated and, for any $\varphi_A \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $\varphi_D \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, functors $(-) \circ \varphi_A \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$, $\varphi_D \circ (-) \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(B, D)$ are exact.

A 2-categorical (A, B)-adjunction $(s, r, \eta, \varepsilon)$ in a 1-triangulated 2-category $\mathcal{C}$ induces 1-endomorphisms c_s, t_s defined via exact triangles:

$$c_s \to \mathrm{Id}_A \xrightarrow{\eta} rs \to c_s[1], \quad sr \xrightarrow{\varepsilon} \mathrm{Id}_B \to t_s \to sr[1]. \tag{C.2}$$

We say that t_s is the *twist* and c_s the *cotwist* for the 2-categorical adjunction $(s, r, \eta, \varepsilon)$.

PROPOSITION C.4. *Let* $(s, r, \eta, \varepsilon)$ *be a 2-categorical* (A, B) *adjunction in a 1-triangulated 2-category* $\mathcal{C}$. *Consider a pair of* (A, A')- *and* (B, B')-*equivalences and the induced pseudo-equivalence* $\theta \colon \mathcal{C}_{A,B} \to \mathcal{C}_{A',B'}$. *Then the twist* $t_{\theta(s)}$ *is isomorphic to* $\theta(t_s)$ *and the cotwist* $c_{\theta(s)}$ *is isomorphic to* $\theta(c_s)$.

Proof. By Proposition C.3 (A, A')- and (B, B')-equivalences induce a pseudo-equivalence $\theta \colon \mathcal{C}_{A,B} \to \mathcal{C}_{A',B'}$ Lemma C.2 implies that $(\theta(s), \theta(r), \eta', \varepsilon')$ is an (A', B')-adjunction and $\theta(\eta) \simeq \eta'$, $\theta(\varepsilon) \simeq \varepsilon'$. The isomorphism of cones follows. □

We say that a 2-categorical (A, B)-adjunction $(s, r, \eta, \varepsilon)$ is a *spherical couple* if the twist t_s, and respectively the cotwist c_s, defined in (C.2), is a (B, B)-, respectively (A, A)-equivalence.

Under the above assumption, octahedron

$$
\begin{array}{ccccc}
sc_s[1] & \longrightarrow & 0 & \longrightarrow & t_s s \\
\uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & s & \xrightarrow{\simeq} & s \\
\uparrow & & \uparrow{\scriptstyle\simeq} & & \uparrow \\
sc_s & \longrightarrow & s & \longrightarrow & srs
\end{array}
$$

implies an isomorphism

$$t_s s \simeq sc_s[2]. \tag{C.3}$$

A 1-morphism $s \colon A \to B$ is said to be *spherical* [AL17] if it has the 2-categorical left and right adjoints and $(s, r, \eta, \varepsilon)$ is a spherical couple. It is convenient to think of twist and cotwist as being assigned to a spherical couple, because their definition needs only one adjoint, not two as in the definition of the spherical functor.

C.2 The bicategory Bimod

Let k be a field. We fix a universe $\mathcal{U}$ and consider the bicategory **Bimod** whose objects are unital DG k-algebras, 1-morphisms $A \to B$ are objects in the derived category $\mathcal{D}_{\mathcal{U}}(A^{\mathrm{op}} \otimes_k B)$ of $A^{\mathrm{op}} \otimes_k B$ DG $\mathcal{U}$-modules and 2-morphisms are morphisms in $\mathcal{D}_{\mathcal{U}}(A^{\mathrm{op}} \otimes_k B)$. Often, we drop the universe $\mathcal{U}$ from the notation. The derived tensor product of bimodules $M \otimes_B^L N$, for $M \in \mathcal{D}(A^{\mathrm{op}} \otimes_k B)$ and $N \in \mathcal{D}(B^{\mathrm{op}} \otimes_k C)$, defines the composition of 1-morphisms in **Bimod**. The derived tensor product is unique up to a canonical isomorphism, i.e. a pair of bimodule isomorphisms $M \xrightarrow{\simeq} M'$, $N \xrightarrow{\simeq} N'$ in the derived categories induces a unique isomorphism $M \otimes_B^L N \xrightarrow{\simeq} M' \otimes_B^L N'$. Various choices involved in the definition of the derived tensor product are controlled by these isomorphisms. In particular, they allow to construct the associators and unitors in the bicategory **Bimod**. We write formulas as if the composition of 1-morphisms in

Bimod was strictly associative and the identity morphisms were strict. If necessary, all morphisms of associativity and unitors can readily find their places in formulas. Equally, one can refer to the fact that every bicategory is biequivalent to a 2-category [MP85].

The tensor product is an exact functor, hence the 2-category **Bimod** is 1-triangulated.

Let $\mathcal{V}$ be a universe such that $\mathcal{D}_{\mathcal{U}}(A^{\mathrm{op}} \otimes_k B)$ is an essentially $\mathcal{V}$-small category, for any pair A, B of unital DG k-algebras, cf. [Mac71]. Let further $\mathbf{Cat}_{\mathcal{V}}$ denote the 2-category of $\mathcal{V}$-small categories, functors and natural transformations. Consider the 2-functor

$$\Phi \colon \mathbf{Bimod} \to \mathbf{Cat}_{\mathcal{V}}, \quad \Phi(A) = \mathcal{D}(A). \tag{C.4}$$

For $M \in \mathcal{D}(A^{\mathrm{op}} \otimes_k B)$, $\Phi_M \colon \mathcal{D}(A) \to \mathcal{D}(B)$ is the functor

$$\Phi_M(-) = (-) \otimes_A^L M.$$

Finally, morphism $\alpha \colon M \to M'$ induces a natural transformation $\Phi_\alpha \colon \Phi_M \to \Phi_{M'}$.

In particular, as Φ is a 2-functor, the tensor product of bimodules corresponds to the composition of tensor functors:

$$\Phi_{M_1} \circ \Phi_{M_2} \simeq \Phi_{M_2 \otimes^L M_1}.$$

We note that the 2-functor Φ is, in general, neither full nor faithful.

Note that by [Toë07, Corollary 7.6], any DG enhanceable (i.e. admitting a lift to a functor of suitable DG enhancements) commuting with direct sums functor $\mathcal{D}(A) \to \mathcal{D}(B)$ is of the form Φ_M, for some bimodule M. We refer to functors of the form Φ_M as *bimodule functors*.

An exact triangle of bimodules in the category $\mathcal{D}(A^{\mathrm{op}} \otimes B)$:

$$M_1 \to M_2 \to M_3 \to M_1[1]$$

induces via the functor Φ a functorial exact triangle

$$\Phi_{M_1}(-) \to \Phi_{M_2}(-) \to \Phi_{M_3}(-) \to \Phi_{M_1}(-)[1]. \tag{C.5}$$

LEMMA C.5. *Functor Φ is conservative on the categories of 1-morphisms in **Bimod**.*

Proof. Let A and B be unital DG algebras and $f \colon M_1 \to M_2$ a morphism in $\mathcal{D}(A^{\mathrm{op}} \otimes B)$ such that $\Phi_f \colon \Phi_{M_1} \to \Phi_{M_2}$ is an isomorphism. Morphism f fits into an exact triangle $M_1 \xrightarrow{f} M_2 \to M_3 \to M_1[1]$. For any $E \in \mathcal{D}(A)$, complex $\Phi_{M_1}(E) \xrightarrow{\Phi_f} \Phi_{M_2}(E) \to \Phi_{M_3}(E) \to \Phi_{M_1}(E)[1]$ is an exact triangle. As Φ_f is an isomorphism, $\Phi_{M_3}(E) \simeq 0$. In particular, $M_3 \simeq \Phi_{M_3}(A) \simeq 0$, which implies that f is an isomorphism. $\qquad\square$

C.3 The bicategory FM

Let k be a field. Consider the bicategory **FM** whose objects are quasi-compact, quasi-separated k-schemes. The category of morphisms $X \to Y$ is the derived category $\mathrm{Hom}_{\mathbf{FM}}(X, Y) = \mathcal{D}_{\mathrm{qc}}(X \times Y)$ of $\mathcal{U}$-quasi-coherent sheaves on $X \times Y$ and the composition is given by the convolution: for schemes X, Y, Z, $K \in \mathcal{D}_{\mathrm{qc}}(X \times Y)$ and $L \in \mathcal{D}_{\mathrm{qc}}(Y \times Z)$ their composition is

$$L * K := \pi_{XZ*}(\pi_{YZ}^* L \otimes \pi_{XY}^* K),$$

where $\pi_{XY} \colon X \times Y \times Z \to X \times Y$, $\pi_{YZ} \colon X \times Y \times Z \to Y \times Z$, $\pi_{XZ} \colon X \times Y \times Z \to X \times Z$ are the projections.

The identity in **FM** is given by the structure sheaf of the diagonal $\Delta \subset X \times X$. For K in $\mathcal{D}_{\mathrm{qc}}(X \times Y)$, the unitor is given by the flat base change and projection formula isomorphisms:

$$\pi_{13*}(\pi_{23}^* K \otimes \pi_{12}^* \mathcal{O}_\Delta) = \pi_{13*}(\pi_{23}^* K \otimes \pi_{12}^* \Delta_* \mathcal{O}_X) \xrightarrow{\simeq} \pi_{13*}(\pi_{23}^* K \otimes (\Delta \times \mathrm{Id}_Y)_* \pi_1^* \mathcal{O}_X)$$

$$\xrightarrow{\simeq} \pi_{13*}(\Delta \times \mathrm{Id}_Y)_*((\Delta \times \mathrm{Id}_Y)^* \pi_{23}^* K \otimes \pi_1^* \mathcal{O}_X) \simeq \mathrm{Id}_{X*}(\mathrm{Id}_X^* K) \otimes \mathcal{O}_{X \times Y}) \simeq K,$$

where π_{12}, π_{13} and π_{23} are projections from $X \times X \times Y$ to the corresponding factors, similarly, π_1 and π_2 are projections from $X \times X$. We denote by $\Delta \colon X \to X \times X$ the diagonal embedding and by $\Delta \times \mathrm{Id}_Y \colon X \times Y \to X \times X \times Y$ the map induced by Δ and the identity on Y. The second unitor is defined analogously.

The associator in **FM** is defined analogously, using the projection formula and the flat base change isomorphism for the projections from $W \times X \times Y \times Z$ to the triple and double products.

Similarly to **Bimod**, we consider **FM** as a 1-triangulated bicategory.

Consider the bifunctor

$$\Xi \colon \mathbf{FM} \to \mathbf{Cat}_{\mathcal{V}}, \quad \Xi(X) = \mathcal{D}_{\mathrm{qc}}(X)$$

which assigns to a 1-morphism $K \in \mathcal{D}_{\mathrm{qc}}(X \times Y)$ the *Fourier–Mukai functor* with kernel K:

$$\Xi_K(-) = Rp_{Y*}(Lp_X^*(-) \otimes K),$$

where $p_X \colon X \times Y \to X$, $p_Y \colon X \times Y \to Y$ are the projections. A 2-morphism $\alpha \colon K \to K'$ induces a natural transformation $\Xi_\alpha \colon \Xi_K \to \Xi_{K'}$.

The isomorphism $\mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(X)} \to \Xi(\mathcal{O}_\Delta)$ is induced by the projection formula

$$(-) = \mathcal{O}_X \otimes (-) = \mathrm{Id}_{X*}(\mathcal{O}_X \otimes \mathrm{Id}_X^*(-)) \simeq \pi_{2*}\Delta_*(\mathcal{O}_X \otimes \Delta^*\pi_1^*(-)) \xrightarrow{\cong} \pi_{2*}(\Delta_*\mathcal{O}_X \otimes \pi_1^*(-)),$$

where $\pi_1, \pi_2 \colon X \times X \to X$ are the projections to the corresponding factors.

For $K \in \mathcal{D}_{\mathrm{qc}}(X \times Y)$, $L \in \mathcal{D}_{\mathrm{qc}}(Y \times Z)$, the composition coherence isomorphism $\Xi(L) \otimes \Xi(K) \xrightarrow{\cong} \Xi(L * K)$ is again given by the flat base change and projection formula:

$$\Xi(L) \circ \Xi(K)(-) = q_{Z*}(L \otimes q_Y^* r_{Y*}(K \otimes r_X^*(-))) \xrightarrow{\cong} q_{Z*}(L \otimes \pi_{YZ*}\pi_{XY}^*(K \otimes r_X^*(-)))$$

$$\xrightarrow{\cong} q_{Z*}\pi_{YZ*}(\pi_{YZ}^*L \otimes \pi_{XY}^*K \otimes \pi_{XY}^*r_X^*(-)) = p_{Z*}\pi_{XZ*}(\pi_{YZ}^*L \otimes \pi_{XY}^*K \otimes \pi_{XZ}^*p_X^*(-))$$

$$\xrightarrow{\cong} p_{Z*}(\pi_{XZ*}(\pi_{YZ}^*L \otimes \pi_{XY}^*K) \otimes p_X^*(-)) = \Xi(L * K)(-),$$

where π_{XY}, π_{XZ} and π_{YZ} are the projections from $X \times Y \times Z$ to the corresponding products $X \times Y$, $X \times Z$ and $Y \times Z$. We denote by r_X, r_Y the projections from $X \times Y$, by p_X, p_Z the projections from $X \times Z$ and by q_Y, q_Z the projections from $Y \times Z$.

An exact triangle in the category $\mathcal{D}_{\mathrm{qc}}(X \times Y)$

$$\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to \mathcal{E}_1[1]$$

induces via the functor Ξ a functorial exact triangle of FM functors:

$$\Xi_{\mathcal{E}_1} \to \Xi_{\mathcal{E}_2} \to \Xi_{\mathcal{E}_3} \to \Xi_{\mathcal{E}_1}[1]. \tag{C.6}$$

C.4 2-adjunctions, spherical couples and spherical 1-morphisms in Bimod

Any 2-functor preserves 2-adjunctions, in particular the 2-functor Φ maps a 2-categorical adjunction $(M, R, \eta, \varepsilon)$ in **Bimod** to a pair of adjoint functors $\Phi_M \dashv \Phi_R$ between the corresponding derived categories.

Let A and B be unital DG algebras. We discuss when a pair of adjoint functors between $\mathcal{D}(A)$ and $\mathcal{D}(B)$ can be lifted to a 2-categorical adjunction in **Bimod**. For $M \in (A^{\mathrm{op}} \otimes_k B)$, define its A and B duals $M^A, M^B \in \mathcal{D}(B^{\mathrm{op}} \otimes_k A)$ (see [AL21]):

$$M^A := R\,\mathrm{Hom}_A(M, A), \quad M^B := R\,\mathrm{Hom}_B(M, B).$$

We have several (derived) evaluation and action morphisms [AL21]:

$$\varepsilon_R \colon M^B \otimes_A^L M \to B, \quad \varepsilon_L \colon M \otimes_B^L M^A \to A,$$
$$\eta_R \colon A \to M \otimes_B^L M^B, \quad \eta_L \colon B \to M^A \otimes_A^L M. \tag{C.7}$$

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Note that η_R is defined if M is B-perfect, i.e. if its image in $\mathcal{D}(B)$ lies in the full subcategory of compact objects. Respectively, η_L is defined if M is A-perfect.

The following proposition allows us to lift suitable pairs of adjoint functors between triangulated categories to 2-categorical adjunctions in **Bimod**.

PROPOSITION C.6. *Let M be in $\mathcal{D}(A^{\mathrm{op}} \otimes_k B)$.*

(i) *Then M is A-perfect if and only if the left adjoint Φ_M^* to Φ_M exists. Under these assumptions functor Φ_M^* is isomorphic to Φ_{M^A} and the evaluation and action maps yield a 2-categorical adjunction $(M^A, M, \eta_L, \varepsilon_L)$ in the bicategory **Bimod**.*

(ii) *Then M is B-perfect if and only if Φ_M maps compact objects to compact ones if and only if the right adjoint $\Phi_M^!$ is isomorphic to Φ_{M^B}. Under these assumptions the evaluation and action maps yield a 2-categorical adjunction $(M, M^B, \eta_R, \varepsilon_R)$.*

Proof. This is basically the statement of [AL21, Propositions 4.2 and 4.7]. In [AL21] object M is lifted to $\overline{M}$ in a suitable (weak, in the sense of Drinfeld [Dri04]) DG enhancement for $\mathcal{D}(A^{\mathrm{op}} \otimes_k B)$. The A and B duals of $\overline{M}$ are defined, which lift M^A and M^B. Also, the evaluation and action maps are constructed on the DG level. With [AL21, Proposition 4.7] the authors check that the composites in (C.1) are identities of objects in the derived categories of bimodules. $\square$

Consider B-perfect $M \in \mathcal{D}(A^{\mathrm{op}} \otimes_k B)$. The 2-categorical adjunction $(M, M^B, \eta_R, \varepsilon_R)$ in **Bimod** (Proposition C.6(ii)) allows us to consider the twist T_B and the cotwist C_A, defined with triangles (C.2). Lemma C.1 implies that T_B and C_A are unique up to isomorphisms. Applying functor Φ as in (C.4) we obtain endo-functors $\Omega_{A,M}$ and $\Theta_{B,M}$ of $\mathcal{D}(A)$, respectively $\mathcal{D}(B)$. They fit into functorial exact triangles:

$$\Omega_{A,M} \to \mathrm{Id}_{\mathcal{D}(A)} \xrightarrow{\eta_R} \Phi_M^! \Phi_M \to \Omega_{A,M}[1],$$

$$\Phi_M \Phi_M^! \xrightarrow{\varepsilon_R} \mathrm{Id}_{\mathcal{D}(B)} \to \Theta_{B,M} \to \Phi\Phi^![1].$$

Uniqueness of T_A and C_B implies that $\Omega_{A,M} = \Phi_{C_A}$ and $\Theta_{B,M} = \Phi_{T_B}$ are defined uniquely by M up to a (non-unique) functorial isomorphism. Note that we do not claim that the cotwist functor $\Omega_{A,M}$ and twist functor $\Theta_{B,M}$ are uniquely defined by the functor Φ_M.

Assume now that M is A-perfect. According to Proposition C.6(i), we can consider the twist T_A and the cotwist C_B for the 2-categorical adjunction $(M^A, M, \eta_L, \varepsilon_L)$. By applying functor Φ as in (C.4), we obtain endo-functors $\Omega_{B,M} = \Phi_{C_B}$ and $\Theta_{A,M} = \Phi_{T_A}$. Those fit into functorial exact triangles:

$$\Omega_{B,M} \to \mathrm{Id}_{\mathcal{D}(B)} \xrightarrow{\eta_L} \Phi_M \Phi_M^* \to \Omega_{B,M}[1],$$

$$\Phi_M^* \Phi_M \xrightarrow{\varepsilon_L} \mathrm{Id}_{\mathcal{D}(A)} \to \Theta_{A,M} \to \Phi_M^* \Phi_M[1].$$

We say that Φ_M is a spherical functor if M is a spherical 1-morphism in **Bimod**.

THEOREM C.7 [AL17, 5.1 and 5.2]. *Let M be an A and B-perfect bimodule. Functor Φ_M is spherical if any two of the following hold:*

(S_1) *$\Theta_{B,M}$ is an equivalence;*
(S_2) *$\Omega_{A,M}$ is an equivalence;*
(S_3) *composite $\Phi_M^* \Theta_{B,M}[-1] \to \Phi_M^* \Phi_M \Phi_M^! \to \Phi_M^!$ is an isomorphism of functors;*
(S_4) *composite $\Phi_M^! \to \Phi_M^! \Phi_M \Phi_M^* \to \Omega_{A,M} \Phi_M^*[1]$ is an isomorphism of functors.*

Then also $\Omega_{B,M}$ and $\Theta_{A,M}$ are also equivalences of categories, quasi-inverse to $\Theta_{B,M}$ and $\Omega_{A,M}$, respectively.

C.5 Lifting push-forwards, pull-backs and the base change to Bimod

We fix a DG enhancement [BK90] for the category $\mathcal{D}_{\mathrm{qc}}(X)$, for example, by h-injective complexes [Spa88, KS06]. Let $P \in \mathcal{D}_{\mathrm{qc}}(X)$ be a compact generator (see [BVdB03]) and A the DG endomorphism algebra of its lift to the DG enhancement. Then $\mathcal{D}(A) \simeq \mathcal{D}_{\mathrm{qc}}(X)$ by [Kel94], which paves the way to performing the necessary constructions in the category **Bimod**. With Proposition C.9 we check that the constructed functorial exact triangles are independent of the choice of P.

Assume that X and Y are Noetherian separated schemes such that any perfect complex on both X and Y is isomorphic to a bounded complex of locally free sheaves. Choose compact generators in $\mathcal{D}_{\mathrm{qc}}(X)$ and in $\mathcal{D}_{\mathrm{qc}}(Y)$ and fix lifts of the generators to DG enhancements of both categories. Denote by A_X and A_Y their DG endomorphism algebras.

By [LS16] there exists a choice of equivalences $\Upsilon_X \colon \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}(A_X)$, $\Upsilon_Y \colon \mathcal{D}_{\mathrm{qc}}(Y) \to \mathcal{D}(A_Y)$ and $\Upsilon_{X,Y} \colon \mathcal{D}_{\mathrm{qc}}(X \times Y) \simeq \mathcal{D}(A_X^{\mathrm{op}} \otimes A_Y)$ which map FM functors $\mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(Y)$ to bimodule functors $\mathcal{D}(A_X) \to \mathcal{D}(A_Y)$. More precisely, for any $\mathcal{E} \in \mathcal{D}_{\mathrm{qc}}(X \times Y)$ the diagram

$$
\begin{array}{ccc}
\mathcal{D}(A_X) & \xrightarrow{\;\Phi_{\Upsilon_{X,Y}(\mathcal{E})}\;} & \mathcal{D}(A_Y) \\[2pt]
{\scriptstyle \Upsilon_X} \big\uparrow {\scriptstyle \simeq} & & {\scriptstyle \Upsilon_Y} \big\uparrow {\scriptstyle \simeq} \\[2pt]
\mathcal{D}_{\mathrm{qc}}(X) & \xrightarrow{\;\Xi_{\mathcal{E}}\;} & \mathcal{D}_{\mathrm{qc}}(Y)
\end{array}
\tag{C.8}
$$

commutes up to a functorial isomorphism.

Commutativity of (C.8) implies that having fixed compact generators for $\mathcal{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}_{\mathrm{qc}}(Y)$, we can view FM functors $\mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(Y)$ as bimodule functors and vice versa.

Fix compact generators for $\mathcal{D}_{\mathrm{qc}}(X)$ and $\mathcal{D}_{\mathrm{qc}}(Y)$ and consider $\mathcal{E} \in \mathcal{D}_{\mathrm{qc}}(X \times Y)$. In view of (C.8), for

$$
M = \Upsilon_{X,Y}(\mathcal{E}),
$$

functors $\Upsilon_Y \Xi_{\mathcal{E}}$ and $\Phi_M \Upsilon_X$ are isomorphic.

LEMMA C.8. *If $\Xi_{\mathcal{E}}$ has left adjoint $\Xi_{\mathcal{E}}^*$, then $\Xi_{\mathcal{E}}^* \simeq \Xi_{\Upsilon_{Y,X}^{-1}(M^{A_X})}$ is an FM functor. If $\Xi_{\mathcal{E}}$ maps compact objects to compact objects, then the right adjoint exists and it is an FM functor, $\Xi_{\mathcal{E}}^! \simeq \Xi_{\Upsilon_{Y,X}^{-1}(M^{A_Y})}$.*

Proof. If $\Xi_{\mathcal{E}}^*$ exists then $\Upsilon_X \Xi_{\mathcal{E}}^* \Upsilon_Y^{-1}$ is left adjoint to Φ_M. Proposition C.6(i) implies that $\Upsilon_X \Xi_{\mathcal{E}}^* \Upsilon_Y^{-1} \simeq \Phi_{M^{A_X}}$. Commutativity of (C.8) implies $\Xi_{\mathcal{E}}^* \simeq \Xi_{\Upsilon_{Y,X}^{-1}(M^{A_X})}$.

If $\Xi_{\mathcal{E}}$ maps compact objects to compact objects then so does $\Phi_M \simeq \Upsilon_Y \Xi_{\mathcal{E}} \Upsilon_X^{-1}$. By Proposition C.6(ii), $\Phi_M^! \simeq \Phi_{M^{A_Y}}$. Moreover, $\Upsilon_X^{-1} \Phi_{M^{A_Y}} \Upsilon_Y$ is right adjoint to $\Xi_{\mathcal{E}}$, i.e. $\Xi_{\mathcal{E}}^! \simeq \Upsilon_X^{-1} \Phi_{M^{A_Y}} \Upsilon_Y$. As above we conclude that $\Xi_{\mathcal{E}}^! \simeq \Xi_{\Upsilon_{Y,X}^{-1}(M^{A_Y})}$. $\qquad\square$

The 2-categorical adjunctions $(M^{A_X}, M, \eta_L, \varepsilon_L)$, $(M, M^{A_Y}, \eta_R, \varepsilon_R)$ in **Bimod** define the twist and the cotwist by triangles (C.2). In view of the isomorphism $\Xi_{\mathcal{E}} \simeq \Phi_M$ and the uniqueness of adjoint functors (see Lemma C.1), the functorial exact triangles read

$$
\begin{aligned}
\Xi_{\mathcal{E}}^* \Xi_{\mathcal{E}} &\to \mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(X)} \to \Theta_{X,\mathcal{E}} \to \Xi_{\mathcal{E}}^* \Xi_{\mathcal{E}}[1], \\
\Omega_{Y,\mathcal{E}} &\to \mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(Y)} \to \Xi_{\mathcal{E}} \Xi_{\mathcal{E}}^* \to \Omega_{Y,\mathcal{E}}[1], \\
\Xi_{\mathcal{E}} \Xi_{\mathcal{E}}^! &\to \mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(Y)} \to \Theta_{Y,\mathcal{E}} \to \Xi_{\mathcal{E}} \Xi_{\mathcal{E}}^![1], \\
\Omega_{X,\mathcal{E}} &\to \mathrm{Id}_{\mathcal{D}_{\mathrm{qc}}(X)} \to \Xi_{\mathcal{E}}^! \Xi_{\mathcal{E}} \to \Omega_{X,\mathcal{E}}[1].
\end{aligned}
\tag{C.9}
$$

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Note that we construct the triangles via the category **Bimod**. We do not check that (C.9) are induced by morphisms between the convolutions of $\mathcal{E}$ with FM kernels of the adjoint functors and the structure sheaf of the diagonal.

PROPOSITION C.9. *Consider $\mathcal{E} \in \mathcal{D}_{qc}(X \times Y)$. Then, up to an isomorphism, the functorial exact triangles of FM functors (C.9) do not depend on the choice of compact generators for $\mathcal{D}_{qc}(X)$ and $\mathcal{D}_{qc}(Y)$.*

Proof. Another choice of generators for $\mathcal{D}_{qc}(X)$ and $\mathcal{D}_{qc}(Y)$ gives equivalences $\Upsilon_X^B \colon \mathcal{D}_{qc}(X) \xrightarrow{\simeq} \mathcal{D}(B_X)$, $\Upsilon_Y^B \colon \mathcal{D}_{qc}(Y) \xrightarrow{\simeq} \mathcal{D}(B_Y)$, for some DG algebras B_X, B_Y. The composite $\Upsilon_X^B \circ \Upsilon_X^{-1}$, respectively $\Upsilon_Y^B \circ \Upsilon_Y^{-1}$, is DG enhanceable and commutes with direct sums, hence it admits a lift to a bimodule functor, cf. [Toë07]. Functor $\Upsilon_X^B \circ \Upsilon_X^{-1}$, respectively $\Upsilon_Y^B \circ \Upsilon_Y^{-1}$, together with its quasi-inverse yields (A_X, B_X)-equivalence, respectively (A_Y, B_Y)-equivalence, in **Bimod**. By Proposition C.4, the induced pseudo-equivalence $\theta \colon \mathbf{Bimod}_{A_X, A_Y} \to \mathbf{Bimod}_{B_X, B_Y}$ preserves the twist and the cotwist of the adjunction. The image of θ under functor Φ yields the required isomorphism of functorial triangles induced by Υ and Υ^B. □

Let $f \colon X \to Y$ be a proper morphism. The FM kernel of the functor $Rf_* \colon \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(Y)$ is the structure sheaf of the graph of f. As functor Lf^*, left adjoint to Rf_*, exists, the functorial exact triangles of endo-functors of $\mathcal{D}_{qc}(X)$ and $\mathcal{D}_{qc}(Y)$ are

$$Lf^*Rf_* \to \mathrm{Id}_{\mathcal{D}_{qc}(X)} \to \Theta_{X,f} \to Lf^*Rf_*[1],$$
$$\Omega_{Y,f} \to \mathrm{Id}_{\mathcal{D}_{qc}(Y)} \to Rf_*Lf^* \to \Omega_{Y,f}[1].$$

Assume further that f is of finite Tor dimension. Then $Rf_* \colon \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(Y)$ maps compact objects to compact objects and we have functorial exact triangles:

$$Rf_*f^! \to \mathrm{Id}_{\mathcal{D}_{qc}(Y)} \to \Theta_{Y,f} \to Rf_*f^![1],$$
$$\Omega_{X,f} \to \mathrm{Id}_{\mathcal{D}_{qc}(X)} \to f^!Rf_* \to \Omega_{X,f}[1].$$

Without the assumption on the Tor-dimension of f the pair $Rf_* \dashv f^!$ admits a lift to DG functors of the categories of ind-coherent sheaves [GR17].

In the following we discuss compatibility of liftings of push-forward functors to **Bimod** with the composition of 1-morphisms.

Let X be a quasi-compact quasi-separated scheme over a field k. Let $\mathscr{C}(X)$ denote the category of complexes of sheaves of $\mathcal{O}_X$-modules with quasi-coherent cohomology and $\mathscr{I}(X)$ its full subcategory of h-injective objects. The category $\mathscr{I}(X)$ is a DG enhancement for $\mathcal{D}_{qc}(X)$. By [Sch18, Corollary 2.3] there exists a DG functor $\iota_X \colon \mathscr{C}(X) \to \mathscr{I}(X)$ together with a natural transformation $\alpha_X \colon \mathrm{Id}_{\mathscr{C}(X)} \to \iota_X$ whose evaluation at every object of $\mathscr{C}(X)$ is a quasi-isomorphism. In fact, functor ι_X is defined in [Sch18] on the bigger category of complexes of $\mathcal{O}_X$-modules, here we denote by the same letter its restriction to $\mathscr{C}(X)$.

Let $f \colon X \to Y$ be a morphism of quasi-compact quasi-separated schemes. The composite

$$\widetilde{f}_* \colon \mathscr{I}(X) \xrightarrow{f_*} \mathscr{C}(Y) \xrightarrow{\iota_Y} \mathscr{I}(Y)$$

defines a DG functor such that the induced functor on homotopy categories is isomorphic to Rf_*. For $g \colon Y \to Z$, the natural transformation $\alpha_Y \colon \mathrm{Id}_{\mathscr{C}(Y)} \to \iota_Y$ yields a morphism

$$\beta \colon \widetilde{gf}_* = \iota_Z g_* f_* \to \iota_Z g_* \iota_Y f_* = \widetilde{g}_* \widetilde{f}_* \tag{C.10}$$

which induces an isomorphism of the induced functors of homotopy categories.

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Let $E_X \in \mathscr{I}(X)$ be an object whose image in the homotopy category is a compact generator. Denote by A_X its DG endomorphism algebra. Then

$$\mathrm{Hom}_{\mathscr{I}(X)}(E_X, -) \colon \mathscr{I}(X) \to \mathrm{DGMod}_{\mathcal{U}}\text{-}A_X$$

induces an equivalence

$$\zeta_X \colon \mathcal{D}_{\mathrm{qc}}(X) \xrightarrow{\simeq} \mathcal{D}(A_X).$$

Analogously, a lift to $\mathscr{I}(Y)$ of a compact generator for $\mathcal{D}_{\mathrm{qc}}(Y)$ to $\mathscr{I}(Y)$ yields an equivalence $\zeta_Y \colon \mathcal{D}_{\mathrm{qc}}(Y) \to \mathcal{D}(A_Y)$. Consider an $A_X^{\mathrm{op}} \otimes A_Y$-bimodule:

$$M_f := \mathrm{Hom}_{\mathscr{I}(Y)}(E_Y, \widetilde{f}_* E_X). \tag{C.11}$$

Then

$$\Phi_{M_f} \circ \zeta_X \simeq \zeta_Y \circ Rf_*. \tag{C.12}$$

PROPOSITION C.10. *Consider morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of quasi-compact separated schemes and denote by A_Z the DG endomorphism algebra of a lift E_Z of a compact generator for $\mathcal{D}_{\mathrm{qc}}(Z)$ to $\mathscr{I}(Z)$. Then there exists a canonical isomorphism in $\mathcal{D}(A_X^{\mathrm{op}} \otimes A_Z)$:*

$$M_f \otimes_{A_Y}^L M_g \to M_{gf} \tag{C.13}$$

Proof. Morphism β (C.10) applied to E_X is a quasi-isomorphism of h-injective complexes, hence it has a homotopy inverse. It follows that morphism of DG bimodules

$$\delta \colon M_{gf} = \mathrm{Hom}_{\mathscr{I}(Z)}(E_Z, \widetilde{gf}_* E_X) \xrightarrow{\beta \circ (-)} \mathrm{Hom}_{\mathscr{I}(Z)}(E_Z, \widetilde{g}_* \widetilde{f}_* E_X) =: N_{gf}$$

has a homotopy inverse, i.e. the induced morphism $\overline{\delta}$ in $\mathcal{D}(A_X^{\mathrm{op}} \otimes A_Z)$ is an isomorphism. In particular, for equivalence $\zeta_Z \colon \mathcal{D}_{\mathrm{qc}}(Z) \to \mathcal{D}(A_Z)$ induced by E_Z, formula (C.12) with f replaced by gf, ζ_Y by ζ_Z, and M_f by M_{gf} implies $\zeta_Z \circ R(gf)_* \simeq \Phi_{N_{gf}} \circ \zeta_X$.

Functor $\widetilde{g}_*$ together with the composition of morphisms in $\mathscr{I}(Z)$ defines a morphism of bimodules

$$\gamma \colon M_f \otimes_{A_Y} M_g = \mathrm{Hom}_{\mathscr{I}(Y)}(E_Y, \widetilde{f}_* E_X) \otimes_{A_Y} \mathrm{Hom}_{\mathscr{I}(Z)}(E_Z, \widetilde{g}_* E_Y)$$
$$\to \mathrm{Hom}_{\mathscr{I}(Z)}(E_Z, \widetilde{g}_* \widetilde{f}_* E_X) =: N_{gf}.$$

Let $\overline{\gamma}$ be the induced morphism in $\mathcal{D}(A_X^{\mathrm{op}} \otimes A_Z)$.

Formula (C.12) and its analogue for $g \colon Y \to Z$ yields isomorphisms $\zeta_Z \circ (Rg_* \circ Rf_*) \simeq \Phi_{M_g} \circ \zeta_Y \circ Rf_* \simeq \Phi_{M_g} \circ \Phi_{M_f} \circ \zeta_X \simeq \Phi_{M_f \otimes_{A_Y}^L M_g} \circ \zeta_X$. Therefore, $\Phi_{N_{gf}}$ and $\Phi_{M_f \otimes_{A_Y}^L M_g}$ correspond via ζ_X and ζ_Z to $R(gf)_*$ and $Rg_* \circ Rf_*$. As these functors are isomorphic, we conclude that applying Φ to $\overline{\gamma}$ yields an isomorphism of functors. As, by Lemma C.5, Φ is conservative, $\overline{\gamma}$ is an isomorphism. Then $\overline{\delta}^{-1} \circ \overline{\gamma}$ is the sought isomorphism (C.13). $\square$

Note that as functor $\Phi_{M_f} \colon \mathcal{D}(A_X) \to \mathcal{D}(A_Y)$ has left adjoint, Proposition C.6(ii) implies a 2-categorical adjunction $(M_f^{A_X}, M_f, \eta_L, \varepsilon_L)$ in the category **Bimod**.

Consider the following commutative diagram of morphisms of quasi-compact quasi-separated schemes.

$$
\begin{array}{ccc}
 & Z & \\
{}^{p}\swarrow & & \searrow^{q} \\
X & & W \\
{}_{f}\searrow & & \swarrow_{g} \\
 & Y &
\end{array}
\tag{C.14}
$$

We search for a 1-morphism in **Bimod** whose image under the functor Φ as in (C.4) is the base change $Lg^*Rf_* \to Rq_*Lp^*$.

Choose $E_X \in \mathscr{I}(X)$, $E_Y \in \mathscr{I}(Y)$, $E_Z \in \mathscr{I}(Z)$ and $E_W \in \mathscr{I}(W)$ whose images in the homotopy categories are compact generators. Denote by A_X, A_Y, A_Z and A_W their DG endomorphism algebras. Let M_f, M_q, M_g and M_p be DG bimodules as in (C.11).

In view of Proposition C.10 equality $f \circ p = g \circ q$ implies an isomorphism $\nu\colon M_p \otimes_{A_X}^L M_f \to M_q \otimes_{A_W}^L M_g$. Consider the composite

$$
\begin{aligned}
\omega\colon M_f \otimes_{A_Y}^L M_g^{A_W} &\to M_p^{A_Z} \otimes_{A_Z}^L M_p \otimes_{A_X}^L M_f \otimes_{A_Y}^L M_g^{A_W} \\
&\to M_p^{A_Z} \otimes_{A_Z}^L M_q \otimes_{A_W}^L M_g \otimes_{A_Y}^L M_g^{A_W} \to M_p^{A_Z} \otimes_{A_Z}^L M_q
\end{aligned}
\tag{C.15}
$$

where the first map is induced by the lift of the $Lp^* \dashv Rp_*$ adjunction unit to the unit $A_X \to M_p^{A_Z} \otimes_{A_Z}^L M_p$ of the 2-categorical (A_X, A_Z)-adjunction, the second by ν and the third by the lift of the $Lg^* \dashv Rg_*$ adjunction counit to the counit $M_g \otimes_{A_Y}^L M_g^{A_W} \to A_W$ of the (A_Y, A_W)-adjunction. Then ω is a lift of the 'base change' $Lg^*Rf_* \to Rq_*Lp^*$ to a 1-morphism in **Bimod**.

The base-change morphism ω (C.15) is the composite of appropriate adjunction units and counits. Hence, Proposition C.9 implies that ω yields a unique up to isomorphism morphism $Lg^*Rf_* \to Rq_*Lp^*$ of FM functors.

C.6 Isomorphisms of functors via the restriction to one object

Throughout the paper we repeatedly use the fact that, for *algebras* A, B and functors $F, G\colon \mathcal{D}(A) \to \mathcal{D}(B)$ defined by pure modules, an isomorphism of F and G on generators yields an isomorphism $F \xrightarrow{\simeq} G$.

LEMMA C.11. *Let A, B be algebras and let M be a complex of $A - B$ bimodules such that $H^i(M) = 0$, for $i \neq j$, for a fixed $j \in \mathbb{Z}$. Let also $\mathcal{E} \subset \mathcal{D}(A)$ be the full subcategory with one object A. Then the quasi-isomorphism class of M is determined by the isomorphism class of the restriction functor $\Phi_M|_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{D}(B)$.*

Proof. By definition, $M \simeq \Phi_M(A)$ as a right B module. Functor $\Phi_M|_{\mathcal{E}}$ gives a map $A \simeq \mathrm{Hom}_A(A, A) \xrightarrow{\beta} \mathrm{Hom}_B(M, M)$. By assumption on M, we have a quasi-isomorphism $M \simeq H^j(M)$, hence $\mathrm{Hom}_B(M, M) \simeq \mathrm{Hom}_B(H^j(M), H^j(M))$. Then morphism β recovers the left A module structure of $H^j(M)$, hence the quasi-isomorphism class of M as an $A - B$ bimodule. $\square$

As an easy corollary, we obtain the following lemma.

LEMMA C.12. *Let A, B be algebras and let M_1, M_2 be complexes of $A - B$ bimodules such that $H^i(M_1) = 0$, for $i \neq j$, for a fixed $j \in \mathbb{Z}$. Let $\mathcal{E} \subset \mathcal{D}(A)$ denote the full subcategory with one object A. Then functors Φ_{M_1} and Φ_{M_2} are isomorphic if and only if $\Phi_{M_1}|_{\mathcal{E}}$ and $\Phi_{M_2}|_{\mathcal{E}}$ are.*

Proof. As $\Phi_{M_1}(A) \simeq M_1$ and $\Phi_{M_2}(A) \simeq M_2$, isomorphism of functors $\Phi_{M_1}|_{\mathcal{E}} \simeq \Phi_{M_2}|_{\mathcal{E}}$ implies a quasi-isomorphism of M_1 and M_2 as complexes of right B modules. Therefore, $H^i(M_2) = 0$, for $i \neq j$. We conclude by Lemma C.11. $\qquad\square$

Appendix D. The structure of the reduced fiber

Let $f: X \to Y$ be a proper morphism with fibers of dimension bounded by one such that $Rf_*(\mathcal{O}_X) = \mathcal{O}_Y$. Let C denote the reduced fiber of f over a closed point of Y.

Applying functor Rf_* to the short exact sequence

$$0 \to I_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

of sheaves on X, we obtain a surjective morphism $R^1 f_* \mathcal{O}_X \to R^1 f_* \mathcal{O}_C$, which implies

$$H^1(\mathcal{O}_C) = 0.$$

Moreover, by [Har77, Corollary II.11.3], C is connected, i.e. $H^0(\mathcal{O}_C) = k$.

Let $C = \bigcup C_i$ be a reduced curve such that all of its irreducible components C_i are smooth. An *incidence graph* is a bipartite graph whose vertices correspond to irreducible components and singular points of C. An edge connects a vertex corresponding to an irreducible component C_i with a vertex corresponding to a singular point $c \in \mathrm{Sing}(C)$ if and only if $c \in C_i$. Note that the incidence graph can be simply connected even though the dual intersection graph might be not, as the example of three curves meeting in one point shows: the incidence graph is a tree, a point connected with three other points, whereas the dual graph is a triangle.

We say that a reduced curve has *normal crossing* singularities if all components of the curve are smooth and the Zariski tangent space at every singular point is the direct sum of tangent subspaces corresponding to components that meet at this point.

THEOREM D.1. *Let C be a reduced proper algebraic curve over field k. Then $H^1(\mathcal{O}_C) = 0$ if and only if the following conditions are satisfied:*

 (i) *every irreducible component C_i of C is a smooth rational curve;*
 (ii) *the incidence graph of C has no cycles;*
 (iii) *the curve has normal crossing singularities.*

Proof. Assume that $H^1(\mathcal{O}_C) = 0$. Let C_i be an irreducible component of C. The restriction morphism $\mathcal{O}_C \to \mathcal{O}_{C_i}$ gives a surjection on cohomology $H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_{C_i})$, hence $H^1(\mathcal{O}_{C_i}) = 0$. Let

$$\pi_i: \widetilde{C_i} \to C_i$$

denote the normalisation of C_i. Consider a short exact sequence of sheaves on C_i:

$$0 \to \mathcal{O}_{C_i} \to \pi_{i*}\mathcal{O}_{\widetilde{C_i}} \to \mathcal{F} \to 0.$$

As $\mathcal{F}$ is supported at singular points of C_i, the group $H^1(\mathcal{F})$ vanishes, hence $H^1(\mathcal{O}_{\widetilde{C_i}}) = 0$. As $\widetilde{C_i}$ is smooth, it is isomorphic to $\mathbb{P}^1_k$.

Sequence

$$0 \to H^0(\mathcal{O}_{C_i}) \to H^0(\mathcal{O}_{\widetilde{C_i}}) \to H^0(\mathcal{F}) \to 0$$

is exact and the first morphism is an isomorphism. It follows that $H^0(\mathcal{F}) = 0$, i.e. sheaf $\mathcal{F}$ is trivial. This proves that C_i is isomorphic to its normalisation $\widetilde{C_i}$, which is a smooth rational curve. This proves condition (i).

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Assume that the incidence graph of C has a cycle. Let $C' \subset C$ be the corresponding (minimal) cycle of smooth rational curves, a subscheme in C with irreducible components $C_1, \ldots, C_l$. Again, we know that $H^1(\mathcal{O}_{C'}) = 0$. Then we have a short exact sequence

$$0 \to \mathcal{O}_{C'} \to \bigoplus_{i=1}^{l} \mathcal{O}_{C_i} \to \bigoplus_{i=1}^{l} \mathcal{O}_{c_i} \to 0,$$

where $c_1, \ldots, c_l$ are singular points of C'. From long exact sequence of cohomology groups it follows that $H^1(\mathcal{O}_{C'}) = k$. This contradiction proves condition (ii).

Note that, for curves C satisfying conditions (i) and (ii), there exists a universal curve $\bar{C}$ and morphism

$$\bar{\pi} : \bar{C} \to C,$$

with the property that $\bar{C}$ has normal crossing singularities and, for every curve D with normal crossing singularities and a map $D \to C$, there is a unique lifting map $D \to \bar{C}$.

The construction of the curve $\bar{C}$ is simple. Consider an affine neighbourhood $U \subset C$ of every point of the curve which does not contain more than one singular point. Assume there is one. The irreducible components that meet at the singular point give some affine subschemes in U. As the category of affine schemes is opposite to the category of unital commutative algebras, we can define the open chart $\bar{U}$ of the universal curve over U as a colimit over the diagram of embedding of the singular point into all these affine subschemes (by taking the spectrum of the limit over the diagram of the corresponding commutative algebras). It has one normal crossing singularity.

If U has no singular points, then we put $\bar{U} = U$. We can glue $\bar{U}$ over C into a curve, $\bar{C}$, with normal crossing singularities. Conditions (i) and (ii) guarantee that $\bar{C}$ has the required universal property. As we do not need this for our purpose, we skip the proof.

Clearly, $\bar{\pi}$ is a set-theoretic isomorphism. We again have a short exact sequence with sheaf $\mathcal{F}$ supported at singular points of C:

$$0 \to \mathcal{O}_C \to \bar{\pi}_* \mathcal{O}_{\bar{C}} \to \mathcal{F} \to 0,$$

whose long exact sequence of cohomology shows that $\mathcal{F}$ is trivial sheaf. This implies that the schematic structure of C and $\bar{C}$ coincide, which proves condition (iii).

Conversely, assume that C is a reduced curve satisfying conditions (i), (ii) and (iii). We can assume that the curve is connected. We proceed by induction on the number of irreducible components of C. The base of induction, the case of a single component, is obvious. As the incidence graph is a tree, then we can choose an irreducible component C_1 which has only one singular point, c. The curve C', the union of the other components, satisfies the same assumptions as C. Hence, $H^1(\mathcal{O}_{C'}) = 0$ by induction hypothesis. As the point c is a normal crossing singularity, then the kernel of the restriction morphism $\mathcal{O}_C \to \mathcal{O}_{C'}$ is easily seen to be $\mathcal{O}_{C_1}(-1)$. Then looking at the cohomology sequence for the short exact sequence

$$0 \to \mathcal{O}_{C_1}(-1) \to \mathcal{O}_C \to \mathcal{O}_{C'} \to 0,$$

we see that $H^1(\mathcal{O}_C) = 0$. $\qquad \square$

Appendix E. Calculation of Ext groups for bounded above complexes

Let $\mathcal{A}$ be an abelian category and $A^{\textbf{\textperiodcentered}}$ a bounded above complex over $\mathcal{A}$. Its 'stupid' truncations $\sigma_{\geqslant i} A$ define direct system $\sigma_{\geqslant i} A^{\textbf{\textperiodcentered}} \to \sigma_{\geqslant i-1} A^{\textbf{\textperiodcentered}}$. Given a complex $B^{\textbf{\textperiodcentered}}$, we are interested in the group

$\mathrm{Hom}^{\textbf{.}}(A^{\textbf{.}}, B^{\textbf{.}})$ in terms of the stupid truncations of both complexes. There is a spectral sequence with E_1-layer:

$$\mathrm{E}_1^{pq} = \bigoplus_{j-i=p} \mathrm{Ext}^q(A^i, B^j). \tag{E.1}$$

If both complexes $A^{\textbf{.}}$, $B^{\textbf{.}}$ have finite number of non-trivial components, then it is well-known that spectral sequence (E.1) converges. If complex $B^{\textbf{.}}$ is infinite then we need to put some conditions on $A^{\textbf{.}}$ in order to guarantee the spectral sequence is converging to $\mathrm{Hom}^{\textbf{.}}(A^{\textbf{.}}, B^{\textbf{.}})$. We show that if, in addition, both complexes are bounded above and $\mathrm{Ext}^q(A^i, B^j) = 0$, for $q > 0$ and $i, j \in \mathbb{Z}$, then we have a graded complex with terms

$$\mathrm{C}^p = \prod_{j-i=p} \mathrm{Hom}(A^i, B^j) \tag{E.2}$$

whose cohomology calculate $\mathrm{Ext}^p(A^{\textbf{.}}, B^{\textbf{.}})$. Here the differentials are the limits of the relevant differentials in the sequence (E.1) when we allow the truncations of $A^{\textbf{.}}$ and $B^{\textbf{.}}$ to go to infinity.

Recall the following fact about cohomology of limits. Let $(M_i)_{i \in \mathbb{Z}}$ be an inverse system of abelian groups and $M = \varprojlim M_i$. We say that system $(M_i)_{i \in \mathbb{Z}}$ satisfies $(\star)$ if morphism $M_i \to M_{i-1}$ is surjective, for any i.

LEMMA E.1 [Spa88, Lemma 0.11]. *Let* $(A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \xrightarrow{h_i} D_i)_{i \in \mathbb{Z}}$ *be an inverse system of complexes of abelian groups such that systems* (A_i), (B_i), (C_i), (D_i) *satisfy* $(\star)$ *for all* i. *Denote by* $(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D)$ *the limit complex. Let* A_i', B_i', C_i' *and* D_i' *be kernels of* $A_i \to A_{i-1}$, $B_i \to B_{i-1}$, $C_i \to C_{i-1}$ *and* $D_i \to D_{i-1}$, *respectively. Assume there is* $j \in \mathbb{Z}$ *such that sequence* $A_i' \to B_i' \to C_i' \to D_i'$ *is exact for all* $i > j$. *Then the natural homomorphism* $\mathrm{Ker}g/\mathrm{Im}f \to \mathrm{Ker}g_j/\mathrm{Im}f_j$ *is an isomorphism.*

LEMMA E.2. *Let* $A^{\textbf{.}}$ *and* $B^{\textbf{.}}$ *be complexes over* $\mathcal{A}$, $A^{\textbf{.}}$ *bounded above. Assume* $l \in \mathbb{Z}$ *such that* $\mathrm{Ext}^j_{\mathcal{D}^b(\mathcal{A})}(A^i, B^{\textbf{.}}) = 0$, *for* $j < l$ *and all* i. *Then the canonical morphism*

$$\mathrm{Ext}^j(A^{\textbf{.}}, B^{\textbf{.}}) \to \varprojlim \mathrm{Ext}^j(\sigma_{\geqslant i} A^{\textbf{.}}, B^{\textbf{.}})$$

is an isomorphism for any $j \in \mathbb{Z}$. *Here* $\sigma_{\geqslant i} A^{\textbf{.}}$ *is the 'stupid' truncation of* $A^{\textbf{.}}$.

Proof. Fix $I^{\textbf{.}}$, an h-injective resolution of $B^{\textbf{.}}$ (see [Spa88]). The direct system $\sigma_{\geqslant i+1} A^{\textbf{.}} \to \sigma_{\geqslant i} A^{\textbf{.}}$ induces an inverse system $\alpha_i \colon \mathrm{Hom}_{\mathrm{Com}}(\sigma_{\geqslant i} A^{\textbf{.}}, I^{\textbf{.}}) \to \mathrm{Hom}_{\mathrm{Com}}(\sigma_{\geqslant i+1} A^{\textbf{.}}, I^{\textbf{.}})$. Morphism α_i is surjective, i.e. property $(\star)$ is satisfied, and its kernel is complex $\mathrm{Hom}_{\mathrm{Com}}(A^i[-i], I^{\textbf{.}})$. Cohomology of this complex is $H^j(\mathrm{Hom}_{\mathrm{Com}}(A^i[-i], I^{\textbf{.}})) \simeq \mathrm{Ext}^{j+i}_{\mathcal{D}^b(\mathcal{A})}(A^i, B^{\textbf{.}})$.

Our assumption implies that, for any j, there exists $N_j \in \mathbb{Z}$ such that, for $i < N_j$, complex $\mathrm{Hom}_{\mathrm{Com}}(A^i[-i], I^{\textbf{.}})$ is exact at degree j. By Lemma E.1, we conclude that

$$H^j(\varprojlim \mathrm{Hom}_{\mathrm{Com}}(\sigma_{\geqslant i} A^{\textbf{.}}, I^{\textbf{.}})) \simeq H^j(\mathrm{Hom}_{\mathrm{Com}}(\sigma_{\geqslant i} A^{\textbf{.}}, I^{\textbf{.}})) \simeq \mathrm{Ext}^j_{\mathcal{D}^b(\mathcal{A})}(\sigma_{\geqslant i} A^{\textbf{.}}, B^{\textbf{.}}). \tag{E.3}$$

In particular, the inverse system $\mathrm{Ext}^j_{\mathcal{D}^b(\mathcal{A})}(\sigma_{\geqslant i} A^{\textbf{.}}, B^{\textbf{.}}) \to \mathrm{Ext}^j_{\mathcal{D}^b(\mathcal{A})}(\sigma_{\geqslant i+1} A^{\textbf{.}}, I^{\textbf{.}})$ stabilises, hence the right-hand side of (E.3) is isomorphic to $\varprojlim \mathrm{Ext}^j_{\mathcal{D}^b(\mathcal{A})}(\sigma_{\geqslant i} A^{\textbf{.}}, B^{\textbf{.}})$, for i sufficiently negative. Moreover, we have $\varprojlim \mathrm{Hom}_{\mathrm{Com}}(\sigma_{\geqslant i} A^{\textbf{.}}, I^{\textbf{.}}) \simeq \mathrm{Hom}_{\mathrm{Com}}(A^{\textbf{.}}, I^{\textbf{.}})$. Hence, isomorphism (E.3) yields

$$\mathrm{Ext}^j_{\mathcal{D}^b(\mathcal{A})}(A^{\textbf{.}}, B^{\textbf{.}}) \simeq H^j(\mathrm{Hom}_{\mathrm{Com}}(A^{\textbf{.}}, I^{\textbf{.}})) \simeq \varprojlim \mathrm{Ext}^j_{\mathcal{D}^b(\mathcal{A})}(\sigma_{\geqslant -i} A^{\textbf{.}}, B^{\textbf{.}}). \qquad \square$$

For the reader's convenience, we also give a proof of the following well-known fact.

LEMMA E.3. *Let* $A^{\textbf{.}}$ *and* $B^{\textbf{.}}$ *be complexes over* $\mathcal{A}$ *such that* $A^{\textbf{.}}$ *is finite and* $B^{\textbf{.}}$ *is bounded above and with bounded cohomology. Assume there exists* $N \in \mathbb{Z}$ *such that* $\mathrm{Ext}^j_{\mathcal{A}}(A^i, A) = 0$, *for all*

$j > N$, $i \in \mathbb{Z}$ and $A \in \mathcal{A}$. Then, there exists a spectral sequence

$$E_1^{p,q} = \bigoplus_k \mathrm{Ext}^p(A^k, B^{k+q}) \Rightarrow \mathrm{Ext}_{\mathcal{D}^b(\mathcal{A})}^{\cdot}(A^{\cdot}, B^{\cdot}). \tag{E.4}$$

Proof. For any $n \in \mathbb{Z}$, injective resolutions $I^{i,\cdot}$ of B^i give an injective resolution $J_n^{\cdot}$ of $\sigma_{\geqslant n} B^{\cdot}$ such that complex $J_n^{\cdot}$ admits a filtration with graded factors $I^{i,\cdot}$. The filtered complex $J_n^{\cdot}$ yields a spectral sequence

$$^nE_1^{p,q} = \bigoplus_{\{k \,|\, k+q \geqslant n\}} \mathrm{Ext}^p(A^k, B^{k+q}) \Rightarrow \mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}^{\cdot}(A^{\cdot}, \sigma_{\geqslant n} B^{\cdot}). \tag{E.5}$$

It follows from our assumptions that $^nE_1^{p,q}$ is stationary for fixed p, q and $n \to -\infty$. Moreover, the term $^nE_1^{p,q}$ is zero, for any $p \notin [0, N]$ or for any sufficiently positive q. This confirms existence of the spectral sequence (E.4) and also implies that the lth layer $^nE_l^{p,q}$ is stationary for fixed p, q and $n \to -\infty$ and $l \to \infty$.

As $B^{\cdot}$ has bounded cohomology, for n sufficiently negative, we have $\tau_{\geqslant n+1} \sigma_{\geqslant n} B^{\cdot} \simeq B^{\cdot}$ and $\tau_{\leqslant n} \sigma_{\geqslant n} B^{\cdot} \simeq \mathcal{H}^n(\sigma_{\geqslant n} B^{\cdot})[-n]$. Then triangle $\tau_{\leqslant n} \sigma_{\geqslant n} B^{\cdot} \to \sigma_{\geqslant n} B^{\cdot} \to \tau_{\geqslant n+1} \sigma_{\geqslant n} B^{\cdot}$ reads

$$\mathcal{H}^n(\sigma_{\geqslant n} B^{\cdot})[-n] \to \sigma_{\geqslant n} B^{\cdot} \to B^{\cdot} \to \mathcal{H}^n(\sigma_{\geqslant n} B^{\cdot})[-n+1].$$

As $A^{\cdot}$ is finite, the assumptions on A^i imply that there exists $\widetilde{N}$ such that $\mathrm{Hom}(A^{\cdot}, C[k]) = 0$, for $k > \widetilde{N}$ and all $C \in \mathcal{A}$. Thus, $\mathrm{Ext}^j(A^{\cdot}, \mathcal{H}^n(\sigma_{\geqslant n} B^{\cdot})[-n]) \simeq \mathrm{Hom}(A^{\cdot}, \mathcal{H}^n(\sigma_{\geqslant n} B^{\cdot})[j-n]) = 0$, for $n < j - \widetilde{N}$. Then, for $n < j - \widetilde{N}$, we have from the above triangle an isomorphism $\mathrm{Ext}_{\mathcal{D}^b(\mathcal{A})}^{j-1}(A^{\cdot}, \sigma_{\geqslant n} B^{\cdot}) \simeq \mathrm{Ext}_{\mathcal{D}^b(\mathcal{A})}^{j-1}(A^{\cdot}, B^{\cdot})$.

This shows that the spectral sequence in (E.4) converges to $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}^{\cdot}(A^{\cdot}, B^{\cdot})$. □

PROPOSITION E.4. *Let $A^{\cdot}$, $B^{\cdot}$ be bounded above complexes of objects in $\mathcal{A}$ with bounded cohomology. Assume that there exists $N \in \mathbb{Z}$ such that $\mathrm{Ext}^j(A^i, A) = 0$, for all $j > N$, $i \in \mathbb{Z}$ and $A \in \mathcal{A}$. Assume further that $\mathrm{Ext}^j(A^i, B^k) = 0$, for $j > 0$ and all $i, k \in \mathbb{Z}$. Then, $\mathrm{Ext}_{\mathcal{D}^b(\mathcal{A})}^j(A^{\cdot}, B^{\cdot})$ is isomorphic to the jth cohomology group of complex (E.2).*

Proof. Lemma E.3 applied to $\sigma_{\geqslant i} A^{\cdot}$ and $B^{\cdot}$, for any $i \in \mathbb{Z}$, implies that the spectral sequence (E.4) reduces to a single row. Hence, we have $\mathrm{Ext}^j(\sigma_{\geqslant i} A^{\cdot}, B^{\cdot}) \simeq H^j(\oplus_{k>i} \mathrm{Hom}(A^k, B^{\cdot+k}))$. The direct system $\sigma_{\geqslant i+1} A^{\cdot} \to \sigma_{\geqslant i} A^{\cdot}$ induces an inverse system of complexes $\bigoplus_{k>i} \mathrm{Hom}(A^k, B^{\cdot+k}) \to \bigoplus_{k>i+1} \mathrm{Hom}(A^k, B^{\cdot+k})$ satisfying condition $(\star)$. The kernel of $\bigoplus_{k>i} \mathrm{Hom}(A^k, B^{\cdot+k}) \to \bigoplus_{k>i+1} \mathrm{Hom}(A^k, B^{\cdot+k})$ is a complex $\mathrm{Hom}(A^i, B^{\cdot})[i]$. By Lemma E.3, we have $H^j(\mathrm{Hom}(A^i, B^{\cdot})[i]) \simeq \mathrm{Ext}^{j+i}(A^i, B^{\cdot})$, thus, for a fixed j and any sufficiently negative i, complex $\mathrm{Hom}(A^i, B^{\cdot})[i]$ is exact at degree j. Lemma E.1 implies that, for such i, we have

$$H^j\left(\prod_k \mathrm{Hom}(A^k, B^{\cdot+k})\right) \simeq H^j\left(\bigoplus_{k>i} \mathrm{Hom}(A^k, B^{\cdot+k})\right) \simeq \mathrm{Ext}^j(\sigma_{\geqslant i} A^{\cdot}, B^{\cdot}).$$

Thus, $\mathrm{Ext}^j(\sigma_{\geqslant i} A^{\cdot}, B^{\cdot})$ is stationary, and $H^j(\prod_k \mathrm{Hom}(A^k, B^{\cdot+k})) \simeq \varprojlim \mathrm{Ext}^j(\sigma_{\geqslant i} A^{\cdot}, B^{\cdot})$. As $B^{\cdot}$ has bounded cohomology, the assumptions of Lemma E.2 are satisfied. It follows that the latter space is isomorphic to $\mathrm{Ext}^j(A^{\cdot}, B^{\cdot})$. □

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