THE APPROXIMATION OF CONTINUOUS FUNCTIONS BY RIESZ TYPICAL MEANS OF THEIR FOURIER SERIES

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(Received 26 July 1966)

1. Introduction

Let $f(x)$ be a continuous function with period 2π. It is well known that the Fourier series of $f(x)$ is summable Riesz of any positive order to $f(x)$. The aim of this paper is the proof of the following theorem.

THEOREM A. *If $f(x)$ is a continuous function with period 2π, k is a positive integer,*

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) = \sum_{\nu=0}^{\infty} s_\nu(x)$$

and

$$R_\lambda^k(x) = \sum_{\nu \leq \lambda} \left(1 - \frac{\nu}{\lambda}\right)^k s_\nu(x),$$

then

$$R_\lambda^k(x) - f(x) = \frac{k}{\pi} \int_a^\infty \frac{\phi_x(t/\lambda)}{t^2}\, dt + O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right),$$

where

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x),$$

$$\omega_2(h, f) = \sup_{|\delta| \leq h} ||\phi_x(\delta)|| = \sup_{|\delta| \leq h} \max_x |\phi_x(\delta)|.$$

2. Lemmas

In the proof of the following lemmas, we assume that $n = [\lambda]$.

LEMMA 1.

$$\int_1^\infty \phi_x\left(\frac{t}{\lambda}\right) \frac{\cos t}{t^2}\, dt = O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right).$$

https://doi.org/10.1017/S144678870000447X Published online by Cambridge University Press

PROOF.

$$\int_1^\infty \phi_x\left(\frac{t}{\lambda}\right)\frac{\cos t}{t^2}\,dt = \int_1^{n\pi}\phi_x\left(\frac{t}{\lambda}\right)\frac{\cos t}{t^2}\,dt + \sum_{\nu=1}^{\infty}\int_{\nu n\pi}^{(\nu+1)n\pi}\phi_x\left(\frac{t}{\lambda}\right)\frac{\cos t}{t^2}\,dt$$

$$= \frac{1}{n}\int_{1/n}^{\pi}\phi_x\left(\frac{nt}{\lambda}\right)\frac{\cos nt}{t^2}\,dt + \frac{1}{n}\int_0^\pi \cos nt \sum_{\nu=1}^{\infty}\frac{(-1)^{n\nu}\phi_x\left(\frac{\nu n\pi+nt}{\lambda}\right)}{(\nu\pi+t)^2}\,dt$$

$$= A_n + B_n.$$

Suppose that $P_n(x)$ is the best approximation trigonometric polynomial of order n to $f(x)$. Then [2]

$$E_n(f) = ||f(x)-P_n(x)|| = O\left(\omega_2\left(\frac{1}{n}, f\right)\right)$$

and

$$(2.1) \qquad ||P_n''(x)|| = O\left(n^2\omega_2\left(\frac{1}{n}, f\right)\right).$$

Hence

$$(2.2) \qquad \phi_x(t) = \psi_x(t) + O\left(\omega_2\left(\frac{1}{n}, f\right)\right),$$

where

$$\psi_x(t) = P_n(x+t)+P_n(x-t)-2P_n(x).$$

Now

$$A_n = \frac{1}{n}\int_{1/n}^{\pi}\phi_x\left(\frac{nt}{\lambda}\right)\frac{\cos nt}{t^2}\,dt$$

$$= \frac{1}{n}\int_{1/n}^{\pi}\psi_x\left(\frac{nt}{\lambda}\right)\frac{\cos nt}{t^2}\,dt + O\left(\omega_2\left(\frac{1}{n}, f\right)\right)$$

$$= \frac{1}{n}\left\{\frac{\psi_x\left(\frac{nt}{\lambda}\right)\sin nt}{nt^2}\right\}_{1/n}^{\pi} - \frac{1}{n^2}\int_{1/n}^{\pi}\frac{\frac{n}{\lambda}\psi_x'\left(\frac{nt}{\lambda}\right)t - 2\psi_x\left(\frac{nt}{\lambda}\right)}{t^3}\sin nt\,dt$$

$$+ O\left(\omega_2\left(\frac{1}{n}, f\right)\right).$$

It follows from (2.1) that

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$$\psi_x\left(\frac{nt}{\lambda}\right) = P_n\left(x+\frac{nt}{\lambda}\right)+P_n\left(x-\frac{nt}{\lambda}\right)-2P_n(x)$$
$$= \frac{n^2t^2}{\lambda^2}P_n''(x+\theta t) = O\left(n^2t^2\omega_2\left(\frac{1}{n}, f\right)\right),$$
$$\psi_x'\left(\frac{nt}{\lambda}\right) = P_n'\left(x+\frac{nt}{\lambda}\right)-P_n'\left(x-\frac{nt}{\lambda}\right) = \frac{2nt}{\lambda}P_n''(x+\theta' t) = O\left(n^2t\omega_2\left(\frac{1}{n}, f\right)\right),$$
$$\psi_n''\left(\frac{nt}{\lambda}\right) = P_n''\left(x+\frac{nt}{\lambda}\right)+P_n''\left(x-\frac{nt}{\lambda}\right) = O\left(n^2\omega_2\left(\frac{1}{n}, f\right)\right).$$

Hence

$$A_n = -\frac{1}{n^2}\int_{1/n}^{\pi}\frac{\frac{n}{\lambda}\psi_n'\left(\frac{nt}{\lambda}\right)t-2\psi_x\left(\frac{nt}{\lambda}\right)}{t^3}\sin nt\,dt+O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right)$$
$$= \left\{\frac{\frac{n}{\lambda}\psi_x'\left(\frac{nt}{\lambda}\right)t-2\psi_x\left(\frac{nt}{\lambda}\right)}{n^3t^3}\cos nt\right\}_{1/n}^{\pi}$$
$$-\frac{1}{n^3}\int_{1/n}^{\pi}\left\{\frac{\frac{n^2t^2}{\lambda^2}\psi_x''\left(\frac{nt}{\lambda}\right)-\frac{nt}{\lambda}\psi_x'\left(\frac{nt}{\lambda}\right)-6\psi_x\left(\frac{nt}{\lambda}\right)}{t^4}\right\}\cos nt\,dt$$
$$+O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right)$$
$$= O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right).$$

By (2.2)

$$B_n = \frac{1}{n}\int_0^{\pi}\cos nt\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu n}\psi_x\left(\frac{\nu n\pi+nt}{\lambda}\right)}{(\nu\pi+t)^2}dt+O\left(\omega_2\left(\frac{1}{n}, f\right)\right)$$
$$= \frac{1}{n}\left\{\frac{\sin nt}{n}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu n}\psi_x\left(\frac{\nu n\pi+nt}{\lambda}\right)}{(\nu\pi+t)^2}\right\}_0^{\pi}$$
$$-\frac{1}{n^2}\int_0^{\pi}\sin nt\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu n}\frac{n}{\lambda}\psi_x'\left(\frac{\nu n\pi+nt}{\lambda}\right)(\nu\pi+t)-2(-1)^{\nu n}\psi_x\left(\frac{\nu n\pi+nt}{\lambda}\right)}{(\nu\pi+t)^3}dt$$
$$+O\left(\omega_2\left(\frac{1}{n}, f\right)\right).$$

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Since ψ'_x and ψ_x are periodic,

$$\psi'_x\left(\frac{\nu n\pi+nt}{\lambda}\right)=O\left(n^2\omega_2\left(\frac{1}{n},f\right)\right),$$
$$\psi_x\left(\frac{\nu n\pi+nt}{\lambda}\right)=O\left(n^2\omega_2\left(\frac{1}{n},f\right)\right).$$

Hence

$$B_n=O\left(\omega_2\left(\frac{1}{n},f\right)\right)=O\left(\omega_2\left(\frac{1}{\lambda},f\right)\right).$$

Thus lemma 1 is proved.

LEMMA 2. *For* $r\geqq 2$,

$$\int_1^\infty\phi_x\left(\frac{t}{\lambda}\right)\frac{1-\cos t}{t^{2r}}\,dt=O\left(\omega_2\left(\frac{1}{\lambda},f\right)\right).$$

PROOF. Since, for $1/n\leqq t\leqq\pi$,

$$\omega_2(t,f)=O\left(n^2t^2\omega_2\left(\frac{1}{n},f\right)\right),$$

$$\int_1^\infty\phi_x\left(\frac{t}{\lambda}\right)\frac{1-\cos t}{t^{2r}}\,dt=\frac{1}{n^{2r-1}}\int_{1/n}^{\pi}\phi_x\left(\frac{nt}{\lambda}\right)\frac{1-\cos nt}{t^{2r}}\,dt$$
$$+\frac{1}{n^{2r-1}}\int_0^\pi\sum_{\nu=1}^\infty\frac{[1-(-1)^{\nu n}\cos nt]\,\phi_x\left(\frac{\nu n\pi+nt}{\lambda}\right)}{(\nu\pi+t)^2}\,dt$$
$$=O\left(\frac{1}{n^{2r-1}}\int_{1/n}^{\pi}\frac{\left|\phi_x\left(\frac{nt}{\lambda}\right)\right|}{t^{2r}}\,dt\right)+O\left(\omega_2\left(\frac{1}{n},f\right)\right)$$
$$=O\left(\frac{1}{n^{2r-1}}\int_{1/n}^{\pi}\frac{\omega_2(t,f)}{t^{2r}}\,dt\right)+O\left(\omega_2\left(\frac{1}{n},f\right)\right)$$
$$=O\left(\frac{\omega_2\left(\frac{1}{n},f\right)}{n^{2r-3}}\int_{1/n}^{\pi}\frac{dt}{t^{2r-2}}\right)+O\left(\omega_2\left(\frac{1}{n},f\right)\right)$$
$$=O\left(\omega_2\left(\frac{1}{n},f\right)\right)=O\left(\omega_2\left(\frac{1}{\lambda},f\right)\right).$$

Thus Lemma 2 is proved.

By the same argument we can prove

LEMMA 3. *For* $r \geqq 2$,

$$\int_1^\infty \phi_x\left(\frac{t}{\lambda}\right)\frac{\sin t}{t^{2r-1}}\,dt = O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right),$$

$$\int_1^\infty \phi_x\left(\frac{t}{\lambda}\right)\frac{\cos t}{t^{2r}}\,dt = O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right).$$

3. Proof of theorem A

We have [1]

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\,\frac{\sin \lambda t}{t}\,dt = \sum_{\nu\leqq\lambda}{}' s_\nu(x),$$

where λ is positive but not necessarily an integer, the integral is defined as $\lim_{T\to\infty}\int_{-T}^{T}$, and the dash indicates that if λ is an integer then the last term of the sum is taken with a factor $\frac{1}{2}$.

Integrating with respect to λ we verify by induction that

$$\sum_{\nu\leqq\lambda}\left(1-\frac{\nu}{\lambda}\right)^k s_\nu(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f\left(x+\frac{t}{\lambda}\right) g(t)dt,$$

where

$$g(t) = \sum_{r=1}^{\left[\frac{k+1}{2}\right]}(-1)^r\binom{k}{2r-1}\frac{d^{(2r-1)}}{dt^{(2r-1)}}\left(\frac{1-\cos t}{t}\right) + \sum_{r=0}^{\left[\frac{k}{2}\right]}(-1)^r\binom{k}{2r}\frac{d^{(2r)}}{dt^{(2r)}}\left(\frac{\sin t}{t}\right),$$

and so

$$R_\lambda^k(x)-f(x) = \frac{1}{\pi}\int_0^\infty \phi_x\left(\frac{t}{\lambda}\right) g(t)dt.$$

By expanding $g(t)$ into a power series of t, we know that it is uniformly bounded when n is fixed and $0 \leqq t \leqq 1/n$. Hence

$$R_\lambda^k(x)-f(x) = \frac{1}{\pi}\int_1^\infty \phi_x\left(\frac{t}{\lambda}\right) g(t)dt + O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right).$$

By Leibniz' theorem,

$$(-1)^r\frac{d^{(2r-1)}}{dt^{(2r-1)}}\left(\frac{1-\cos t}{t}\right) = -\frac{\sin t}{t} - (2r-1)\frac{\cos t}{t^2} + \frac{(2r-1)(2r-2)}{2!}\frac{\sin t}{t^3}$$
$$-\cdots+(-1)^{r-1}(2r-1)!\,\frac{1-\cos t}{t^{2r}}$$

$$(-1)^r\frac{d^{(2r)}}{dt^{(2r)}}\left(\frac{\sin t}{t}\right) = \frac{\sin t}{t} + 2r\frac{\cos t}{t^2} - \frac{2r(2r-1)}{2!}\frac{\sin t}{t^3}$$
$$+\cdots+(-1)^r(2r)!\,\frac{\sin t}{t^{2r+1}},$$

and hence

$$g(t) = \sum_{r=1}^{\left[\frac{k+1}{2}\right]} (-1)^{r-1} \binom{k}{2r-1} \frac{1-\cos t}{t^{2r}} + \sum_{r=1}^{\left[\frac{k}{2}\right]} p_{2r}(k) \frac{\cos t}{t^{2r}} + \sum_{r=1}^{\left[\frac{k}{2}\right]} p_{2r-1} \frac{\sin t}{t^{2r+1}},$$

where $p_1(k), p_2(k), \cdots, p_{\left[\frac{k}{2}\right]}(k)$ are integers depending only on k. Thus by the lemmas,

$$R_\lambda^k(x) - f(x) = \frac{k}{\pi} \int_1^\infty \frac{\phi_x\left(\frac{t}{\lambda}\right)}{t^2} dt + O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right).$$

Clearly we have

$$\max_{|t| \leqq \max(1,a)} \left|\phi_x\left(\frac{t}{\lambda}\right)\right| \leqq \sup_{|t| \leqq \max(1,a)} \max_x \left|\phi_x\left(\frac{t}{\lambda}\right)\right| = O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right),$$

and so

$$\left|\int_a^1 \frac{\phi_x(t/\lambda)}{t^2} dt\right| \leqq \left|\int_a^1 \frac{|\phi_x(t/\lambda)|}{t^2} dt\right| = O\left(\omega_2\left(\frac{1}{\lambda}, f\right) \int_a^1 \frac{dt}{t^2}\right)$$
$$= O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right).$$

Finally,

$$R_\lambda^k(x) - f(x) = \frac{k}{\pi} \int_a^\infty \frac{\phi_x(t/\lambda)}{t^2} dt + O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right),$$

and the theorem is therefore proved.

References

[1] A. Zygmund, *Trigonometric series* (Cambridge, 1959).

[2] S. B. Steckin, 'On the order of the best approximations of continuous functions', *Izv. Akad. Nauk USSR Ser. Mat.* 15 (1951), 219–242 (Russian).

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https://doi.org/10.1017/S144678870000447X Published online by Cambridge University Press