Can. J. Math., Vol. XXXII, No. 6, 1980, pp. 1411–1422

THE HOMOTOPY OF SIMPLICIAL ALGEBRAS

HARRY LAKSER

1. Introduction. In [**6**] Walter Taylor investigated the relationship between the algebraic structure of a topological algebra A and the group structure of its fundamental group $\pi_1(A)$ and of the higher homotopy groups $\pi_n(A)$, $n > 1$. The main result is that a variety $\mathscr{V}$ satisfies a group law λ in homotopy (that is, π_1) if and only if every group in the idempotent reduct of $\mathscr{V}$ obeys λ. (The relevant definitions are in [**6**] and also § 2 of this paper.) A similar result is stated for the higher homotopy groups. As Taylor points out in the introduction, the hard part of the theorem is constructing a topological algebra in $\mathscr{V}$ whose fundamental group may fail to obey λ; indeed, in [**6**] this is only done in detail for the commutative law, and the proof is rather computational. The basic technique used is the Swierczkowski topology.

In this paper we interpose the simplicial category between the topological and algebraic. The bulk of this paper treats simplicial algebras, and, using standard results of the theory of simplicial sets, we derive our result with a minimum of computation. For each $N \geqq 1$ we construct a simplicial algebra in the variety whose π_N is the free (abelian if $N > 1$) group in the idempotent reduct of the variety. The geometric realization is then the desired topological algebra in the variety in those cases where the realization preserves products, for instance, when the simplicial set is countable. Thus we can get Taylor's result only if there are only countably many fundamental operations (see § 7). In this case our result is slightly better than Taylor's; the relevant homotopy group is the free group in the idempotent reduct, and the topological algebra is a CW complex with the algebraic operations cellular maps.

There is unfortunately very little standardized notation in the theory of simplicial sets. As our standard of notation and definitions we in general use [**1**], and those results from [**2**] and [**5**] that we need will be transcribed into the notation of [**1**].

2. Homotopy groups of Kan algebras. A *simplicial set* is a graded set indexed on the non-negative integers; that is, a sequence of sets $(A_n | n \geqq 0)$, together with maps $d_i : A_n \to A_{n-1}$ and $s_i : A_n \to A_{n+1}$,

Received March 8, 1979 and in revised form June 25, 1979. This research was supported by a grant from the Natural Science and Engineering Research Council of Canada.

$0 \leqq i \leqq n$, for each non-negative integer n, which satisfy the following identities:

(i) $d_i d_j = d_{j-1} d_i$ if $i < j$.
(ii) $s_i s_j = s_{j+1} s_i$ if $i \leqq j$.
(iii) $d_i s_j = s_{j-1} d_i$ if $i < j$,
$d_j s_j = \text{identity} = d_{j+1} s_j$,
$d_i s_j = s_j d_{i-1}$ if $i > j + 1$.

The elements of A_n are called *n-simplices*. The d_i and s_i are called *face* and *degeneracy* operators respectively.

Let A be a simplicial set and let $\mathscr{V}$ be a class of algebras of fixed similarity type. We say that A is a *simplicial $\mathscr{V}$-algebra* if, for each $n \geqq 0$, the set A_n of n-simplices of A has the structure of a $\mathscr{V}$-algebra and the face operators d_i and degeneracy operators s_i are homomorphisms. Thus for each r-ary operation symbol **f** of the type of $\mathscr{V}$ we have a simplicial map $f:A^r \rightarrow A$. A *homomorphism* $\varphi:A \rightarrow B$ of simplicial $\mathscr{V}$-algebras is a simplicial map such that, for each n, $\varphi:A_n \rightarrow B_n$ is a homomorphism of algebras.

If the simplicial $\mathscr{V}$-algebra A is a Kan complex, that is, if A satisfies the Kan extension condition [**1**, p. 114], we say that A is a Kan $\mathscr{V}$-algebra. A direct definition of the homotopy groups exists only for Kan complexes. In § 3 we describe a functor due to Kan [**2**] that associates with each simplicial set a Kan complex, and so enables one to define homotopy groups for a general simplicial set.

A *pointed simplicial $\mathscr{V}$-algebra* $(A, *)$ is a simplicial $\mathscr{V}$-algebra A with base-point $*$ such that $f(*, \ldots, *) = *$ for all operations f of $\mathscr{V}$. Thus the simplicial map $f: (A, *)^r \rightarrow (A, *)$ is a simplicial map of pointed simplicial sets.

If $\mathscr{V}$ is a class of algebras then by a *group G in $\mathscr{V}$* is meant an algebra G in $\mathscr{V}$ that also has a group structure, composition denoted $\cdot$, inverse denoted $^{-1}$, identity denoted e, such that each operation $f:A^r \rightarrow A$ of $\mathscr{V}$ is a group homomorphism [**6**, p. 499].

If $(A, *)$ is a pointed Kan $\mathscr{V}$-algebra and f is an r-ary operation of $\mathscr{V}$ then the simplicial map $f: (A, *)^r \rightarrow (A, *)$ yields, for each $n \geqq 1$, a group homomorphism

$$\pi_n(f): \pi_n(A, *)^r \rightarrow \pi_n(A, *).$$

Thus $\pi_n(A, *)$ inherits the structure of an algebra of the type of $\mathscr{V}$.

Recall that a class of algebras is said to be a *variety* (also called an "equational class") if it is the class of all algebras satisfying a set of identities. The fundamental theorem of G. Birkhoff states that a class $\mathscr{V}$ of algebras is a variety if and only if it is closed under the operations of taking products of algebras in $\mathscr{V}$, subalgebras of $\mathscr{V}$, and homomorphic images of algebras in $\mathscr{V}$.

LEMMA 1. *Let $\mathscr{V}$ be a variety of algebras and let $n \geqq 1$. The functor π_n is a functor from the category of pointed Kan $\mathscr{V}$-algebras to the category of groups in $\mathscr{V}$.*

Proof. Let f be an operation of the type of $\mathscr{V}$. Then $f: (A, *)^r \rightarrow (A, *)$ is simplicial and thus $\pi_n(f): \pi_n(A, *)^r \rightarrow \pi_n(A, *)$ is a group homomorphism. To show that $\pi_n(A, *)$ is a group in $\mathscr{V}$ we thus need only show that $\pi_n(A, *)$ is an algebra in $\mathscr{V}$ under the operations $\pi_n(f)$. It is most convenient to use the definition of the homotopy groups in [**5**, Definitions 3.1 and 3.6] throughout this paper. The subset

$$\tilde{A}_n = \{x \in A_n | d_0 x = d_1 x = \ldots = d_n x = *\}$$

is a subalgebra of A_n since the d_i are homomorphisms and $f(*, \ldots, *) = *$ for all operations f of $\mathscr{V}$. We next observe that the homotopy relation $\simeq$ on $\tilde{A}_n$ is a congruence relation. Let f be r-ary and let $x_i, y_i \in \tilde{A}_n$, $i = 1, \ldots, r$ with each $x_i \simeq y_i$. This means that there are $w_1, \ldots, w_r \in A_{n+1}$ with $d_n w_i = x_i$, $d_{n+1} w_i = y_i$ and $d_j w_i = *$, $0 \leqq j < n$. Then $f(w_1, \ldots, w_r)$ is a homotopy from $f(x_1, \ldots, x_r)$ to $f(y_1, \ldots, y_r)$;

$$d_n f(w_1, \ldots, w_r) = f(x_1, \ldots, x_r),$$
$$d_{n+1} f(w_1, \ldots, w_r) = f(y_1, \ldots, y_r) \quad \text{and}$$
$$d_j f(w_1, \ldots, w_r) = f(*, \ldots, *) = *, \ 0 \leqq j < n,$$

since the d_i are homomorphisms. It thus follows that $\pi_n(A, *) \in \mathscr{V}$.

We note finally that if $(A, *)$, $(B, *)$ are pointed Kan $\mathscr{V}$-algebras and $\varphi: (A, *) \rightarrow (B, *)$ is a simplicial homomorphism then, for any r-ary operation of $\mathscr{V}$, the diagram

$$\begin{array}{ccc} (A, *)^r & \xrightarrow{\varphi^r} & (B, *)^r \\ \downarrow f & & \downarrow f \\ (A, *) & \xrightarrow{\varphi} & (B, *) \end{array}$$

commutes. Consequently

$$\begin{array}{ccc} \pi_n(A, *)^r & \xrightarrow{\pi_n(\varphi)^r} & \pi_n(B, *)^r \\ \downarrow \pi_n(f) & & \downarrow \pi_n(f) \\ \pi_n(A, *) & \xrightarrow{\pi_n(\varphi)} & \pi_n(B, *) \end{array}$$

commutes, and thus $\pi_n(\varphi)$ preserves the algebraic operations. That $\pi_n(\varphi)$ preserves the group structure is standard; thus $\pi_n(\varphi)$ is a homomorphism of groups in $\mathscr{V}$, concluding the proof of the lemma.

3. The functor Ex^∞. In [**5**] Kan defined a functor Ex^∞ from the category of simplicial sets to the subcategory of Kan complexes, thereby enabling one to define homotopy groups for an arbitrary simplicial set. The functor Ex^∞ is a direct limit of iterates of a functor Ex. We review these definitions and show how they give rise to a functor from the category of simplicial $\mathscr{V}$-algebras to the subcategory of Kan $\mathscr{V}$-algebras.

For each $n \geqq 0$ let $\Delta[n]$ be the standard n-simplex [**1**, p. 112]. The barycentric subdivision $\Delta'[n]$ of $\Delta[n]$ has as q-simplices $(q + 1)$-tuples $(\sigma_0, \ldots, \sigma_q)$ where the σ_i are nondegenerate simplices of $\Delta[n]$ and, for each i, $0 \leqq i < q$, σ_1 is a face of σ_{i+1}. That is, there is a sequence, possibly empty, $d_{i_0}, \ldots, d_{i_k}$ with

$$\sigma_i = d_{i_0}, \ldots, d_{i_k}\sigma_{i+1}.$$

If $f:\Delta[m] \rightarrow \Delta[n]$ is a simplicial map and σ is a simplex of $\Delta[m]$ denote by $\tilde{f}\sigma$ the (unique) nondegenerate simplex of $\Delta[n]$ determined by $f\sigma$. We then have a simplicial map $f':\Delta'[m] \rightarrow \Delta'[n]$ with

$$f'(\sigma_0, \ldots, \sigma_q) = (\tilde{f}\sigma_0, \ldots, \tilde{f}\sigma_q).$$

For a simplicial set A the simplicial set $\mathrm{Ex}\,A$ has $(\mathrm{Ex}\,A)_n$ as the set of all simplicial maps $\Delta'[n] \rightarrow A$. The face operators are defined by setting, for each $x:\Delta'[n] \rightarrow A$, $d_i x = x \circ \epsilon_i'$ where $\epsilon_i:\Delta[n-1] \rightarrow \Delta[n]$ is the inclusion i-face [**1**, p. 115]. The degeneracy operator $s_i x = x \circ \eta_i'$ where $\eta_i:\Delta(n+1] \rightarrow \Delta[n]$ is the ith projection [**1**, p. 115]. If $f:A \rightarrow B$ is a simplicial map then $\mathrm{Ex}\,f:\mathrm{Ex}\,A \rightarrow \mathrm{Ex}\,B$ is defined by setting $\mathrm{Ex}\,f(x) = f \circ x$.

Observe that the functor Ex preserves direct products; if $p_1:A \times B \rightarrow A$ and $p_2:A \times B \rightarrow B$ are the natural projections then $\mathrm{Ex}\,p_1$ and $\mathrm{Ex}\,p_2$ give rise to a natural isomorphism

$$\mathrm{Ex}(A \times B) \rightarrow \mathrm{Ex}\,A \times \mathrm{Ex}\,B.$$

Thus if $f:A^r \rightarrow A$ is a simplicial map we get

$$\mathrm{Ex}\,f: (\mathrm{Ex}\,A)^r \rightarrow \mathrm{Ex}\,A,$$

and so if $\mathscr{V}$ is a variety of algebras and A is a simplicial $\mathscr{V}$-algebra then $\mathrm{Ex}\,A$ is a simplicial algebra of the type of $\mathscr{V}$. If $x:\Delta'[n] \rightarrow A$ is an n-simplex of $\mathrm{Ex}\,A$ then x is determined by its values on the n-simplices of $\Delta'[n]$. Let $f:A^r \rightarrow A$ be a basic operation and let

$$x_1, \ldots, x_r:\Delta'[n] \rightarrow A$$

be n-simplices of $\mathrm{Ex}\,A$. Let σ be an n-simplex of $\Delta'[n]$. Then

$$((\mathrm{Ex}\,f)(x_1, \ldots, x_r))(\sigma) = f(x_1(\sigma), \ldots, x_r(\sigma)).$$

We see thus that $(\mathrm{Ex}\,A)_n$ is a subalgebra of a power of A_n, and conclude that if $\mathscr{V}$ is a variety of algebras and A is a simplicial $\mathscr{V}$-algebra then so is $\mathrm{Ex}\,A$. If $\varphi:A \rightarrow B$ is a homomorphism of $\mathscr{V}$-algebras then $\mathrm{Ex}\,\varphi$, given by $\operatorname{Ex}\varphi(x) = \varphi \circ x$, is clearly also a homomorphism. We thus have the following lemma.

LEMMA 2. *If* $\mathscr{V}$ *is a variety of algebras then* Ex *is a functor from the category of simplicial* $\mathscr{V}$*-algebras to itself.*

There is a simplicial map $\delta[n]:\Delta'[n] \to \Delta[n]$ determined as follows [**2**, p. 452]; if $\sigma = (n_0, \ldots, n_q) \in \Delta[n]_q$ is a vertex of $\Delta'[n]$ then $\delta[n]\sigma = (n_q)$, a vertex of $\Delta[n]$. Kan defines a natural embedding e with $e_A:A \to \operatorname{Ex} A$ determined by setting

$$e_A x = f_x \circ \delta[n],$$

where x is any n-simplex of A and f_x is the simplicial map $f_x:\Delta[n] \to A$ determined by x [**4**, p. 112]. Since e preserves products we conclude that, if A is a simplicial $\mathscr{V}$-algebra, then e_A is an embedding of simplicial $\mathscr{V}$-algebra.

The functor Ex^∞ is defined as the direct limit of

$$A \xrightarrow{e_A} \operatorname{Ex} A \xrightarrow{e_{\operatorname{Ex} A}} \operatorname{Ex}^2 A \longrightarrow \cdots$$

and we have the natural limit embedding e^∞. It turns out that $\operatorname{Ex}^\infty A$ is always a Kan complex [**2**, Theorem (4.2)]. Since varieties are closed under direct limits we get the following lemma.

LEMMA 3. *If* $\mathscr{V}$ *is a variety of algebras then* Ex^∞ *is a functor from the category of simplicial* $\mathscr{V}$*-algebras to the category of Kan* $\mathscr{V}$*-algebras, and* e^∞ *is a natural embedding.*

We observe that if $(A, *)$ is a pointed simplicial set then $\operatorname{Ex} A$ has a natural base-point e_A* and $\operatorname{Ex}^\infty A$ has a natural base-point $e_A{}^\infty *$; as usual, we denote these simply by $*$. From Lemma 3 we see that, via Ex^∞, we can define homotopy groups for any simplicial $\mathscr{V}$-algebra.

4. Homotopy groups of simplicial $\mathscr{V}$-algebras. Let $\mathscr{V}$ be a variety of algebras and let $\mathscr{V}_0$ be the idempotent reduct of $\mathscr{V}$ [**6**, p. 499]. Let I be the functor that associates with each algebra A in $\mathscr{V}$ the algebra IA defined on the same set but using only the operations of the similarity type of $\mathscr{V}_0$.

If A is a simplicial $\mathscr{V}$ algebra and $* \in A_0$, then the pointed simplicial set $(A, *)$ will in general not be a pointed simplicial $\mathscr{V}$-algebra but, because of the idempotence of $\mathscr{V}_0$, $(IA, *)$, with $(IA)_n = IA_n$, is a pointed simplicial $\mathscr{V}_0$-algebra. We conclude that $(\operatorname{Ex}^\infty IA, *)$ is a pointed Kan $\mathscr{V}_0$-algebra and if A itself is a Kan complex then $e_A{}^\infty$ is a homotopy equivalence, [**2**, Theorem (4.5)]. We can thus define, without ambiguity, the nth homotopy group functor π_n as $\pi_n \circ \operatorname{Ex}^\infty \circ I$. By Lemma 1 and Lemma 3 we get the following theorem.

THEOREM 1. *Let $\mathscr{V}$ be a variety of algebras and let $n \geqq 1$. The nth homotopy group functor $\pi_n \circ \mathrm{Ex}^\infty \circ I$ is a functor from the category of simplicial $\mathscr{V}$-algebras with base-point to the category of groups in the idempotent reduct of $\mathscr{V}$.*

When the context is clear, we often will write $\pi_n(A, *)$ rather than $\pi_n(\mathrm{Ex}^\infty IA, *)$.

5. The Eilenberg-MacLane complexes. Let $\mathscr{V}$ be a variety of algebras and let G be a simplicial group in $\mathscr{V}$, that is, each G_n is a group in $\mathscr{V}$ and the face and degeneracy operators are group homomorphisms and $\mathscr{V}$-homomorphisms. The functor $\bar{W}$ [**1**, p. 136] yields a Kan complex $\bar{W}(G)$; $\bar{W}(G)_0$ consists of a single simplex ( ) and

$$\bar{W}(G)_n = G_{n-1} \times \ldots \times G_0 \text{ for } n > 0.$$

The face and degeneracy operators are defined as

$$d_0(g_{n-1}, \ldots, g_0) = (g_{n-2}, \ldots, g_0),$$
$$s_0(g_{n-1}, \ldots, g_0) = (e, g_{n-1}, \ldots, g_0), \quad e \text{ the identity of } G_n,$$

and, for $i > 0$,

$$d_i(g_{n-1}, \ldots, g_0) = (d_{i-1}g_{n-1}, d_{i-2}g_{n-2}, \ldots,$$
$$d_0 g_{n-i} \cdot g_{n-i-1}, g_{n-i-2}, \ldots, g_0),$$
$$s_i(g_{n-1}, \ldots, g_0) = (s_{i-1}g_{n-1}, s_{i-2}g_{n-2}, \ldots, s_0 g_{n-i}, e, g_{n-i-1}, \ldots, g_0).$$

Since $\bar{W}(G)_n$ is a product of $\mathscr{V}$-algebras, $\bar{W}(G)_n$ has the structure of a $\mathscr{V}$-algebra, with $\bar{W}(G)_0$ the trivial (one-element) $\mathscr{V}$-algebra. The face and degeneracy operators are then homomorphisms; this is clear for d_0 and the s_i's, and for d_i, $i > 0$, follows from the fact that the G_n are groups in $\mathscr{V}$, that is, if f is r-ary then

$$f(d_0 g_1 \cdot h_1, \ldots, d_0 g_r \cdot h_r) = f(d_0 g_1, \ldots, d_0 g_r) f(h_1, \ldots, h_r)$$
$$= d_0 f(g_1, \ldots, g_r) f(h_1, \ldots, h_r).$$

We then have the following lemma.

LEMMA 4. *$\bar{W}$ is a functor from the category of simplicial groups in $\mathscr{V}$ to the category of Kan $\mathscr{V}$-algebras.*

Now let G be a group in $\mathscr{V}$. The functor $K(\cdot, 0)$ associates with G the simplicial group $K(G, 0)$ in $\mathscr{V}$ [**1**, p. 137]; $K(G, 0)_n = G$ for all $n \geqq 0$ and the face and degeneracy operators are the identity map. Let $N \geqq 1$, and let G be abelian if $N > 1$. Then the N-fold iterate $K(G, N) = \bar{W} \circ \ldots \circ \bar{W}K(G, 0)$, the Eilenberg–MacLane complex, gives rise to a functor $K(\cdot, N)$ from the category of groups in $\mathscr{V}$ to the category of Kan $\mathscr{V}$-algebras (of simplicial groups in $\mathscr{V}$ if G is abelian). Note that for $n < N$ $K(G, N)_n$ is the trivial $\mathscr{V}$-algebra, and that $K(G, N)_N \cong G$ as a

$\mathscr{V}$-algebra (as a group in $\mathscr{V}$ if G is abelian). We denote the unique element ( ) of $K(G, N)_0$ by $*$.

LEMMA 5. *The group in* $\mathscr{V}$ $\pi_N(K(G, N), *)$ *is naturally isomorphic to* G *under the isomorphism* $\rho: G \to \pi_N(K(G, N), *)$ *with* $\rho g = [g]$ (*identifying* G *with* $K(G, N)_N$).

Proof. That ρ is a group isomorphism is Theorem 23.2 of [**5**, p. 98]. By the definition of the $\mathscr{V}$-structure of $K(G, N)$ it is immediate that ρ also preserves the operations of $\mathscr{V}$.

In [**5**] it is also proved that $\pi_n(K(G, N), *)$ is trivial if $n \neq N$, but this is irrelevant to our purposes.

6. The free group in $\mathscr{V}_0$. Let $\mathscr{V}$ be a variety of algebras and let $N \geqq 1$. Let X be a countable set. We construct a simplicial set ${}_NA$. We set $({}_NA)_0 = \{*\}$ and, for $n < N$, $({}_NA)_n$ consists of a single n-simplex, the degenerate n-simplex $s_0{}^n*$, also denoted $*$, following the convention in [**1**, p. 113]. We let $({}_NA)_N$ consist of X and the degenerate N-simplex $*$ determined by $* \in ({}_NA)_0$. The face operators

$$d_i: ({}_NA)_N \to ({}_NA)_{N-1}$$

are defined by setting $d_i y = *$ for all $y \in ({}_NA)_N$. For $n > N$, $({}_NA)_n$ consists of the degenerate n-simplices determined by $({}_NA)_N$ in the freest possible manner.

We now let ${}_NF$ be the simplicial $\mathscr{V}$-algebra obtained by letting $({}_NF)_n$ be the free $\mathscr{V}$-algebra generated by $({}_NA)_n$, $n \geqq 0$, and determining the face and degeneracy operators from those on ${}_NA$ by freeness.

Observe that ${}_NA$ is the analogue of the topological join of countably many N-spheres, and that ${}_NF$ is the analogue of the free $\mathscr{V}$-algebra generated by this join.

We let ${}_NF'$ be the connected component of $*$ in ${}_NF$. Let $\mathscr{V}_0$ be the idempotent reduct of $\mathscr{V}$ and recall the functor I replacing a $\mathscr{V}$-algebra by its $\mathscr{V}_0$-reduct.

LEMMA 6. *For each* $n \geqq 0$, $({}_NF')_n$ *is a subalgebra of* $I({}_NF)_n$ *and is the free* $\mathscr{V}_0$*-algebra generated by* $({}_NA)_n$.

Proof. The n-simplex $x \in {}_NF$ lies in ${}_NF'$ if and only if there is a sequence $x_1, \ldots, x_k \in ({}_NF)_1$ with

$$d_0{}^n x = d_0 x_1,\ d_1 x_k = *, \quad \text{and} \quad d_1 x_i = d_0 x_{i+1}, \quad 1 \leqq i < k.$$

Since $({}_NF)_1$ is the free $\mathscr{V}$-algebra generated by $({}_NA)_1$ there are $a_1, \ldots, a_r \in ({}_NA)_1$ and r-ary polynomials $p_1, \ldots, p_k$ with

$$x_i = p_i(a_1, \ldots, a_r), \quad 1 \leqq i \leqq k.$$

We then get

$$d_0^n x = p_1(*, \ldots, *) = \ldots = p_k(*, \ldots, *) = *.$$

Thus $x \in {}_NF'$ if and only if $d_0^n x = *$. By the freeness of $({}_NF)_n$ there is an r-ary $\mathscr{V}$-polynomial p and $a_1, \ldots, a_r \in ({}_NA)_n$ with $x = p(a_1, \ldots, a_r)$. Then

$$d_0^n x = p(d_0^n a_1, \ldots, d_0^n a_r) = p(*, \ldots, *).$$

Consequently $x \in {}_NF'$ if and only if $p(*, \ldots, *) = *$ and thus, since $({}_NF)_0$ is free on $\{*\}$, if and only if p is idempotent. The lemma then follows.

THEOREM 2. *Let* $\mathscr{V}$ *be a variety of algebras.*

(a) $\pi_1({}_1F, *)$ *is the free group in the idempotent reduct of* $\mathscr{V}$ *generated by* $\{[e_{{}_1F}^\infty x] | x \in X\}$.

(b) *If* $N > 1$ *then* $\pi_N({}_NF, *)$ *is the free abelian group in the idempotent reduct of* $\mathscr{V}$ *generated by* $\{[e_{{}_NF}^\infty x] | x \in X\}$.

Proof. Let $\mathscr{V}_0$ denote the idempotent reduct of $\mathscr{V}$, let G be a group in $\mathscr{V}_0$, abelian if $N > 1$, and let $\varphi: X \to G$ be any mapping.

Since the embeddings

$$e_{{}_NF'}^\infty: ({}_NF')_0 \to (\mathrm{Ex}^\infty {}_NF')_0 \quad \text{and}$$
$$e_{{}_NF}^\infty: ({}_NF)_0 \to (\mathrm{Ex}^\infty {}_NF)_0$$

are bijective it follows that $\mathrm{Ex}^\infty {}_NF'$ is the component of $*$ in $\mathrm{Ex}^\infty {}_NF$. Thus $\pi_N({}_NF, *)$ (which is $\pi_N(\mathrm{Ex}^\infty I({}_NF), *)$ by our convention) is $\pi_N(\mathrm{Ex}^\infty {}_NF', *)$ as a group in $\mathscr{V}_0$.

We now recall the Eilenberg-MacLane complex $K(G, N)$. There is a pointed simplicial map

$$\psi: ({}_NA, *) \to (K(G, N), *)$$

with $\psi x = \varphi x$ for $x \in ({}_NA)_N - \{*\}$ (recall that $K(G, N)_N = G$). By Lemma 6 we can extend ψ to a homomorphism, also denoted ψ, of simplicial $\mathscr{V}_0$-algebras $\psi: {}_NF' \to K(G, N)$. We thus have

$$\pi_N(\mathrm{Ex}^\infty \psi): \pi_N(\mathrm{Ex}^\infty {}_NF', *) \to \pi_N(\mathrm{Ex}^\infty K(G, N), *),$$

a homomorphism of groups in $\mathscr{V}_0$ with

$$\pi_N(\mathrm{Ex}^\infty \psi)\,[e_{{}_NF}^\infty x] = [e_{K(G,N)}^\infty \varphi(x)] \quad \text{for } x \in X \subseteq ({}_NA)_N.$$

Since $K(G, N)$ is a Kan complex the embedding of simplicial $\mathscr{V}_0$-algebras

$$e_{K(G,N)}^\infty: K(G, N) \to \mathrm{Ex}^\infty K(G, N)$$

is a homotopy equivalence [**2**, Theorem (4.5)] and thus

$$\pi_N(e_{K(G,N)}^\infty): \pi_N(K(G, N), *) \to \pi_N(\mathrm{Ex}^\infty K(G, N), *)$$

is an isomorphism. Then the map

$$\bar{\psi} = \rho^{-1} \circ \pi_N(e_{K(G,N)}{}^{\infty})^{-1} \circ \pi_N(\mathrm{Ex}^{\infty}\,\psi)$$

is a homomorphism of groups in $\mathscr{V}_0$, $\bar{\psi}:\pi_N({}_NF, *) \to G$ with $\bar{\psi}[e_{{}_NF}{}^{\infty}\,x] = \varphi(x)$ for $x \in X$.

To complete the proof of the theorem we need only show that $\pi_N({}_NF, *)$ is generated as a group in $\mathscr{V}_0$ by the set $\{[e_{{}_NF}{}^{\infty}\,x]|x \in X\}$. This could be done directly but we prefer to minimize the computations and so refer the reader to the Corollary to Lemma **7** in the appendix, § 8.

7. The geometric realization. If $\mathscr{V}$ is a variety of algebras and A is a topological algebra in $\mathscr{V}$ then the singular simplicial functor S, [**1**, p. 107], associates a Kan complex SA with A. Then SA is a simplicial $\mathscr{V}$-algebra under the operations Sf, f an operation on the algebra A. The homotopy groups of A are just the homotopy groups of SA. By Theorem 1 we get the following theorem.

THEOREM 3. (Walter Taylor) *Let $\mathscr{V}$ be a variety of algebras and let $n \geqq 1$. The functor π_n is a functor from the category of topological algebras in $\mathscr{V}$ with base-point to the category of groups in the idempotent reduct of $\mathscr{V}$.*

Since SA is Kan, we do not need the full force of Theorem 2. Indeed the direct proof presented in [**6**] is really as simple.

To proceed in the opposite direction we recall the geometrical realization functor R, [**1**, (1.29)] and, for more details, [**5**, Chapter III]. To each simplicial set A is associated a CW complex RA and to each simplicial map f a continuous function, indeed a cellular map, Rf. If A is a Kan complex then each homotopy group of A is naturally isomorphic to the corresponding homotopy group of RA. If A is countable then the functor R commutes with direct products and so if A is a countable simplicial $\mathscr{V}$-algebra then RA is a topological algebra in $\mathscr{V}$. If the type of $\mathscr{V}$ is countable then ${}_NF$ and $\mathrm{Ex}^{\infty}\,{}_NF$ are countable and so R_NF and $R\,\mathrm{Ex}^{\infty}{}_NF$ are topological algebras in $\mathscr{V}$. The map $R\,e_{{}_NF}{}^{\infty}$ is a homotopy equivalence between R_NF and $R\,\mathrm{Ex}^{\infty}{}_NF$ [**2**, Lemma (6.5)]; thus the Nth homotopy group of R_NF is isomorphic, as a group in the idempotent reduct of $\mathscr{V}$, to the Nth homotopy group of $\mathrm{Ex}^{\infty}{}_NF$. We consequently have the following theorem.

THEOREM 4. *Let $\mathscr{V}$ be a variety of countable type.*

(a) *There is a* CW *topological algebra in $\mathscr{V}$ whose fundamental group (that is, π_1) is the free group on $\aleph_0$ generators in the idempotent reduct of $\mathscr{V}$.*

(b) *For $N > 1$ there is a* CW *topological algebra in $\mathscr{V}$ whose Nth homotopy group is the free abelian group on $\aleph_0$ generators in the idempotent reduct of $\mathscr{V}$.*

COROLLARY. (Walter Taylor) (a) *For a fixed group law* λ *and variety* $\mathscr{V}$ *of countable type every group in the idempotent reduct of* $\mathscr{V}$ *obeys* λ *if and only if* $\mathscr{V}$ *obeys* λ *in homotopy.*

(b) *For any* $N \geqq 2$, *any variety* $\mathscr{V}$ *of countable type, and any abelian group law* λ, *every abelian group in the idempotent reduct of* $\mathscr{V}$ *obeys* λ *if and only if* $\mathscr{V}$ *obeys* λ *in* N*-homotopy.*

Recall [**6**] that we say a variety $\mathscr{V}$ obeys the group law (group identity) λ in N-homotopy if the Nth homotopy group of each topological algebra in $\mathscr{V}$ satisfies λ. If $N = 1$, we simply say that $\mathscr{V}$ obeys λ in homotopy, that is, the fundamental group of each topological algebra in $\mathscr{V}$ satisfies λ.

Taylor's results do not actually involve any restriction on the type of $\mathscr{V}$. To get these full results from our work would entail having a geometrical realization functor that preserves products in general.

In our case, though, the general result holds if we change the definition of topological algebra so that the algebraic operations need only be continuous on compact sets, the Kelley topology [**3**, p. 42] and [**4**, p. 56].

8. Appendix. Let $(A, *)$ be a pointed simplicial set. We say that A is N-reduced, $n \geqq 1$, if $A_n = \{*\}$ for $n < N$.

LEMMA 7. *If* $N \geqq 1$ *and* $(A, *)$ *is* N*-reduced then the group* $\pi_N(\mathrm{Ex}^\infty A, *)$ *is generated by the set* $\{[e_A{}^\infty x] | x \in A_N - \{*\}\}$.

Proof. If $x \in A_N$ then

$$d_i e_A{}^\infty x = e_A{}^\infty d_i x = e_A{}^\infty * = * \quad \text{for each } i.$$

Thus

$$\{[e_A{}^\infty x] | x \in A_N - \{*\}\} \subseteq \pi_N(\mathrm{Ex}^\infty A, *).$$

To show that $\{[e_A{}^\infty x] | x \in A_N - \{*\}\}$ generates we recall the simplicial group GA [**1**, Definition (3.15)]. For each $n \geqq 0$ $(GA)_n$ is the group freely generated by a set $\{\bar{x} | x \in A_{n+1}\}$ subject to the relation $\overline{s_0 x} = e$, the group identity, for each $x \in A_n$; thus the set $\{\bar{x} | x \in A_{n+1} - \{*\}\}$ is a generating set for $(GA)_n$. The mapping $t: A \to GA$ given by $tx = \bar{x}$ is a twisting function and we can consider the simplicial set $EA = GA \times_t A$. Then

$$GA \xrightarrow{\iota} EA \xrightarrow{p} A$$

is a fibration, where $\iota g = (g, *)$ and $p(g, x) = x$. Then

$$\mathrm{Ex}^\infty GA \xrightarrow{\mathrm{Ex}^\infty \iota} \mathrm{Ex}^\infty EA \xrightarrow{\mathrm{Ex}^\infty p} \mathrm{Ex}^\infty A$$

is also a fibration [**2**, Theorem (4.3)] and the diagram

$$\begin{array}{ccccc} GA & \xrightarrow{\iota} & EA & \xrightarrow{p} & A \\ \Big\downarrow{\scriptstyle e_{GA}^{\infty}} & & \Big\downarrow{\scriptstyle e_{EA}^{\infty}} & & \Big\downarrow{\scriptstyle e_{A}^{\infty}} \\ \mathrm{Ex}^{\infty}GA & \xrightarrow{\mathrm{Ex}^{\infty}\iota} & \mathrm{Ex}^{\infty}EA & \xrightarrow{\mathrm{Ex}^{\infty}p} & \mathrm{Ex}^{\infty}A \end{array}$$

commutes.

Since GA is a simplicial group it is a Kan complex and so $e_{GA}{}^{\infty}$ is a homotopy equivalence; thus

$$\pi_n\, e_{GA}{}^{\infty}:\pi_n(GA, e) \to \pi_n(\mathrm{Ex}^{\infty}\, GA, e_{GA}{}^{\infty}\, e)$$

is an isomorphism for all n. Now REA, the geometrical realization of EA, is contractible [**5**, Theorem 26.6] and, since

$$R\, e_{EA}{}^{\infty}:REA \to R\ \mathrm{Ex}^{\infty}EA$$

is a homotopy equivalence, $R\ \mathrm{Ex}^{\infty}EA$ is a contractible space. Thus the Kan complex $\mathrm{Ex}^{\infty}EA$ is contractible. From the homotopy sequence [**1**, Theorem (2.8)] of the fibration

$$\mathrm{Ex}^{\infty}GA \to \mathrm{Ex}^{\infty}EA \to \mathrm{Ex}^{\infty}A$$

we then get an isomorphism

$$\partial:\pi_N(\mathrm{Ex}^{\infty}A, *) \to \pi_{N-1}(\mathrm{Ex}^{\infty}GA, e_{GA}{}^{\infty}\, e).$$

Since A is N-reduced $\tilde{A}_N = A_N$. Let $x \in A_N$ and compute $\partial[e_A{}^{\infty}x]$. The N-simplex $e_{EA}{}^{\infty}(e, x)$ of $\mathrm{Ex}^{\infty}EA$ satisfies

$$d_i\, e_{EA}{}^{\infty}(e, x) = e_{EA}{}^{\infty}d_i(e, x) = e_{EA}{}^{\infty}(e, *) = * \quad \text{for } i > 0 \quad \text{and}$$
$$d_0\, e_{EA}{}^{\infty}(e, x) = e_{EA}{}^{\infty}d_0(e, x) = e_{EA}{}^{\infty}(\bar{x}, *) = e_{EA}{}^{\infty}\iota\bar{x} = (\mathrm{Ex}^{\infty}\iota)(e_{GA}{}^{\infty}\bar{x});$$

since $(\mathrm{Ex}^{\infty}p)(e, x) = e_A{}^{\infty}x$ we get

$$\partial[e_A{}^{\infty}x] = [e_{GA}{}^{\infty}\bar{x}].$$

Thus the isomorphism

$$(\pi_{N-1}\, e_{GA}{}^{\infty})^{-1} \circ \partial:\pi_N(\mathrm{Ex}^{\infty}A, *) \to \pi_{N-1}(GA, e)$$

maps $[e_A{}^{\infty}x]$ to $[\bar{x}]$. For each $x \in A_n$ $\bar{x} \in (\widetilde{GA})_{N-1}$ and, since $(\widetilde{GA})_{N-1}$ is generated by $\{\bar{x}|x \in A_N - \{*\}\}$, we see that $\pi_{N-1}(GA, e)$ is generated by $\{[\bar{x}]|x \in A_N - \{*\}\}$. Then $\pi_N(\mathrm{Ex}^{\infty}A, *)$ is generated by $\{[e_A{}^{\infty}x]|x \in A_N - \{*\}\}$, proving the lemma.

COROLLARY. *As a group in the idempotent reduct of* $\mathscr{V}$, $\pi_N({}_NF, *)$ *is generated by the set* $\{[e_{{}_NF}{}^{\infty}x]|x \in X\}$.

Proof. By Lemma 6 ${}_NF'$ is N-reduced. Thus, as a group, $\pi_N({}_NF, *)$ is generated by the set

$$\{[e_{{}_NF}{}^\infty y] | y \in ({}_NF')_N - \{*\}\}.$$

As a $\mathscr{V}_0$-algebra, $({}_NF')_N$ is generated by the set $X \cup \{*\}$. Since $[*]$ is the group identity of $\pi_N({}_NF, *)$ we conclude that, as a group in $\mathscr{V}_0$, $\pi_N({}_NF, *)$ is generated by the set $\{[e_{{}_NF}{}^\infty x] | x \in X\}$.

References

1. E. B. Curtis, *Simplicial homotopy theory*, Advances in Mathematics *6* (1971), 107–209.

2. D. M. Kan, *On c.s.s. complexes*, Amer. J. Math. *79* (1957), 449–476.

3. K. Lamotke, *Semisimpliziale algebraische Topologie*, Die Grundlehren der mathematischen Wissenschaften, Band *147* (Springer-Verlag, Berlin, Heidelberg, New York, 1968).

4. A. T. Lundell and S. Weingram, *The topology of CW complexes*, University Series in Higher Mathematics (Van Nostrand Reinhold, New York, 1969).

5. J. P. May, *Simplicial objects in algebraic topology*, Van Nostrand Mathematical Studies #11 (D. Van Nostrand, Princeton, N.J., 1967).

6. Walter Taylor, *Varieties obeying homotopy laws*, Can. J. Math. *29* (1977), 498–527.

University of Manitoba,
Winnipeg, Manitoba