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ON LEGENDRE CURVES IN CONTACT PSEUDO-HERMITIAN 3-MANIFOLDS

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Abstract

We find necessary and sufficient conditions for a Legendre curve in a Sasakian manifold to have: (i) a pseudo-Hermitian parallel mean curvature vector field; (ii) a pseudo-Hermitian proper mean curvature vector field in the normal bundle.

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1. Introduction

Given a contact structure η, we have two compatible structures. One is a Riemannian structure (or metric) g, and then we call $(M; \eta, g)$ a *contact Riemannian manifold*. The other is an *almost CR-structure* (η, L), where L is the *Levi form* associated with an endomorphism J on D such that $J^2 = -I$. In particular, if J is integrable, then we call it the (integrable) CR-structure. The associated almost CR-structure is said to be *pseudo-Hermitian, strongly pseudo-convex* if the Levi form is Hermitian and positive definite. We call such a manifold a *contact strongly pseudo-convex pseudo-Hermitian* (or *almost CR-*) *manifold*. There is a one-to-one correspondence between the two associated structures given by the relation

$$g = L + \eta \otimes \eta,$$

where we denote by the same letter L the natural extension of the Levi form to a (0, 2)-tensor field on M. From this point of view, we have two geometries for a given contact structure, that is, one is formed by the Levi-Civita connection ∇, the other is derived by the *Tanaka–Webster connection* $\hat{\nabla}$ (or the *pseudo-Hermitian connection*), which is a canonical affine connection on a strongly pseudo-convex CR-manifold.

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In the present paper, we study the contact pseudo-Hermitian geometry in the three-dimensional contact Riemannian manifold with respect to the Tanaka–Webster connection $\hat{\nabla}$. Corresponding to the Laplacian mean curvature vector with respect to the Levi-Civita connection ∇ (see [3, 4, 7]), we investigate the following for the Tanaka–Webster connection $\hat{\nabla}$:

$$\begin{aligned} \hat{\Delta}\hat{H} &= \lambda \hat{H}, \\ \hat{\Delta}^{\perp}\hat{H} &= \lambda \hat{H}, \end{aligned}$$

where λ is a function, $\hat{H}$ is the pseudo-Hermitian mean curvature vector and $\hat{\nabla}^{\perp}$ denotes the normal connection in the normal bundle.

A curve γ satisfying the first equation in the three-dimensional contact Riemannian manifold M is called a *curve with pseudo-Hermitian proper mean curvature vector field*. A curve γ satisfying the second equation in the three-dimensional contact Riemannian manifold M is called a *curve with pseudo-Hermitian proper mean curvature vector field in the normal bundle*. In Section 3.1 we study Legendre curves with pseudo-Hermitian minimal and parallel pseudo-Hermitian mean curvature vector in Sasakian manifolds. In Section 3.2 we find necessary and sufficient conditions for a Legendre curve with pseudo-Hermitian harmonic mean curvature vector field and proper pseudo-Hermitian mean curvature vector field in Sasakian manifolds. In Section 3.3 we briefly study curves of $AW(k)$ type from the viewpoint of pseudo-Hermitian geometry.

2. Preliminaries

2.1. Contact Riemannian manifolds. A three-dimensional smooth manifold M^3 is called a *contact manifold* if it admits a global 1-form η such that $\eta \wedge d\eta \neq 0$ everywhere on M. This 1-form η is called the *contact form* on M.

Given a contact form η, we have a unique vector field ξ, which is called the *characteristic vector field* of (M, η), satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X.

A Riemannian metric g on M is said to be an *associated metric* to a contact structure η if there exists an endomorphism field φ satisfying

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

where X and Y are vector fields on M. From (2.1), it follows that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

A Riemannian manifold M equipped with the structure tensors (η, ξ, φ, g) satisfying (2.1) is said to be a *contact Riemannian manifold*. We denote it by $M = (M, \eta; \xi, \varphi, g)$. Given a contact Riemannian manifold M, we define an endomorphism field h by $\mathrm{h} = \frac{1}{2} L_\xi \varphi$, where L_ξ denotes Lie differentiation in the characteristic direction ξ. The endomorphism field h is called the *structural operator* of $(M, \eta; \varphi, \xi, g)$.

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Then we may observe that h is symmetric and satisfies

$$\mathrm{h}\xi = 0, \quad \mathrm{h}\varphi = -\varphi\mathrm{h},$$
$$\nabla_X \xi = -\varphi X - \varphi \mathrm{h} X, \tag{2.2}$$

where ∇ is the Levi-Civita connection of (M, g).

For a three-dimensional contact Riemannian manifold M^3, one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(\varphi X - f\xi, \eta(X)\frac{d}{dt}\Big),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, then the contact Riemannian manifold M is said to be a *Sasakian manifold*.

PROPOSITION 2.1. *Let $(M^3, \eta; \xi, \varphi, g)$ be a contact Riemannian 3-manifold. Then the following three conditions are mutually equivalent.*

(1) *The characteristic vector field ξ is a Killing vector field, that is, $\nabla\xi = -\varphi$.*
(2) $\mathrm{h} = 0$.
(3) *M is Sasakian.*

On a Sasakian 3-manifold, the covariant derivative $\nabla\varphi$ is given by

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in \mathfrak{X}(M). \tag{2.3}$$

Let (T, N, B) be the Frenet frame field along γ. Then the Frenet frame satisfies the following *Frenet–Serret* equations:

$$\begin{cases} \nabla_T T = & \kappa N, \\ \nabla_T N = -\kappa T & \quad + \tau B, \\ \nabla_T B = & -\tau N, \end{cases} \tag{2.4}$$

where $\kappa = |\mathcal{T}(\gamma)| = |\nabla_T T|$ is the *geodesic curvature* of γ and τ its *geodesic torsion*.

2.2. Pseudo-Hermitian structure and Tanaka–Webster connection. For a three-dimensional contact Riemannian manifold $M = (M^3, \eta; \xi, \varphi, g)$, the tangent space T_pM of M at a point $p \in M$ can be decomposed as

$$T_pM = D_p \oplus \mathbb{R}\xi_p, \quad D_p = \{v \in T_pM \mid \eta(v) = 0\}$$

as the direct sum of linear subspaces. Then $D : p \longmapsto D_p$ defines a two-dimensional distribution orthogonal to ξ, which is called the *contact distribution*. We see that the restriction $J = \varphi|_D$ of φ to D defines an almost complex structure on D. Then the associated almost CR-structure of the contact Riemannian manifold M is given by the holomorphic subbundle

$$\mathcal{H} = \{X - iJX \mid X \in D\}$$

of the complexified tangent bundle $TM^{\mathbb{C}}$. Then we see that each fiber $\mathcal{H}_p$ is of complex dimension 1, $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$, and $D \otimes \mathbb{C} = \mathcal{H} \oplus \overline{\mathcal{H}}$. Furthermore, the associated almost CR-structure is always *integrable*, that is, the space $\Gamma(\mathcal{H})$ of all smooth sections of $\mathcal{H}$ satisfies the integrability condition:

$$[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H}).$$

For $\mathcal{H}$ we define the *Levi form* L by

$$L : \Gamma(D) \times \Gamma(D) \to \mathfrak{F}(M), \quad L(X, Y) = -d\eta(X, JY),$$

where $\mathfrak{F}(M)$ denotes the algebra of smooth functions on M. Then we see that the Levi form is Hermitian and positive definite. We call the pair (η, L) a *contact strongly pseudo-convex pseudo-Hermitian structure* on M.

Now, we recall the *Tanaka–Webster connection* [8, 10] on a contact strongly pseudo-convex pseudo-Hermitian manifold $M = (M, \eta, L)$ with the associated contact Riemannian structure (η, ξ, φ, g). The Tanaka–Webster connection $\hat{\nabla}$ is defined by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X, Y on M. Together with (2.2), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + A(X, Y), \tag{2.5}$$

where we have put

$$A(X, Y) = \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi \mathrm{h} X) - g(\varphi X + \varphi \mathrm{h} X, Y)\xi. \tag{2.6}$$

We see that the Tanaka–Webster connection $\hat{\nabla}$ has the torsion

$$\hat{T}(X, Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi \mathrm{h} X - \eta(X)\varphi \mathrm{h} Y.$$

In particular, for Sasakian manifolds, (2.6) and the above equation are reduced to:

$$\begin{aligned} A(X, Y) &= \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi, \\ \hat{T}(X, Y) &= 2g(X, \varphi Y)\xi. \end{aligned} \tag{2.7}$$

Furthermore, the following proposition was proved in [9].

PROPOSITION 2.2. *The Tanaka–Webster connection $\hat{\nabla}$ on a three-dimensional contact Riemannian manifold $M = (M^3; \eta, \varphi, \xi, g)$ is the unique linear connection satisfying the following conditions:*

(1) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(2) $\hat{\nabla} g = 0$, $\hat{\nabla}\varphi = 0$;
(3) $\hat{T}(X, Y) = -\eta([X, Y])\xi$, $X, \ Y \in \Gamma(D)$;
(4) $\hat{T}(\xi, \varphi Y) = -\varphi \hat{T}(\xi, Y)$, $Y \in \Gamma(D)$.

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3. Legendre curves in pseudo-Hermitian geometry

Let $\gamma : I \to M^3$ be a curve parameterized by arc-length in a contact Riemannian 3-manifold M^3. We may define a Frenet frame field (T, N, B) along γ with respect to the Tanaka–Webster connection $\hat{\nabla}$. This satisfies the following *Frenet–Serret equations* for $\hat{\nabla}$:

$$\begin{cases} \hat{\nabla}_T T = \hat{\kappa} N \\ \hat{\nabla}_T N = -\hat{\kappa} T + \hat{\tau} B \\ \hat{\nabla}_T B = -\hat{\tau} N \end{cases} \tag{3.1}$$

where $\hat{\kappa} = |\hat{\nabla}_T T|$ is the *pseudo-Hermitian curvature* of γ and $\hat{\tau}$ its *pseudo-Hermitian torsion*. A *pseudo-Hermitian helix* is a curve whose pseudo-Hermitian curvature and pseudo-Hermitian torsion are constants. In particular, curves with constant nonzero pseudo-Hermitian curvature and zero pseudo-Hermitian torsion are called *pseudo-Hermitian circles*. Note that *pseudo-Hermitian geodesics* are regarded as pseudo-Hermitian helices whose pseudo-Hermitian curvature and pseudo-Hermitian torsion are zero.

Blair and Baikoussis introduced the notion of Legendre curves in a contact Riemannian manifold. A one-dimensional integral submanifold in the contact subbundle is called a *Legendre curve* (see [2]).

3.1. Parallel pseudo-Hermitian mean curvature vector. The pseudo-Hermitian mean curvature vector field $\hat{H}$ of a curve γ in three-dimensional contact Riemannian manifolds is defined by

$$\hat{H} = \hat{\nabla}_{\dot{\gamma}} \dot{\gamma} = \hat{\kappa} N.$$

In particular, for a Legendre curve γ we get

$$\hat{H} = \hat{\nabla}_{\dot{\gamma}} \dot{\gamma} = \hat{\kappa} \varphi \dot{\gamma}. \tag{3.2}$$

Differentiating $\varphi \dot{\gamma}$ along γ for $\hat{\nabla}$, we get $\hat{\tau} = 0$ and this proves the following proposition.

PROPOSITION 3.1 [5, 6]. *If a nongeodesic curve in a three-dimensional contact Riemannian manifold for Tanaka–Webster connection $\hat{\nabla}$ is a Legendre curve, then $\hat{\tau} = 0$.*

For a curve γ in a three-dimensional Sasakian manifold M, from (2.5) and (2.7) we get

$$\hat{\nabla}_{\dot{\gamma}} \dot{\gamma} = \nabla_{\dot{\gamma}} \dot{\gamma} + 2\eta(\dot{\gamma}) \varphi \dot{\gamma}, \tag{3.3}$$

and for a Legendre curve γ in a three-dimensional Sasakian manifold M we can see that $\hat{\nabla}_{\dot{\gamma}} \dot{\gamma} = \nabla_{\dot{\gamma}} \dot{\gamma}$. So we have the following proposition.

PROPOSITION 3.2 [5, 6]. *For a Legendre curve γ in a three-dimensional Sasakian manifold M, γ is pseudo-Hermitian minimal if and only if it is minimal.*

First, we consider Legendre curves γ in a three-dimensional contact Riemannian manifold with respect to the Levi-Civita connection ∇. If we define a parallel mean curvature vector field by $\nabla^{\perp}_{\dot\gamma} H = 0$, then we get the following lemma.

LEMMA 3.3 [7]. *For Legendre curves γ in three-dimensional Sasakian manifolds (with respect to the Levi-Civita connection ∇), γ has a parallel mean curvature vector field if and only if it is minimal.*

PROOF. Using the Frenet–Serret equation for the Levi-Civita connection ∇,

$$\nabla_{\dot\gamma} H = \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = -\kappa^2\dot\gamma + \kappa'\varphi\dot\gamma + \kappa\xi.$$

Since γ has a parallel mean curvature vector field, we can see that $\kappa = 0$. The converse is straightforward. □

In pseudo-Hermitian geometry, we investigate the following definition.

DEFINITION 3.4. In three-dimensional contact Riemannian manifolds M^3 with respect to the Tanaka–Webster connection $\hat\nabla$, a vector field X normal to curve γ is said to be *pseudo-Hermitian parallel* if $\hat\nabla^{\perp}_{\dot\gamma} X = 0$.

On differentiating (3.2),

$$\hat\nabla_{\dot\gamma}\hat H = -\hat\kappa^2\dot\gamma + \hat\kappa'\varphi\dot\gamma. \tag{3.4}$$

Using (3.4) and Definition 3.4, we get that $\hat\kappa$ is a constant. Thus, from Proposition 3.1, we can see that $\hat\tau = 0$ for a Legendre curve in a three-dimensional contact Riemannian manifold. So we obtain the following theorem.

THEOREM 3.5. *For a Legendre curve γ in a three-dimensional Sasakian manifold M, γ is a curve with pseudo-Hermitian parallel mean curvature vector if and only if γ is a pseudo-Hermitian circle.*

3.2. Proper pseudo-Hermitian mean curvature vector field. For a curve γ in a three-dimensional contact Riemannian manifold with respect to the Tanaka–Webster connection $\hat\nabla$,

$$\hat\Delta\hat H = -\hat\nabla_{\dot\gamma}\hat\nabla_{\dot\gamma}\hat\nabla_{\dot\gamma}\dot\gamma,$$

where $\hat H$ is the pseudo-Hermitian mean curvature vector. Moreover, the Laplacian of the pseudo-Hermitian mean curvature vector in the normal bundle is defined by

$$\hat\Delta^{\perp}\hat H = -\hat\nabla^{\perp}_{\dot\gamma}\hat\nabla^{\perp}_{\dot\gamma}\hat\nabla^{\perp}_{\dot\gamma}\dot\gamma,$$

where $\hat\nabla^{\perp}$ denotes the normal connection in the normal bundle.

A curve γ in three-dimensional contact Riemannian manifold M is called a *curve with pseudo-Hermitian proper mean curvature vector field* if $\hat\Delta\hat H = \lambda\hat H$, where λ is a function. In particular, if $\hat\Delta\hat H = 0$ then it reduces to a *curve with pseudo-Hermitian harmonic mean curvature vector field.*

A curve γ is called a *curve with pseudo-Hermitian proper mean curvature vector field in the normal bundle* if $\hat{\Delta}^{\perp}\hat{H} = \lambda\hat{H}$, where $\hat{\Delta}^{\perp}\hat{H}$ is the Laplacian of the pseudo-Hermitian mean curvature vector in the normal bundle and λ is a function. In particular, if $\hat{\Delta}^{\perp}\hat{H} = 0$ then it reduces to a *curve with pseudo-Hermitian harmonic mean curvature vector field in the normal bundle* (see [7]).

Using (3.1), we have the following lemma.

LEMMA 3.6. *Let γ be a Legendre curve in a three-dimensional Sasakian manifold M. Then*

$$\hat{\nabla}_{\dot{\gamma}}\hat{\nabla}_{\dot{\gamma}}\hat{\nabla}_{\dot{\gamma}}\dot{\gamma} = -3\hat{\kappa}\hat{\kappa}'\dot{\gamma} + (\hat{\kappa}'' - \hat{\kappa}^3)\varphi\dot{\gamma}, \tag{3.5}$$

$$\hat{\nabla}^{\perp}_{\dot{\gamma}}\hat{\nabla}^{\perp}_{\dot{\gamma}}\hat{\nabla}^{\perp}_{\dot{\gamma}}\dot{\gamma} = \hat{\kappa}''\varphi\dot{\gamma}. \tag{3.6}$$

First, we study the pseudo-Hermitian mean curvature vector field.

THEOREM 3.7. *Let γ be a Legendre curve in a three-dimensional Sasakian manifold M. Then γ has pseudo-Hermitian proper mean curvature vector field if and only if γ is a minimal or pseudo-Hermitian circle satisfying $\hat{\kappa}^2 = \lambda$ for nonzero constant $\hat{\kappa}$.*

PROOF. From (3.5), the condition $\hat{\Delta}\hat{H} = \lambda\hat{H}$ gives

$$3\hat{\kappa}\hat{\kappa}'\dot{\gamma} - (\hat{\kappa}'' - \hat{\kappa}^3)\varphi\dot{\gamma} = \lambda\hat{\kappa}\varphi\dot{\gamma},$$

which implies that $\hat{\kappa} = 0$ or $\hat{\kappa}^2 - \lambda = 0$ for a nonzero constant $\hat{\kappa}$. The converse follows easily. □

In particular, for the case of $\lambda = 0$ we have the following corollary.

COROLLARY 3.8. *Let γ be a Legendre curve in a three-dimensional Sasakian manifold M. Then $\hat{\Delta}\hat{H} = 0$ if and only if γ is minimal.*

Next, we study pseudo-Hermitian mean curvature vector fields in the normal bundle.

THEOREM 3.9. *Let γ be a Legendre curve in a three-dimensional Sasakian manifold M and suppose that λ is a nonzero constant. Then γ has a pseudo-Hermitian proper mean curvature vector field in the normal bundle if and only if $\hat{\kappa}(s) = \cos(\pm\sqrt{\lambda}s + c)$, where c is a constant.*

PROOF. In view of (3.6), the condition $\hat{\Delta}^{\perp}\hat{H} = \lambda\hat{H}$ gives

$$-\hat{\kappa}''\varphi\dot{\gamma} = \lambda\hat{\kappa}\varphi\dot{\gamma},$$

which implies that $\kappa'' + \lambda\hat{\kappa} = 0$. Since λ is a nonzero constant, we find that $\hat{\kappa}(s) = \cos(\pm\sqrt{\lambda}s + c)$, where c is a constant. The converse is straightforward. □

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For the case of $\lambda = 0$, we have the following corollary.

COROLLARY 3.10. *Let γ be a Legendre curve in a three-dimensional Sasakian manifold M. Then $\hat{\Delta}^{\perp}\hat{H} = 0$ if and only if $\hat{\kappa}(s) = as + b$, where a and b are constants.*

PROOF. From (3.6), the condition $\hat{\Delta}^{\perp}\hat{H} = 0$ gives

$$\kappa''(s) = 0,$$

which implies that $\hat{\kappa}(s) = as + b$, where a and b are constants. The converse follows easily. □

Next, from the study of the Levi-Civita connection ∇, we can see that pseudo-Hermitian geometry is different from Riemmanian geometry. For Legendre curves γ in three-dimensional contact Riemannian manifolds with respect to the Levi-Civita connection ∇, if we define a harmonic mean curvature vector field by $\Delta^{\perp}H = 0$, then we get the following lemma.

LEMMA 3.11 [7]. *For Legendre curves γ in three-dimensional Sasakian manifolds (with respect to the Levi-Civita connection ∇), γ has harmonic a mean curvature vector field if and only if it is minimal.*

PROOF. Using the Frenet–Serret equation for the Levi-Civita connection ∇,

$$\Delta^{\perp}H = \nabla^{\perp}_{\dot{\gamma}}\nabla^{\perp}_{\dot{\gamma}}H = (\kappa'' + \kappa' - \kappa)\varphi\dot{\gamma} + 2\kappa'\xi.$$

Since γ has harmonic mean curvature vector field, we can see that $\kappa = 0$. □

3.3. On curves of $AW(k)$ type. In [1] Arslan and Ozgur studied curves of $AW(k)$ type. In this section, we investigate curves of $AW(k)$ type from the viewpoint of pseudo-Hermitian geometry and we find necessary and sufficient conditions for them.

DEFINITION 3.12. Let M be a three-dimensional contact Riemannian manifold with respect to the Tanaka–Webster connection $\hat{\nabla}$. Curves are of pseudo-Hermitian $AW(1)$ type if they satisfy

$$(\hat{\nabla}_{\dot{\gamma}}\hat{\nabla}_{\dot{\gamma}}\hat{\nabla}_{\dot{\gamma}}\dot{\gamma})^{\perp} = 0,$$

of pseudo-Hermitian $AW(2)$ type if they satisfy

$$(\hat{\nabla}_{\dot{\gamma}}\hat{\nabla}_{\dot{\gamma}}\hat{\nabla}_{\dot{\gamma}}\dot{\gamma})^{\perp} \wedge (\hat{\nabla}_{\dot{\gamma}}\hat{\nabla}_{\dot{\gamma}}\dot{\gamma})^{\perp} = 0,$$

and of pseudo-Hermitian $AW(3)$ type if they satisfy

$$(\hat{\nabla}_{\dot{\gamma}}\hat{\nabla}_{\dot{\gamma}}\hat{\nabla}_{\dot{\gamma}}\dot{\gamma})^{\perp} \wedge (\hat{\nabla}_{\dot{\gamma}}\dot{\gamma})^{\perp} = 0.$$

LEMMA 3.13. *Let γ be a Legendre curve in a three-dimensional Sasakian manifold M. Then we have following.*

(i) *γ is of the pseudo-Hermitian $AW(1)$ type if and only if $\hat{\kappa}(s) = \pm\sqrt{2}/(s + c)$, where c is a constant.*

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(ii) *For a function* $\hat{\kappa}$, γ *always satisfies the condition for the pseudo-Hermitian* $AW(2)$ *or* $AW(3)$ *type.*

In [7] Ozgur and Tripathi studied Legendre curves of $AW(k)$ type in Sasakian manifolds for the Levi-Civita connection in detail.

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