ARITHMETICAL FUNCTIONS OF A GREATEST COMMON DIVISOR, III. CESÀRO'S DIVISOR PROBLEM

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1. Introduction. Let $\sigma_t(n)$ denote the sum of the tth powers of the divisors of n, $\sigma(n) = \sigma_1(n)$. Also place

$$v(x) = x^2\{\log x + 2\gamma - \tfrac{1}{2} - \tfrac{1}{2}\zeta(2)\}, \quad \Delta(x) = \sum_{a,\, b \leqq x} \sigma((a, b)) - v(x), \tag{1.1}$$

where γ is Euler's constant, $\zeta(s)$ is the Riemann ζ-function and $x \geqq 2$. The function $\Delta(x)$ is the remainder term arising in the divisor problem for $\sigma((m, n))$. Cesàro proved originally [**1**], [**6**, p. 328] that $\Delta(x) = o(x^2 \log x)$. More recently in I [**2**, (3.14)] it was shown by elementary methods that $\Delta(x) = O(x^{3/2} \log x)$. This estimate was later improved to $O(x^{3/2})$ in II [**3**, (3.7)]. In the present paper (§ 3) we obtain a much more substantial reduction in the order of $\Delta(x)$, by showing that $\Delta(x)$ can be expressed in terms of the remainder term in the classical Dirichlet divisor problem. On the basis of well known results for this problem, it follows easily that $\Delta(x) = O(x^{4/3})$. The precise statement of the result for $\sigma((m, n))$ is contained in (3.2).

The analogous problem for $\sigma_2((m, n))$ is also considered in § 3. Place

$$\Delta'(x) = \sum_{a,\, b \leqq x} \sigma_2((a, b)) - \frac{x^3}{3}\{2\zeta(2) - \zeta(3)\}. \tag{1.2}$$

It was shown in [**2**, (3.13)], [**3**, (3.5)] that $\Delta'(x) = O(x^2 \log x)$. In this paper, we express $\Delta'(x)$ in terms of the remainder term in the divisor problem for $\sigma(n)$, obtaining as a consequence a material improvement in the order of $\Delta'(x)$. The main result in the case of $\sigma_2((m, n))$ is found in (3.9).

In § 4 we consider average values in two classes of functions generalizing $\sigma((m, n))$ and $\sigma_2((m, n))$, respectively. The results are analogous to those obtained for $\Delta(x)$ and $\Delta'(x)$ in § 3, and furnish improvements on estimates proved in [**2**, Theorem ($\alpha = 1$, $\alpha = 2$)], (also see [**3**, Theorem 4.1]). The corollaries of § 4 contain estimates for the special functions $\phi((m, n))$ and $\phi_2((m, n))$, where $\phi_t(n)$ denotes the generalized totient function.

The method of this paper is essentially a refinement of that employed in II.

2. Preliminary details. We collect in this section a number of known miscellaneous facts that will be needed in the later discussion. Denoting by $[x]$ the integral part of x, we place $\psi(x) = x - [x] - \tfrac{1}{2}$ and write

$$\rho(x) = \sum_{n \leqq \sqrt{x}} \psi\left(\frac{x}{n}\right), \quad \rho'(x) = \sum_{n \leqq x} \frac{1}{n} \psi\left(\frac{x}{n}\right). \tag{2.1}$$

The functions $\rho(x)$ and $\rho'(x)$ occur in the remainder terms of the average-value problems for $\tau(n) = \sigma_0(n)$ and $\sigma(n)$, respectively. In particular, we recall that [**8**, p. 16], [**5**, p. 37],

$$\delta(x) = -2\rho(x)+O(1), \quad \delta'(x) = -x\rho'(x)+O(x), \tag{2.2}$$

where $\delta(x)$ and $\delta'(x)$ are the " remainders ",

$$\delta(x) = \sum_{n\leqq x} \tau(n)-x(\log x+2\gamma-1), \quad \delta'(x) = \sum_{n\leqq x} \sigma(n)-\tfrac{1}{2}x^2\zeta(2).$$

It is also remarked that, if α and β are the least values such that for all $\varepsilon > 0$,

$$\rho(x) = O(x^{\alpha+\varepsilon}), \quad \rho'(x) = O\{(\log x)^{\beta+\varepsilon}\}, \tag{2.3}$$

then $\alpha \leqq 27/82$ [**4**], $\beta \leqq 4/5$ [**5**]. While these estimates have been improved, the exact values of α and β are still unknown. It is easily verified that $\alpha > 0$, $\beta \geqq 0$ (compare [**9**, p. 187]).

The following classical estimates will also be required. In particular [**8**, p. 15],

$$\sum_{n\leqq x} \frac{1}{n} = \log x+\gamma-\frac{\psi(x)}{x}+O\left(\frac{1}{x^2}\right); \tag{2.4}$$

moreover [**7**, (2), p. 26)]

$$\sum_{n\leqq x} \frac{1}{n^\alpha} = \zeta(\alpha)-\frac{1}{(\alpha-1)x^{\alpha-1}}+O\left(\frac{1}{x^\alpha}\right) \quad (\alpha > 1). \tag{2.5}$$

In addition, we note the elementary fact that

$$\sum_{n>x} \frac{\log n}{n^2} = O\left(\frac{\log x}{x}\right). \tag{2.6}$$

Adopting the notation

$$S_1^*(x) = \sum_{a,\, b\leqq x} \sigma((a, b)), \quad S_2^*(x) = \sum_{a,\, b\leqq x} \sigma_2((a, b)), \tag{2.7}$$

we remark [**3**, (3.9)] that

$$S_1^*(x) = \sum_{n\leqq \sqrt{x}} (2n-\tfrac{1}{2})\left[\frac{x}{n}\right]^2+\sum_{n\leqq\sqrt{x}} (n-\tfrac{1}{2})\left[\frac{x}{n}\right]-\tfrac{1}{2}[\sqrt{x}]^4-\tfrac{1}{2}[\sqrt{x}]^3; \tag{2.8}$$

moreover

$$S_2^*(x) = \tfrac{1}{3}\sum_{n\leqq x} (2n-1)\left[\frac{x}{n}\right]^3+\tfrac{1}{2}\sum_{n\leqq x} (2n-1)\left[\frac{x}{n}\right]^2+\sum_{n\leqq x} (2n-1)\left[\frac{x}{n}\right], \tag{2.9}$$

the latter relation following from [**3**, (3.8)] in conjunction with the identity

$$\sum_{a\leqq n} a^2 = (2n^3+3n^2+n)/6.$$

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The next lemma was proved first in II. Let $g(n)$ and $h(n)$ denote arbitrary arithmetical functions and place

$$f(n) = \sum_{d\delta=n} g(d)h(\delta), \tag{2.10}$$

$$k(n) = \sum_{d \mid n} h(d), \quad q(n) = \sum_{d \mid n} g(d)\mu\left(\frac{n}{d}\right), \tag{2.11}$$

where $\mu(n)$ denotes the Möbius function. Furthermore, let $K^*(x)$ denote the summatory function of $k((m, n))$, $K^*(x) = \sum_{a, b \leqq x} k((a, b))$. We have [3, (4. 2)]

LEMMA 2.1. *On the basis of the above notation*

$$\sum_{a, b \leqq x} f((a, b)) = \sum_{n \leqq x} q(n)K^*\left(\frac{x}{n}\right). \tag{2.12}$$

Finally, we mention the following estimate proved in II:

LEMMA 2.2 [3, Lemma 4.2]. *If $g(n)$ is bounded then $q(n) = O(n^{\xi})$ for all $\xi > 0$.*

3. The average order of $\sigma((m, n))$ and $\sigma_2((m, n))$. We consider first the function $\sigma((m, n))$.

THEOREM 3.1. *If $\rho(x)$ is defined by* (2.1), *then*

$$\Delta(x) = -4x\rho(x) + O(x \log x); \tag{3.1}$$

moreover, for all $\varepsilon > 0$,

$$\Delta(x) = O(x^{1+\alpha+\varepsilon}), \tag{3.2}$$

where α is defined by (2.3).

Proof. Denote by S_1, S_2, S_3, S_4 respectively, the four terms arising in (2.8). Applying (2.4) and (2.5), one obtains

$$S_1 = \sum_{n \leqq \sqrt{x}} (2n-\tfrac{1}{2})\left\{\frac{x}{n} - \left(\psi\left(\frac{x}{n}\right)+\tfrac{1}{2}\right)\right\}^2 = \sum_{n \leqq \sqrt{x}} (2n-\tfrac{1}{2})\left\{\frac{x^2}{n^2} - \frac{2x}{n}\left(\psi\left(\frac{x}{n}\right)+\tfrac{1}{2}\right)+O(1)\right\}$$

$$= 2x^2 \sum_{n \leqq \sqrt{x}} \frac{1}{n} - \tfrac{1}{2}x^2 \sum_{n \leqq \sqrt{x}} \frac{1}{n^2} - 4x \sum_{n \leqq \sqrt{x}} \left(\psi\left(\frac{x}{n}\right)+\tfrac{1}{2}\right) + x \sum_{n \leqq \sqrt{x}} \frac{1}{n}\left\{\psi\left(\frac{x}{n}\right)+\tfrac{1}{2}\right\} + O\left\{\sum_{n \leqq \sqrt{x}} n\right\}$$

$$= 2x^2\left\{\frac{\log x}{2} + \gamma - \frac{\psi(\sqrt{x})}{\sqrt{x}} + O\left(\frac{1}{x}\right)\right\} - \tfrac{1}{2}x^2\left\{\zeta(2) - \frac{1}{\sqrt{x}} + O\left(\frac{1}{x}\right)\right\}$$

$$-4x \sum_{n \leqq \sqrt{x}} \psi\left(\frac{x}{n}\right) - 2x\{\sqrt{x}+O(1)\} + O\left\{x \sum_{n \leqq \sqrt{x}} \frac{1}{n}\right\} + O(x),$$

from which it follows that

$$S_1 = x^2\{\log x + 2\gamma - \tfrac{1}{2}\zeta(2)\} - 4x\rho(x) - 2x^{3/2}\psi(\sqrt{x}) - \tfrac{3}{2}x^{3/2} + O(x \log x). \tag{3.3}$$

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As for S_2 we have

$$S_2 = \sum_{n \leqq \sqrt{x}} (n-\tfrac{1}{2})\left[\frac{x}{n}\right] = \sum_{n \leqq \sqrt{x}} (n-\tfrac{1}{2})\left(\frac{x}{n}+O(1)\right) = x(\sqrt{x}+O(1)) - \tfrac{1}{2}x \sum_{n \leqq \sqrt{x}} \frac{1}{n} + O\left\{\sum_{n \leqq x} n\right\},$$

from which one deduces that

$$S_2 = x^{3/2} + O(x \log x). \tag{3.4}$$

Regarding S_3, one obtains

$$S_3 = -\tfrac{1}{2}\{\sqrt{x} - (\psi(\sqrt{x})+\tfrac{1}{2})\}^4 = -\tfrac{1}{2}\{x^2 - 4x^{3/2}(\psi(\sqrt{x})+\tfrac{1}{2}) + O(x)\},$$

from which it follows that that

$$S_3 = -\tfrac{1}{2}x^2 + 2x^{3/2}\psi(\sqrt{x}) + x^{3/2} + O(x). \tag{3.5}$$

Also it is clear that

$$S_4 = -\tfrac{1}{2}x^{3/2} + O(x). \tag{3.6}$$

Since $S_1{}^*(x) = S_1 + S_2 + S_3 + S_4$, it follows from (3.3), (3.4), (3.5) and (3.6) that

$$S_1{}^*(x) = x^2\{\log x + 2\gamma - \tfrac{1}{2} - \tfrac{1}{2}\zeta(2)\} - 4x\rho(x) + O(x \log x), \tag{3.7}$$

which is equivalent to (3.1). (3.2) results immediately from (3.1) and (2.3), because $\alpha \geqq 0$. This proves the theorem.

Remark. We note that if, in calculating $\Delta(x)$, the third expressions obtained for S_1 and S_2 are used, then (by [**5**, Lemma 8]) the formula (3.1) is replaced by

$$\Delta(x) = -4x\rho(x) + x\rho'(x) + O(x), \tag{3.1a}$$

so that, by (2.3) and the fact that $\beta \geqq 0$,

$$\Delta(x) = -4x\rho(x) + O\{x(\log x)^{\beta+\varepsilon}\} \quad (\varepsilon > 0). \tag{3.1b}$$

However, this result leads to no improvement over (3.2).

We now consider $\sigma_2((m, n))$, proving

THEOREM 3.2.
$$\Delta'(x) = -2x^2\rho'(x) + O(x^2), \tag{3.8}$$

where $\rho'(x)$ *is defined by* (2.1); *moreover, for all* $\varepsilon > 0$,

$$\Delta'(x) = O\{x^2(\log x)^{\beta+\varepsilon}\}, \tag{3.9}$$

where β *is defined by* (2.3).

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Proof. Denote the three terms of (2.9) by T_1, T_2, T_3 respectively. Then by the estimates of § 2, one obtains

$$T_1 = \frac{1}{3}\sum_{n \leqq x}(2n-1)\left\{\frac{x}{n}-\left(\psi\left(\frac{x}{n}\right)+\frac{1}{2}\right)\right\}^3$$
$$= \frac{1}{3}\sum_{n \leqq x}(2n-1)\left\{\frac{x^3}{n^3}-\frac{3x^2}{n^2}\left(\psi\left(\frac{x}{n}\right)+\frac{1}{2}\right)+O\left(\frac{x}{n}\right)\right\}$$
$$= \frac{2x^3}{3}\sum_{n \leqq x}\frac{1}{n^2}-\frac{x^3}{3}\sum_{n \leqq x}\frac{1}{n^3}-2x^2\rho'(x)-x^2\sum_{n \leqq x}\frac{1}{n}+O(x^2),$$

from which it follows that

$$T_1 = \tfrac{1}{3}x^3\{2\zeta(2)-\zeta(3)\}-2x^2\rho'(x)-x^2\log x+O(x^2). \tag{3.10}$$

In the case of T_2 we have

$$T_2 = \frac{1}{2}\sum_{n \leqq x}(2n-1)\left\{\frac{x^2}{n^2}+O\left(\frac{x}{n}\right)\right\} = x^2\sum_{n \leqq x}\frac{1}{n}+O(x^2),$$

so that

$$T_2 = x^2\log x+O(x^2). \tag{3.11}$$

Also, evidently

$$T_3 = O(x^2). \tag{3.12}$$

Since $S_2^*(x) = T_1+T_2+T_3$, it follows from (3.10), (3.11) and (3.12) that

$$S_2^*(x) = \tfrac{1}{3}x^3\{2\zeta(2)-\zeta(3)\}-2x^2\rho'(x)+O(x^2), \tag{3.13}$$

which can be restated as (3.8). The result (3.9) is a consequence of (3.8), (2.3) and the fact that $\beta \geqq 0$. This completes the proof.

4. The general functions $f_1((m, n))$ and $f_2((m, n))$. As in I and II we define $f_t(n)$ by

$$f_t(n) = \sum_{d\delta = n} g(d)\,\delta^t. \tag{4.1}$$

It is noted, on the basis of Lemma 2.1 with $h(n) = n^t$, that

$$F_t^*(x) \equiv \sum_{a,\, b \leqq x} f_t((a, b)) = \sum_{n \leqq x} q(n)S_t^*\left(\frac{x}{n}\right), \tag{4.2}$$

where $S_t^*(x)$ is the summatory function of $\sigma_t((m, n))$. It follows then from (4.2) and the definition of $q(n)$ that

$$F_t^*(x) \equiv \sum_{d \leqq x} g(d) \sum_{\delta \leqq x/d} \mu(\delta)S_t^*\left(\frac{x}{d\delta}\right). \tag{4.3}$$

As in I and II we place

$$L(s, g) = \sum_{n=1}^{\infty}\frac{g(n)}{n^s} \quad (s > 1), \tag{4.4}$$

and denote its derivative by $L'(s, g)$. We now consider the average order of $f_t((m, n))$ in the

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cases $t = 1$ and $t = 2$, with a boundedness restriction on $g(n)$. It is convenient to introduce the following notation in considering $f_1((m, n))$,

$$C_1 = 2\gamma - \tfrac{1}{2} - \tfrac{1}{2}\zeta(2), \tag{4.5}$$

$$R(x) = \frac{x^2}{\zeta(2)}\left\{L(2, g)\left(\log x + C_1 - \frac{\zeta'(2)}{\zeta(2)}\right) + L'(2, g)\right\}. \tag{4.6}$$

We prove now

THEOREM 4.1. *If $g(n)$ is bounded, then*

$$F_1^*(x) = R(x) - 4x \sum_{n \leqq x} \frac{g(n)}{n} \rho\left(\frac{x}{n}\right) + O(x \log^3 x); \tag{4.7}$$

moreover, for all $\varepsilon > 0$,

$$F_1^*(x) = R(x) + O(x^{1+\alpha+\varepsilon}), \tag{4.8}$$

where α *is defined by* (2.3).

Proof. By (3.7) and (4.3) with $t = 1$, it follows that

$$F_1^* = S_1 + S_2 + S_3 + S_4 + S_5, \tag{4.9}$$

where

$$S_1 = x^2(\log x + C_1) \sum_{d \leqq x} \frac{g(d)}{d^2} \sum_{\delta \leqq x/d} \frac{\mu(\delta)}{\delta^2}, \tag{4.10}$$

$$S_2 = -x^2 \sum_{d \leqq x} \frac{g(d) \log d}{d^2} \sum_{\delta \leqq x/d} \frac{\mu(\delta)}{\delta^2}, \tag{4.11}$$

$$S_3 = -x^2 \sum_{d \leqq x} \frac{g(d)}{d^2} \sum_{\delta \leqq x/d} \frac{\mu(\delta) \log \delta}{\delta^2}, \tag{4.12}$$

$$S_4 = -4x \sum_{d \leqq x} \frac{g(d)}{d} \sum_{\delta \leqq x/d} \frac{\mu(\delta)}{\delta} \rho\left(\frac{x}{d\delta}\right), \tag{4.13}$$

$$S_5 = O\left\{x \sum_{d \leqq x} \frac{|g(d)|}{d} \sum_{\delta \leqq \sqrt{x/d}} \frac{|\mu(\delta)|}{\delta} \log\left(\frac{x}{d\delta}\right)\right\} + O\left\{\sum_{d \leqq x} \frac{|g(d)|}{d} \sum_{\delta \leqq x/d} \frac{|\mu(\delta)|}{\delta}\right\}. \tag{4.14}$$

By (4.10) one obtains

$$S_1 = x^2(\log x + C_1) \sum_{d \leqq x} \frac{g(d)}{d^2}\left\{\frac{1}{\zeta(2)} + O\left(\frac{d}{x}\right)\right\}$$

$$= \frac{x^2}{\zeta(2)}(\log x + C_1)\left\{L(2, g) + O\left(\frac{1}{x}\right)\right\} + O\left\{x \log x \sum_{d \leqq x} \frac{1}{d}\right\},$$

from which it follows that

$$S_1 = \frac{x^2(\log x + C_1)L(2, g)}{\zeta(2)} + O(x \log^2 x). \tag{4.15}$$

By (4.11) and (2.6) one deduces that

$$S_2 = -x^2 \sum_{d \leqq x} \frac{g(d) \log d}{d^2}\left\{\frac{1}{\zeta(2)} + O\left(\frac{d}{x}\right)\right\}$$

$$= \frac{x^2}{\zeta(2)}\left\{L'(2, g) + O\left(\sum_{d > x} \frac{\log d}{d^2}\right)\right\} + O\left\{x \sum_{d \leqq x} \frac{\log d}{d}\right\},$$

so that

$$S_2 = \frac{x^2}{\zeta(2)} L'(2, g) + O(x \log^2 x). \tag{4.16}$$

By (4.12) and (2.6) we have

$$S_3 = x^2 \sum_{d \leqq x} \frac{g(d)}{d^2}\left\{L'(2, \mu) + O\left(\frac{d}{x} \log \frac{x}{d}\right)\right\}$$

$$= -\frac{x^2\zeta'(2)}{\zeta^2(2)}\left\{L(2, g) + O\left(\frac{1}{x}\right)\right\} + O\left\{x \log x \sum_{d \leqq x} \frac{1}{d}\right\},$$

and therefore

$$S_3 = -\frac{x^2\zeta'(2)L(2, g)}{\zeta^2(2)} + O(x \log^2 x). \tag{4.17}$$

Placing $d\delta = n$ in (4.13) one may write, on the basis of (2.11),

$$S_4 = -4x \sum_{n \leqq x} \frac{q(n)}{n} \rho\left(\frac{x}{n}\right). \tag{4.18}$$

As for S_5, it follows from (4.14) that

$$S_5 = O\left(x \log x \sum_{d \leqq x} \frac{1}{d} \sum_{\delta \leqq x/d} \frac{1}{\delta}\right) = O(x \log^3 x), \tag{4.19}$$

and (4.7) results on combining (4.9), (4.15), (4.16), (4.17), (4.18) and (4.19).

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We deduce now (4.8), recalling first that $\alpha > 0$. By Lemma 2.2 and (2.3), if ξ is chosen such that $0 < \xi \leqq \alpha$, then for all $\varepsilon > 0$,

$$\sum_{n \leqq x} \frac{q(n)}{n} \rho\left(\frac{x}{n}\right) = O\left\{x^{\alpha+\varepsilon} \sum_{n \leqq x} \frac{1}{n^{1+\alpha+\varepsilon-\xi}}\right\} = O(x^{\alpha+\varepsilon}).$$

Hence (4.8) results from (4.7) and the theorem is proved.

Placing $g(n) = \mu(n)$ in (4.8), we obtain the following corollary.

COROLLARY 4.1. *For all* $\varepsilon > 0$,

$$\sum_{a,\, b \leqq x} \phi((a, b)) = \frac{x^2}{\zeta^2(2)}\left\{\log x + 2\gamma - \tfrac{1}{2} - \tfrac{1}{2}\zeta(2) - \frac{2\zeta'(2)}{\zeta(2)}\right\} + O(x^{1+\alpha+\varepsilon}). \tag{4.20}$$

It is convenient in considering $f_2((m, n))$ to write

$$C_2 = \tfrac{1}{3}\{2\zeta(2) - \zeta(3)\}, \quad R'(x) = \frac{x^3 L(3, g) C_2}{\zeta(3)}. \tag{4.21}$$

THEOREM 4.2. *If* $g(n)$ *is bounded, then*

$$F_2{}^*(x) = R'(x) - 2x^2 \sum_{n \leqq x} \frac{q(n)}{n^2} \rho'\left(\frac{x}{n}\right) + O(x^2); \tag{4.22}$$

if β *is defined as in* (2.3), *then for all* $\varepsilon > 0$,

$$F_2{}^*(x) = R'(x) + O\{x^2 (\log x)^{\beta+\varepsilon}\}. \tag{4.23}$$

Proof. By (4.2) with $t = 2$, in conjunction with (3.13), we may write

$$F_2{}^*(x) = T_1 + T_2 + T_3, \tag{4.24}$$

where

$$T_1 = C_2 x^3 \sum_{n \leqq x} \frac{q(n)}{n^3}, \tag{4.25}$$

$$T_2 = -2x^2 \sum_{n \leqq x} \frac{q(n)}{n^2} \rho'\left(\frac{x}{n}\right) \tag{4.26}$$

and where, by Lemma 2.2,

$$T_3 = O\left\{x^2 \sum_{d \leqq x} \frac{|q(d)|}{d^2}\right\} = O(x^2). \tag{4.27}$$

From (4.25) one obtains

$$T_1 = C_2 x^3 \sum_{n=1}^{\infty} \frac{q(n)}{n^3} + O\left\{x^3 \sum_{n > x} \frac{|q(n)|}{n^3}\right\},$$

so that by (2.11) and Lemma 2.2

$$T_1 = R'(x) + O(x^{1+\xi}), \quad 0 < \xi \leqq 1. \tag{4.28}$$

Combination of (4.24), (4.26), (4.27) and (4.28) leads to (4.22).

We recall that the quantity β defined by (2.3) is non-negative. Hence if ξ is any positive number less than 1, it follows by Lemma 2.2 that for all $\varepsilon > 0$,

$$\sum_{n \leqq x} \frac{q(n)}{n^2} \rho'\left(\frac{x}{n}\right) = O\left\{\sum_{n \leqq \sqrt{x}} \frac{1}{n^{2-\xi}}\left(\log \frac{x}{n}\right)^{\beta+\varepsilon}\right\} + O(1)$$

$$= O\left\{(\log x)^{\beta+\varepsilon} \sum_{n \leqq x} \frac{1}{n^{2-\xi}}\right\} = O\{(\log x)^{\beta+\varepsilon}\}. \tag{4.29}$$

Thus (4.23) results from (4.22), and the theorem is proved.

The case $g(n) = \mu(n)$ in (4.23) yields the following special result.

COROLLARY 4.2. *For all* $\varepsilon > 0$,

$$\sum_{a, b \leqq x} \phi_2((a, b)) = \frac{x^3}{3\zeta^2(3)}\{2\zeta(2) - \zeta(3)\} + O\{x^2(\log x)^{\beta+\varepsilon}\}. \tag{4.30}$$

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