INTEGRATED SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS WITH ADDITIVE NOISE

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We investigate the existence of a solution to the abstract stochastic evolution equation with additive noise:

\[ dX(t) = AX(t)\, dt + B\, dW(t), \quad X(0) = \xi, \]

in the case when \( A \) is the generator of an \( n \)-times integrated semigroup.

1. INTRODUCTION

Let \( H \) and \( U \) be real separable Hilbert spaces. We consider the stochastic differential equation

\[ dX(t) = AX(t)\, dt + B\, dW(t), \quad X(0) = \xi, \]

where \( A : D(A) \subset H \to H \) is a closed linear operator and \( B : U \to H \) is a bounded linear operator. \( W(\cdot) \) is an \( U \)-valued cylindrical Wiener process in a probability space \( (\Omega, \mathcal{F}, P) \) adapted to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) and \( \xi \) is an \( H \)-valued random variable.

Equation (1) was studied by many authors (see [2] and [3] and references therein) in the case when \( A \) is the generator of a \( C_0 \)-semigroup. The novelty of this note is that we study this equation in the case when the operator \( A \) is the generator of an \( n \)-times integrated semigroup. We prove the existence of a weak \( n \)-integrated solution to (1) and discuss the existence a continuous version of this solution. Finally, we use the stochastic wave equation to illustrate our results.

2. PRELIMINARIES

By \( H \)-valued random variable, we understand an \( H \)-valued mapping \( \xi : \Omega \to H \) which is measurable from \((\Omega, \mathcal{F})\) to \((H, \mathcal{B}(H))\), where \( \mathcal{B}(H) \) is the smallest \( \sigma \)-field containing all closed (or open) subsets of \( H \). A stochastic process \( X(\cdot) \) is said to be adapted to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) if, for any \( t \geq 0 \), \( X(t) \) is \( \{\mathcal{F}_t\} \)-measurable. A stochastic process \( X(\cdot) \) is called \( H \)-valued predictable if \( X : [0, \infty) \times \Omega \to H \) (or \( X : [0, T] \times \Omega \to H \) is
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$\mathcal{P}_\infty$-measurable (respectively $\mathcal{P}_T$-measurable), where $\mathcal{P}_\infty$ is a $\sigma$-field generated by sets of the form:

$(s, t] \times F, \ 0 \leq s < t, \ F \in \mathcal{F}_s$ and $\{0\} \times F, \ F \in \mathcal{F}_0$,

and $\mathcal{P}_T$ is the restriction of $\mathcal{P}_\infty$ to $[0, T]$.

We denote by $L^2(\Omega; H)$ the Banach space of all $H$-valued square integrable mappings endowed with the norm

$$\|X\|_2 := \left( E\left[ \|X\|^2 \right] \right)^{1/2},$$

and by $C_W([0,T];H)$ the Banach space of all mappings $X : [0, T] \to L^2(\Omega; H)$, that are continuous and adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, endowed with the norm

$$\|X(\cdot)\|_{C_W([0,T];H)} := \sup_{t \in [0,T]} \left( E\left[ \|X(t)\|^2 \right] \right)^{1/2}.$$

Let furthermore $\{e_k\}_{k \in \mathbb{N}}$ be a complete orthonormal system in $U$ and $\{\beta_k(\cdot)\}_{k \in \mathbb{N}}$ be a sequence of independent real Brownian motions on $(\Omega, \mathcal{F}, P)$ adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. For all $y \in U$ and $t \geq 0$ one can define the following random variables

$$\langle W(t), y \rangle = \sum_{k=1}^{\infty} \beta_k(t) \langle e_k, y \rangle,$$

which clearly belong to $L^2(\Omega)$. The formal sum

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k, \quad t \geq 0,$$

is called an $U$-valued cylindrical Wiener process. Note that the series in (2) is not convergent in $L^2(\Omega; U)$.

We now give a very brief summary of basic facts about integrated semigroups, which can be found, for example, in [1] and [6].

**Definition 1.** Let $n \in \mathbb{N}$. A one-parameter family of bounded linear operators $\{V_n(t) \in \mathcal{L}(H) : t \in [0, \infty)\}$ is called an $n$-times integrated exponentially bounded semigroup if the following conditions hold

(a) $\left( \frac{1}{(n-1)!} \right) \int_0^s [(s-r)^{n-1}V_n(t+r) - (t+s-r)^{n-1}V_n(r)] \, dr = V_n(t)V_n(s), \quad s, t \geq 0$;

(b) $V_n(t)$ is strongly continuous with respect to $t \geq 0$;

(c) $\exists C > 0, \ a \in \mathbb{R} : \|V_n(t)\| \leq Ce^{at}, \ t \geq 0$.

The semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0, \infty)\}$ is called non-degenerate if

$$\forall t \geq 0, \ V_n(t)x = 0 \Rightarrow x = 0.$$
If the semigroup is non-degenerate, then $V_n(0) = 0$ and the operator

$$R(\lambda) := \int_0^\infty \lambda^n e^{-\lambda t} V_n(t) \, dt, \quad \text{Re} \lambda > a$$

is invertible. The operator $A$ defined by

$$(\lambda I - A)^{-1} x = \int_0^\infty \lambda^n e^{-\lambda t} V_n(t)x \, dt, \quad x \in H$$

with the domain equal to the range of $(\lambda I - A)^{-1}$, is called the generator of $\{V_n(t) \in \mathcal{L}(H) : t \in [0, \infty)\}$.

**Proposition 1.** Let $A$ be a densely defined linear operator on $H$ with nonempty resolvent set. Then the following statements are equivalent:

1. $A$ is the generator of an $n$-times integrated semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0, \infty)\}$;
2. for any $x \in \mathcal{D}(A^{n+1})$ the Cauchy problem

$$u'(t) = Au(t), \quad t \geq 0, \quad u(0) = x,$$

has a unique solution $u(\cdot) \in C([0, \infty], \mathcal{D}(A)) \cap C^1([0, \infty], H)$ satisfying

$$\exists K > 0, \quad a \in \mathbb{R} : \|u(t)\| \leq K e^{at}\|x\|_{A^n},$$

where $\|x\|_{A^n} := \|x\| + \|Ax\| + \ldots + \|A^n x\|.$

In this case the solution of (3) has the form

$$u(t) = V_n^{(n)}(t)x, \quad x \in \mathcal{D}(A^{n+1}).$$

Assume that the operator $A$ in problem (1) generates an exponentially bounded $n$-times integrated semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0, \infty)\}$. We consider the stochastic convolution

$$W_n(t) := \int_0^t V_n(t-s)BdW(s) = \sum_{k=1}^\infty \int_0^t V_n(t-s)B_k d\beta_k(s).$$

The series in (4) is convergent in $L^2(\Omega, H)$ due to the following lemma, which is an obvious generalisation of the corresponding result for the generators of $C_0$-semigroups, which can be found in [2].

**Lemma 1.** Assume that $K(\cdot)x \in C([0, T]; H)$ for any $x \in H$, and that the linear operator

$$L_t x := \int_0^t K(s)BB^* K^*(s)x \, ds, \quad x \in H,$$
is of trace class:

$$\text{Tr } L_t = \sum_{k=1}^{\infty} \int_0^t \left\| K(s) B e_k \right\|^2_H ds < \infty.$$  

Then for all $t > 0$ the series

$$W_K(t) = \int_0^t K(t-s) B dW(s) = \sum_{k=1}^{\infty} \int_0^t K(t-s) B e_k d\beta_k(t)$$

is convergent on $L^2(\Omega; H)$ to a Gaussian random variable $W_K(t)$ with mean zero and covariance operator $L_t$. Moreover $W_K(\cdot)$ belongs to $C_w([0,T]; H)$ for any $T > 0$.

By Lemma 1, we also have that

$$\int_0^t \frac{(t-s)^n}{n!} B dW(s)$$

is a Gaussian random variable.

3. MAIN RESULTS

**Definition 2.** An $H$-valued predictable process $X(t)$ is said to be a weak $n$-integrated solution of (1) if the trajectories of $X(\cdot)$ are $P$-almost surely Bochner integrable and if for all $\nu \in D(A^*)$ and $t \in [0,T]$ the equality

$$(6) \quad \langle X(t), \nu \rangle = \left\langle \frac{t^n}{n!} \xi, \nu \right\rangle + \left\langle \int_0^t X(s) ds, A^* \nu \right\rangle + \left\langle \int_0^t \frac{(t-s)^n}{n!} B dW(s), \nu \right\rangle,$$

holds $P$-almost surely.

**Theorem 1.** Let $A$ be the generator of an $n$-times integrated semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$ and let the operator $L_t$, defined by (5) with $K = V_n$, be of trace class. Then

$$X(t) = V_n(t)\xi + \int_0^t V_n(t-s) B dW(s)$$

is a weak $n$-integrated solution of (1).

**Proof:** Without loss of generality assume that $\xi = 0$. We show that equation (6) is satisfied by

$$W_n(t) = \int_0^t V_n(t-s) B dW(s).$$

Fix $t \in [0,T]$ and let $\nu \in D(A^*)$. Note that

$$\int_0^t \left\langle A^* \nu, W_n(s) \right\rangle ds = \int_0^t \left\langle A^* \nu, \int_0^s \chi_{[0,t]}(r) V_n(s-r) B dW(r) \right\rangle ds.$$
Hence by the stochastic Fubini theorem, we have
\[
\int_0^t \langle A^* \nu, W_n(s) \rangle ds = \int_0^t \left\langle A^* \nu, \int_0^t \chi_{[0,s]}(r) V_n(s-r) B dW(r) \right\rangle ds \\
= \int_0^t \left\langle \int_0^t \chi_{[0,s]}(r) B^* V_n^*(r-s) A^* \nu ds, dW(r) \right\rangle \\
= \int_0^t \left\langle \int_0^t B^* V_n^*(s-r) A^* \nu ds, dW(r) \right\rangle.
\]

Since \( A \) generates an \( n \)-times integrated semigroup \( \{ V_n(t) \in \mathcal{L}(H) : t \in [0,\infty) \} \), \( A^* \) generates the \( n \)-times integrated semigroup \( \{ V_n^*(t) \in \mathcal{L}(H^*) : t \in [0,\infty) \} \), where \( H^* \) is the dual space of \( H \). Since \( A^* \) and \( V_n^*(t) \) commute, using the properties of the \( n \)-times integrated semigroup \( \{ V^*_n(t) \in \mathcal{L}(H^*) : t \in [0,\infty) \} \) we obtain
\[
\int_0^t \langle A^* \nu, W_n(s) \rangle ds = \int_0^t \left\langle \int_0^t \chi_{[0,s]}(r) B^* A^* V_n^*(r-s) \nu ds, dW(r) \right\rangle \\
= \int_0^t \left\langle \int_0^t \left( B^* \frac{d}{ds} V^*(s-r) \nu - B^* \frac{(s-r)^{n-1}}{(n-1)!} \nu \right) ds, dW(r) \right\rangle \\
= \int_0^t \left\langle B^* V^*(t-r) \nu - B^* \frac{(t-r)^n}{n!} \nu, dW(r) \right\rangle \\
= \left\langle \nu, \int_0^t V^*(t-r) B dW(r) \right\rangle - \left\langle \nu, \int_0^t \frac{(t-r)^n}{n!} B dW(r) \right\rangle.
\]

Therefore \( W_n(t) = \int_0^t V_n(t-s) B dW(s) \) is a weak \( n \)-integrated solution of (1).

However, the solution \( X \) does not necessarily have a continuous version. The purpose of our next discussion is to find conditions under which the solutions have continuous versions.

Let \( A \) be the generator of an exponentially bounded \( n \)-times integrated semigroup \( \{ V_n(t) \in \mathcal{L}(H) : t \in [0,\infty) \} \). Hence \( A \) is also the generator of an exponentially bounded \( (n+j) \)-times integrated semigroups \( \{ V_{n+j}(t) \in \mathcal{L}(H) : t \in [0,\infty) \} \) for \( j = 1, 2, \ldots \). In particular, \( A \) generates an exponentially bounded \( 2n \)-times integrated semigroup \( \{ V_{2n}(t) \in \mathcal{L}(H) : t \in [0,\infty) \} \). It is shown in [5], that those semigroups satisfy the relation
\[
V_{2n}(t+s) = V_n(t)V_n(s) + \sum_{j=0}^{n-1} \frac{1}{j!} (s^j V_{2n-j}(t) + t^j V_{2n-j}(s)). 
\]

Define
\[
W_{2n}(t) := \int_0^t V_{2n}(t-s) B dW(s) = \sum_{k=1}^{\infty} \int_0^t V_{2n}(t-s) B e_k d\beta_k(s).
\]

By Lemma 1, \( W_{2n}(t) \) is a Gaussian random variable with the law \( \mathcal{N}(0, L_t^{2n}) \), where
\[
L_t^{2n} x := \int_0^t V_{2n}(s) B B^* V_{2n}^*(s) x ds, \quad x \in H,
\]
given that \( L_t^{2n} \) is of trace class. We now show that \( W_{2n} \) has a continuous version. The following theorem is a generalisation of the corresponding result of Da Prato and Zabczyk [2] for generators of \( C_0 \)-semigroups.

**Theorem 2.** Assume that there is \( \alpha \in (0, 1/2) \) and \( T \in (0, \infty) \) such that

\[
\int_0^T s^{-2\alpha} \text{Tr} \left[ V_n(s) BB^* V_n^*(s) \right] ds = C_{\alpha,T}^n < \infty,
\]

and for \( j = 0, 1, 2, \ldots, n - 1 \),

\[
\int_0^T s^{-2\alpha} \text{Tr} \left[ V_{2n-j}(s) BB^* V_{2n-j}(s) \right] ds = C_{\alpha,T}^{2n-j} < \infty.
\]

Then \( W_{2n}(t) \) defined by (8) has a continuous version.

**Proof:** Using (7) we can write

\[
W_{2n}(t) = \int_0^t V_n(t - \sigma + \sigma - s) BdW(s) \\
= \int_0^t V_n(t - \sigma) V_n(\sigma - s) BdW(s) \\
+ \int_0^t \left( \sum_{j=0}^{n-1} \frac{1}{j!}(\sigma - s)^j V_{2n-j}(t - \sigma) + (t - \sigma)^j V_{2n-j}(\sigma - s) \right) BdW(s).
\]

Using the factorisation formula

\[
\frac{\pi}{\sin (\pi \alpha)} = \int_s^t (t - \sigma)^{-\alpha}(\sigma - s)^{-\alpha} d\sigma, \quad \alpha \in (0, 1), \quad 0 \leq s \leq t,
\]

we obtain

\[
W_{2n}(t) = \frac{\sin (\pi \alpha)}{\pi} \int_s^t (t - \sigma)^{-\alpha}(\sigma - s)^{-\alpha} d\sigma \int_0^t V_n(t - \sigma) V_n(\sigma - s) BdW(s) \\
+ \frac{\sin (\pi \alpha)}{\pi} \int_s^t (t - \sigma)^{-\alpha}(\sigma - s)^{-\alpha} d\sigma \sum_{j=0}^{n-1} \frac{1}{j!} (\sigma - s)^j V_{2n-j}(t - \sigma) BdW(s) \\
+ \frac{\sin (\pi \alpha)}{\pi} \int_s^t (t - \sigma)^{-\alpha}(\sigma - s)^{-\alpha} d\sigma \sum_{j=0}^{n-1} \frac{1}{j!} (t - \sigma)^j V_{2n-j}(\sigma - s) BdW(s).
\]

By the stochastic Fubini theorem, we have

\[
W_{2n}(t) = \frac{\sin (\pi \alpha)}{\pi} \int_0^t \int_0^t \frac{1}{(t - s)^{\alpha} - s^{-\alpha}} V_n(t - \sigma)(t - \sigma)^{-\alpha} V_n(\sigma - s)(\sigma - s)^{-\alpha} BdW(s) d\sigma
\]

\[
+ \frac{\sin (\pi \alpha)}{\pi} \sum_{j=0}^{n-1} \frac{1}{j!} \int_0^t V_{2n-j}(t - \sigma)(t - \sigma)^{-\alpha} \int_0^\sigma (\sigma - s)^j BdW(s) d\sigma \\
+ \frac{\sin (\pi \alpha)}{\pi} \sum_{j=0}^{n-1} \frac{1}{j!} \int_0^t (t - \sigma)^{j+\alpha-1} \int_0^\sigma V_{2n-j}(\sigma - s)(\sigma - s)^{-\alpha} BdW(s) d\sigma.
\]
Writing
\[ U_n(\sigma) = \int_{0}^{\sigma} V_n(\sigma - s)(\sigma - s)^{-\alpha}BdW(s), \]
and for \( j = 0, 1, 2, \ldots, n - 1 \)
\[ U_j(\sigma) = \int_{0}^{\sigma} (\sigma - s)^{j-\alpha}BdW(s), \]
\[ U_{2n-j}(\sigma) = \int_{0}^{\sigma} V_{2n-j}(\sigma - s)(\sigma - s)^{-\alpha}BdW(s), \]
and
\[ P_n(t) = \frac{\sin \left( \frac{\pi \alpha}{\pi} \right)}{\pi} \int_{0}^{t} V_n(t - \sigma)(t - \sigma)^{\alpha-1}U_n(\sigma) d\sigma, \]
\[ P_{2n-j}(t) = \frac{\sin \left( \frac{\pi \alpha}{\pi} \right)}{\pi} \int_{0}^{t} \frac{1}{j!} V_{2n-j}(t - \sigma)(t - \sigma)^{\alpha-1}U_j(\sigma) d\sigma, \]
\[ P_j(t) = \frac{\sin \left( \frac{\pi \alpha}{\pi} \right)}{\pi} \int_{0}^{t} \frac{1}{j!}(t - \sigma)^{j+\alpha-1}U_{2n-j}(\sigma) d\sigma, \]
we can write \( W_{2n}(t) \) as
\[ W_{2n}(t) = P_n(t) + \sum_{j=0}^{n-1} P_{2n-j}(t) + \sum_{j=0}^{n-1} P_j(t). \]
As in Lemma 1 \( U_n(\sigma) \) is a Gaussian random variable \( \mathcal{N}(0, S_n^\alpha) \) for all \( \sigma \in [0, T] \), where
\[ S_n^\alpha x := \int_{0}^{\sigma} s^{-2\alpha}V_n(s)BB^*V^*(s)x ds. \]
Accordingly, for all \( j = 0, 1, 2, \ldots, n - 1, \) \( U_{2n-j}(\sigma) \) and \( U_j(\sigma) \) are Gaussian random variables \( \mathcal{N}(0, S_{2n-j}^\alpha) \) and \( \mathcal{N}(0, S_j^\alpha) \) respectively, where
\[ S_{2n-j}^\alpha x := \int_{0}^{\sigma} s^{-2\alpha}V_{2n-j}BB^*V_{2n-j}^*(s)x ds \]
\[ S_j^\alpha x := \int_{0}^{t} s^{2j-2\alpha}BB^*x ds. \]
By (9), for any \( m > 0 \), there exists a constant \( D_{m,a}^n \) such that for all \( \sigma \in [0, T] \) we have
\[ E\left[ \|U_n(\sigma)\|^{2m} \right] \leq D_{m,a}^n \sigma^m. \]
By (10), for \( j = 0, 1, 2, \ldots, n - 1 \), there exist constants \( D_{m,a}^{2n-j} \) and \( D_{m,a}^j \) such that for all \( \sigma \in [0, T] \) we have
\[ E\left[ \|U_{2n-j}(\sigma)\|^{2m} \right] \leq D_{m,a}^{2n-j} \sigma^m, \]
\[ E\left[ \|U_j(\sigma)\|^{2m} \right] \leq D_{m,a}^j \sigma^m. \]
This implies
\[ \int_0^T E \left[ \left\| U_n(\sigma) \right\|^{2m} \right] d\sigma \leq \frac{D_{m,\alpha}^n T^{m+1}}{m+1}, \]
and for \( j = 0, 1, 2, \ldots, n - 1 \)
\[ \int_0^T E \left[ \left\| U_{2n-j}(\sigma) \right\|^{2m} \right] d\sigma \leq \frac{D_{m,\alpha}^{2n-j} T^{m+1}}{m+1}, \]
\[ \int_0^T E \left[ \left\| U_j(\sigma) \right\|^{2m} \right] d\sigma \leq \frac{D_{m,\alpha}^j T^{m+1}}{m+1}. \]
Therefore \( U_n(\cdot)\omega, U_{2n-j}(\cdot)\omega \) and \( U_j(\cdot)\omega \) are in \( L^{2m}(\mathbb{R}; H) \) for almost all \( \omega \in \Omega \) and \( j = 0, 1, 2, \ldots, n - 1 \). Furthermore, by Hölder's inequality and taking into account the exponential boundedness of \( V_1(t) \) we have
\[
\left\| P_n(t) \right\| \leq \frac{M_T}{\pi^j} \left( \int_0^t [(t - \sigma)^{a-1}]^{2m/(2m-1)} d\sigma \right)^{(2m-1)/2m} \left\| U_n \right\|_{L^{2m}(\mathbb{R}; H)}
=
\frac{M_T}{\pi^j} \left( \frac{2m - 1}{2m\alpha - 1} \right)^{(2m-1)/2m} \left\| U_n \right\|_{L^{2m}(\mathbb{R}; H)},
\]
where \( M_T = \sup_{t \in [0,T]} \left\| V_1(t) \right\| \). Accordingly, for all \( j = 0, 1, 2, \ldots, n - 1 \) we have
\[
\left\| P_{2n-j}(t) \right\| \leq \frac{M_T^j}{\pi^j} \left( \int_0^t [(t - \sigma)^{a-1}]^{2m/(2m-1)} d\sigma \right)^{(2m-1)/2m} \left\| U_{2n-j} \right\|_{L^{2m}(\mathbb{R}; H)}
=
\frac{M_T^j}{\pi^j} \left( \frac{2m - 1}{2m\alpha - 1} \right)^{(2m-1)/2m} \left\| U_{2n-j} \right\|_{L^{2m}(\mathbb{R}; H)},
\]
where \( M_T^j = \sup_{t \in [0,T]} \left\| V_{2n-j}(t) \right\| \), and furthermore
\[
\left\| P_j(t) \right\| \leq \frac{1}{\pi^j} \left( \int_0^t [(t - \sigma)^{j+\alpha-1}]^{2m/(2m-1)} d\sigma \right)^{(2m-1)/2m} \left\| U_{2n-j} \right\|_{L^{2m}(\mathbb{R}; H)}
=
\frac{1}{\pi^j} \left( \frac{2m - 1}{2m\alpha - 1} \right)^{(2m-1)/2m} \left\| U_{2n-j} \right\|_{L^{2m}(\mathbb{R}; H)}.
\]
Hence \( P_n(\cdot)\omega \in C([0,T]; H) \) for almost all \( \omega \in \Omega \) and for all \( j = 0, 1, 2, \ldots, n - 1 \), \( P_{2n-j}(\cdot)\omega, P_j(\cdot)\omega \in C([0,T]; X) \) for almost all \( \omega \in \Omega \). Thus
\[
W_{2n}(\cdot)\omega = \left( P_n + \sum_{j=0}^{n-1} P_{2n-j} + \sum_{j=0}^{n-1} P_j \right)(\cdot)\omega \in C([0,T]; H)
\]
for almost all \( \omega \in \Omega \) and (11) defines a continuous version of \( W_{2n} \).

**Corollary 1.** Let \( A \) be the generator of an \( n \)-times integrated semigroup \( \{V_n(t) \in \mathcal{L}(H); t \in \mathbb{R} \} \) and all the assumptions in Theorem 2 hold. Then
\[
X(t) = V_{2n}(t) \xi + \int_0^t V_{2n}(t-s) BdW(s)
\]
is a weak \( 2n \)-integrated solution of (1) which has a continuous version.
4. Example

Consider the stochastic wave equation

\[ dY_t(t,x) = \frac{d^2}{dx^2} Y(t,x) \, dt + dW(t,x), \quad t \in [0,T], \ x \in \Omega = (0,1), \]

\[ Y(t,0) = Y(t,1) = 0, \quad t \in [0,T], \]

\[ Y(0,x) = Y_0(x), \ Y'_t(0,x) = Y_1(x), \ x \in \Omega, \]

where \( dW(t,x) \) is white noise. Define the operator \( A = \frac{d^2}{dx^2} \) in \( L^2(\Omega) \) with the domain

\[ D(A) = H^2(\Omega) \cap H^1_0(\Omega), \]

where \( H^2(\Omega) \) and \( H^1_0(\Omega) \) are the classical Sobolev spaces. The operator \(-A\) has a self-adjoint compact inverse and therefore its spectrum consists of discrete eigenvalues. Eigenfunctions and eigenvalues of \(-A\) can be obtained by solving

\[ \frac{d^2 e_k}{dx^2} = -\mu_k e_k, \quad e_k(0) = e_k(1) = 0, \ k \in \mathbb{N}, \]

which gives

\[ \mu_k = k^2 \pi^2 > 0, \ e_k = \sqrt{2} \sin k\pi x, \ k \in \mathbb{N}. \]

Note that \( \{e_k\}_{k=1}^{\infty} \) forms an orthonormal basis in \( L^2(\Omega) \). Denote \( L^2(\Omega) =: U, \ L^2(\Omega) \times L^2(\Omega) =: H \), and let \( W(\cdot) \) be a \( U \)-valued cylindrical Wiener process. Setting

\[ X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}, \ Y(t,x) = \begin{pmatrix} Y(t,x) \\ Y'_t(t,x) \end{pmatrix}, \ X_0 = \begin{pmatrix} Y_0(x) \\ Y_1(x) \end{pmatrix}, \]

where \( X(t), X_0 \in H \), we can rewrite the wave equation in the form (1):

\[ dX(t) = AX(t) \, dt + BdW(t), \quad X(0) = X_0. \]

The operator \( A \) is defined by

\[ AX = \begin{pmatrix} X_2 \\ A X_1 \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \]

\[ D(A) = D(A) \times L^2(\Omega) \subset H, \]

and \( B \in \mathcal{L}(U,H) \) is defined by

\[ Bu = \begin{pmatrix} 0 \\ u \end{pmatrix}. \]
Then (see, for example, [4]) $A$ generates an exponentially bounded non-degenerate 1-time integrated semigroup

$$V(t) = \left( \begin{array}{c} S(t) \\ C(t) - I \end{array} \right) \int_0^t S(s) \, ds, \quad t \geq 0,$$

on $H$. Here $C$ and $S$ are cosine and sine operator-functions defined by

$$C(t) := \sum_{k=1}^{\infty} \cos (\sqrt{\mu_k}t) \, v_k e_k, \quad S(t) := \sum_{k=1}^{\infty} \frac{\sin (\sqrt{\mu_k}t)}{\sqrt{\mu_k}} \, v_k e_k,$$

where $v \in L^2(\Omega)$ and $v_k = \langle v, e_k \rangle_{L^2(\Omega)}$.

Note that if we consider a smaller space $H_1 := H_0^1(\Omega) \times L^2(\Omega)$, then the operator $A$ with the domain $D(A) = D(A) \times H_0^1(\Omega) \subset H_1$, is the generator of a $C_0$-semigroup on $H_1$.

Now, since we have that

$$\sum_{k=1}^{\infty} \int_0^t \|V(s)Be_k\|^2 \, ds \leq C(T) \sum_{k=1}^{\infty} \frac{1}{\mu_k} < \infty,$$

then the process

$$X(t) = V(t)\xi + \int_0^t V(t-s)BdW(s)$$

is a weak 1-integrated solution of (1), that is,

$$\langle X(t), \nu \rangle = \langle tX_0, \nu \rangle + \left\langle \int_0^t X(s) \, ds, A^* \nu \right\rangle + \left\langle \int_0^t (t-s)BdW(s), \nu \right\rangle, \quad \nu \in D(A^*).$$

REFERENCES