

# DUALITY FOR CLANS: AN EXTENSION OF GABRIEL–ULMER DUALITY

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**Abstract.** *Clans* are categorical representations of generalized algebraic theories that contain more information than the finite-limit categories associated to the locally finitely presentable categories of models via Gabriel–Ulmer duality. Extending Gabriel–Ulmer duality to account for this additional information, we present a duality theory between clans and locally finitely presentable categories equipped with a weak factorization system of a certain kind.

**§1. Introduction.** *Gabriel–Ulmer duality* [16] is a contravariant biequivalence

$$\mathbf{FL} \overset{\text{op}}{\simeq} \mathbf{LFP}$$

between the 2-category  $\mathbf{FL}$  of *small finite-limit categories*, and the 2-category  $\mathbf{LFP}$  of *locally finitely presentable categories*, i.e., locally small cocomplete categories admitting a dense set of compact (a.k.a. finitely presentable) objects. The duality assigns to every small finite-limit category  $\mathcal{C}$  the category  $\mathbf{FL}(\mathcal{C}, \mathbf{Set})$  of finite-limit preserving functors to  $\mathbf{Set}$ , and conversely it associates to every locally finitely presentable  $\mathfrak{X}$  the category  $\mathbf{LFP}(\mathfrak{X}, \mathbf{Set})$  of finitary right adjoints to  $\mathbf{Set}$ , which is equivalent to the opposite of the full subcategory  $\mathbf{comp}(\mathfrak{X}) \subseteq \mathfrak{X}$  of compact objects<sup>1</sup>.

We view Gabriel–Ulmer duality as a *theory-model duality*: small finite-limit categories  $\mathcal{C}$  are viewed as theories (which we call “finite-limit theories”), and—in the spirit of Lawverian functorial semantics [27]—the functor category  $\mathbf{FL}(\mathcal{C}, \mathbf{Set})$  is viewed as the category of *models* of the finite-limit theory  $\mathcal{C}$ .

It is well known that finite-limit theories are equally expressive as various syntactically defined classes of theories, including

- (1) Freyd’s *essentially algebraic theories* [15], which permit a controlled form of partiality,
- (2) Cartmell’s *generalized algebraic theories (GATs)* [12, 13], which extend algebra by “dependent sorts,
- (3) Johnstone’s *cartesian theories* [25, Definition D1.3.4], which permit a limited form of existential quantification, and

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<sup>1</sup>Strictly speaking we have to choose a small category which is *equivalent* to  $\mathbf{comp}(\mathfrak{X})^{\text{op}}$ , since the latter is only *essentially small* in general.



- (4) Palmgren–Vickers’ *Partial Horn theories* [34], which are based on a calculus of first order logic with partial terms and also admit relation symbols,

in the sense that for any theory  $\mathbb{T}$  from one of these classes, the category  $\mathbb{T}\text{-Mod}$  of models is locally finitely presentable, and conversely for every locally finitely presentable category  $\mathfrak{X}$  there exists a theory from that class whose category of models is equivalent to  $\mathfrak{X}$ . While from a certain perspective this means that the classes (1)–(4) of theories are all equivalent to finite-limit theories, the syntactic representations of theories contain additional information that is not reflected in the categories of models, nor in the finite-limit theories. This “abstracting away” of syntactic details is typically viewed as a *strength* of the categorical/functorial approach, and indeed in mathematical practice we no more want to distinguish between the classical axiomatization of groups and Higman–Neumann’s axiomatization in terms of one operation and one equation [20], than we want to distinguish between the symmetric group  $S_3$  and the dihedral group  $D_3$ .

However, it turns out that in the case of GATs, the theories contain additional information that is not reflected in the corresponding finite-limit category, but nevertheless goes beyond mere syntactic details. This information is related to the structure of *sort dependency* in the theories, and we show here that it is reflected by certain weak factorization systems on the l.f.p. categories of models. For example, the 2-sorted theory of graphs

$$\begin{aligned} &\vdash V \\ &\vdash E \\ x : E &\vdash s(x) : V \\ x : E &\vdash t(x) : V \end{aligned}$$

with sorts  $V$  and  $E$  and source and target operations  $s$  and  $t$ , has the same category of models as the dependently sorted theory

$$\begin{aligned} &\vdash V \\ x\ y : V &\vdash E(x, y) \end{aligned}$$

with a base sort  $V$  of vertices and a dependent sort  $E$  of edges (and *no* operations). But the syntactic categories of the theories are different, and this is reflected by different weak factorization systems on the categories of models: in the first case, the w.f.s. is cofibrantly generated by the initial inclusions  $\emptyset \hookrightarrow (\bullet)$  and  $\emptyset \hookrightarrow (\bullet \rightarrow \bullet)$  of the free graphs on one vertex and one edge, respectively, whereas in the second case the w.f.s. is generated by the inclusions  $\emptyset \hookrightarrow (\bullet)$  and  $(\bullet \bullet) \hookrightarrow (\bullet \rightarrow \bullet)$ . The non-trivial domain of the second generator reflects the dependency of the sort of edges on the sort of vertices.

Concretely, the present work gives in Theorem 6.19 a duality

$$\text{Clan}_{\text{cc}} \overset{\text{op}}{\simeq} \text{ClanAlg} \quad (1.1)$$

of 2-categories which extends Gabriel–Ulmer duality by incorporating this additional structure. On the right we have the 2-category of *clan-algebraic categories*, which are locally finitely presentable categories equipped with a well-behaved kind of weak factorization system (Definition 6.1), while on the left we have a 2-category

of Cauchy-complete *clans* (Definition 2.1). These are categorical representations of GATs which can be viewed as a non-strict variant of Cartmell’s *contextual categories* (Definition 10.2), and are given by small categories equipped with a class of “display maps” representing type families, admitting certain (but not all) finite limits.

Besides extending Gabriel–Ulmer duality, the duality (1.1) recovers Adámek–Rosický–Vitale’s duality between *algebraic theories* and *algebraic categories* [3, Theorem 9.15] as a special case, and the latter duality was in fact inspirational for the present work. See Remark 6.20(a).

**1.1. Structure of the paper.** §2 introduces clans (Definition 2.1), the category of models of a clan  $\mathcal{T}\text{-Mod}$  (Definition 2.6), and the extension–full weak factorization system on models (Definition 2.10).

§3 gives a characterization of  $\mathcal{T}\text{-Mod}$  as a kind of cocompletion of  $\mathcal{T}^{\text{op}}$  (Theorem 3.3), and uses this to give presentations of slice categories  $\mathcal{T}\text{-Mod}/A$ , and of coslice categories  $H(\Gamma)/\mathcal{T}\text{-Mod}$  under representable models, as categories of models of derived clans (Propositions 3.5 and 3.6).

§4 introduces in Definition 4.1 the auxiliary notion of  $(E, F)$ -category (a l.f.p. category with a w.f.s.  $(E, F)$ ), and shows that the mapping  $\mathcal{T} \mapsto \mathcal{T}\text{-Mod}$  gives rise to a contravariant 2-functor from clans to  $(E, F)$ -categories which admits a left biadjoint (Proposition 4.6).

§5 shows that this biadjunction is idempotent, and that its fixed points in clans are precisely the *Cauchy complete* clans (Definition 5.1). For this we use the notion of *flat model* (Definition 5.3), and the *fat small object argument*, a Corollary of which we state in Corollary 5.5, but whose systematic treatment we defer to Appendix 11. Lemma 5.7 is an argument about compact objects in coslice categories which was not found in the literature.

§6 characterizes the fixed points of the biadjunction among  $(E, F)$ -categories as *clan-algebraic categories*, which are  $(E, F)$ -categories satisfying a density and an exactness condition (Definition 6.1). The characterization is given by Theorems 6.2 and 6.18, where the proof of the latter requires a quite lot of machinery including a Reedy-like resolution argument. As a consequence, we obtain our main result in Theorem 6.19. As an application, §6.1 gives additional clan-algebraic w.f.s.s on the example  $\text{Cat}$ , which by the duality result correspond to additional clan-representations of  $\text{Cat}$ .

§7 contains a common counterexample to two natural questions about clan-algebraic w.f.s.s, and §8 discusses  $\infty$ -models of clans in higher types.

Appendix 9 contains basic facts about locally finitely presentable categories, weak factorization systems, and Quillen’s small-object argument, and Appendix 10 is an informal introduction to Cartmell’s generalized algebraic theories.

Finally, Appendix 11 contains a careful development of the *fat small object argument* for clans. The fat small object is a variant of Quillen’s small object argument due to Makkai, Rosický, and Vokrinek [33] (based on ideas by Lurie), which allows a more fine grained analysis of the process of saturating a class of maps. We use it to show that *0-extensions are flat* (Corollary 11.9), which is needed in Sections 5 and 6.

## §2. Clans.

DEFINITION 2.1. A *clan* is a small category  $\mathcal{T}$  with a distinguished class  $\mathcal{T}_\dagger$  of arrows called *display maps*, such that:

- (i) Pullbacks of display maps along arbitrary maps exist and are display maps, i.e., if  $p : \Gamma' \rightarrow \Gamma$  is a display map and  $s : \Delta \rightarrow \Gamma$  is arbitrary, then there exists a pullback square

$$\begin{array}{ccc} \Delta' & \xrightarrow{s'} & \Gamma' \\ q \downarrow & \lrcorner & \downarrow p \\ \Delta & \xrightarrow{s} & \Gamma \end{array} \quad (2.1)$$

where  $q$  is a display map.

- (ii) Isomorphisms and compositions of display maps are display maps.
- (iii)  $\mathcal{T}$  has a terminal object, and terminal projections are display maps.

A *clan morphism* is a functor between clans which preserves display maps, pullbacks of display maps, and the terminal object. We write  $\mathbf{Clan}$  for the 2-category of clans, clan-morphisms, and natural transformations.

REMARKS 2.2. (a) Definition 2.1 (apart from the smallness condition), and the term “display map”, were introduced by Taylor in his thesis [41, §4.3.2], the explicit link to Cartmell’s work was made in his textbook [42, Chapter VIII]. The name “clan” is due to Joyal [26, Definition 1.1.1].

- (b) Following Cartmell, we use the arrow symbol  $\rightarrow$  for display maps.
- (c) We have defined clans to be *small* by default, since this fits with our point of view of clans as theories, and makes the duality theory work.

However, it is also reasonable to consider non-small, “semantic” clans, and we will mention them occasionally (e.g., in Example 2.3(c) below), using the term *large clan* in this case.

EXAMPLES 2.3. (a) Small finite-limit categories can be viewed as clans where *all* morphisms are display maps. We call such clans *finite-limit clans*.

- (b) Small finite-product categories can be viewed as clans where the display maps are the morphisms that are (isomorphic to) product projections. We call such clans *finite-product clans*.

- (c) Kan is the *large* clan whose underlying category is the full subcategory of the category  $[\Delta^{\text{op}}, \text{Set}]$  of simplicial sets on *Kan complexes*, and whose display maps are the *Kan fibrations*.

- (d) The syntactic category of every *generalized algebraic theory* in the sense of Cartmell [12, 13] is a clan. This is explained in §10, and we discuss the example of the clan for categories in greater detail in §2.2 below.

Since it seems to lead to a more readable exposition, we introduce explicit notation and terminology for the dual notion.

DEFINITION 2.4. A *coclans* is a small category  $\mathcal{C}$  with a distinguished class  $\mathcal{C}_\dagger$  of arrows called *codisplay maps* satisfying the dual axioms of clans. The 2-category  $\mathbf{CoClan}$  of coclans is defined dually to that of clans, i.e.,

$$\mathbf{CoClan}(\mathcal{C}, \mathcal{D}) = \mathbf{Clan}(\mathcal{C}^{\text{op}}, \mathcal{D}^{\text{op}})^{\text{op}}$$

for coclans  $\mathcal{C}, \mathcal{D}$ . We use the arrow symbol  $\rightrightarrows$  for co-display maps.

REMARK 2.5. Coclans appear under the name *cofibration categories* in [19, Definition 2.1.5]. This is however in conflict with Baues’ notion of cofibration category, which also includes a notion of weak equivalence [6, Section I.1]. See also Remark 2.11(d) below.

## 2.1. Models.

DEFINITION 2.6. A *model* of a clan  $\mathcal{T}$  is a functor  $A : \mathcal{T} \rightarrow \mathbf{Set}$  which preserves the terminal object and pullbacks of display maps. We write  $\mathcal{T}\text{-Mod}$  for the category of models of  $\mathcal{T}$ , viewed as a full subcategory of the functor category  $[\mathcal{T}, \mathbf{Set}]$ .

REMARK 2.7. In other words, a model of a clan  $\mathcal{T}$  is a clan morphism into the large clan with underlying category  $\mathbf{Set}$  and the maximal (i.e., finite-limit) clan structure.

In the spirit of functorial semantics, it is possible to consider models of clans in other categories than sets, and even in other (typically large) clans. However, the duality theory presented here is about models in  $\mathbf{Set}$  and we don’t consider any other kind (apart from some speculations about  $\infty$ -categorical models in §8).

- EXAMPLES 2.8. (a) If  $\mathcal{C}$  is a finite-limit clan (Example 2.3(a)) then  $\text{Mod}(\mathcal{C})$  coincides with the category  $\text{FL}(\mathcal{C}, \mathbf{Set})$  of finite-limit preserving functors into  $\mathbf{Set}$ , which we also view as category of models of  $\mathcal{C}$  qua finite-limit theory. This means that it makes sense to view finite-limit theories as a special case of clans.
- (b) If  $\mathcal{C}$  is a finite-product clan, then  $\text{Mod}(\mathcal{C})$  is the category  $\text{FP}(\mathcal{C}, \mathbf{Set})$  of finite-product preserving functors into  $\mathbf{Set}$ .

In Adámek, Rosický and Vitale’s textbook [3, Definition 1.1], small finite-product categories are called *algebraic theories*, and models of algebraic theories are defined to be finite-product preserving functors into  $\mathbf{Set}$ . Thus, we recover their notions as a special case, i.e., finite-product clans correspond to algebraic theories, and models correspond to algebras. To emphasize the analogy to the finite-limit case, we refer to algebraic theories also as *finite-product theories*.

- (c) If  $\mathbb{T}$  is a GAT, then the category of models of its syntactic category  $\mathcal{C}[\mathbb{T}]$  (with the clan structure described in §10.1) is equivalent to the models of  $\mathbb{T}$ , which Cartmell defines<sup>2</sup> to be the category  $\text{ConFunc}(\mathcal{C}[\mathbb{T}], \mathbf{Fam})$  of contextual functors and natural transformations into the contextual category  $\mathbf{Fam}$  of iterated families of sets.

The following remarks discuss some categorical properties of the category  $\mathcal{T}\text{-Mod}$  of models of a clan, establishing in particular that it is locally finitely presentable. We refer to §9 for the relevant definitions.

- REMARKS 2.9. (a) As a category of models of a finite-limit sketch,  $\mathcal{T}\text{-Mod}$  is reflective (and therefore closed under arbitrary limits) in  $[\mathcal{T}, \mathbf{Set}]$ , and moreover it is closed under filtered colimits [2, Section 1.C]. In particular,  $\mathcal{T}\text{-Mod}$  is locally finitely presentable.

<sup>2</sup>Strictly speaking, Cartmell does not “define” the models of  $\mathbb{T}\text{-Mod}$  to be  $\text{ConFunc}(\mathcal{C}[\mathbb{T}], \mathbf{Fam})$  but ‘asserts’ that the categories are equivalent [12, p. 2.77]. But since he refrains from giving a formal definition of  $\mathbb{T}\text{-Mod}$ —writing only “It should be quite clear what we mean by model” [12, p. 1.45]—we take the assertion as a definition.

- (b) The representable functors  $\mathcal{T}(\Gamma, -) : \mathcal{T} \rightarrow \mathbf{Set}$  are models of  $\mathcal{T}$  for all  $\Gamma \in \mathcal{T}$ , thus the Yoneda embedding  $\mathcal{Y} : \mathcal{T}^{\text{op}} \rightarrow [\mathcal{T}, \mathbf{Set}]$  lifts along the inclusion  $\mathcal{T}\text{-Mod} \hookrightarrow [\mathcal{T}, \mathbf{Set}]$  to a fully faithful functor  $H : \mathcal{T}^{\text{op}} \rightarrow \mathcal{T}\text{-Mod}$ .

$$\begin{array}{ccc} & & \mathcal{T}\text{-Mod} \\ & \nearrow H & \downarrow \\ \mathcal{T}^{\text{op}} & \xrightarrow{\mathcal{Y}} & [\mathcal{T}, \mathbf{Set}] \end{array}$$

- (c) For  $\Gamma \in \mathcal{T}$ , the representable functor

$$\mathcal{T}\text{-Mod}(H(\Gamma), -) : \mathcal{T}\text{-Mod} \rightarrow \mathbf{Set}$$

is isomorphic to the evaluation functor  $A \mapsto A(\Gamma)$ , hence it preserves filtered colimits as those are computed in  $[\mathcal{T}, \mathbf{Set}]$  and therefore pointwise. This means that  $H(\Gamma)$  is *compact*<sup>3</sup> in  $\mathcal{T}\text{-Mod}$ .

**2.2. The clan for categories.** §10.1 describes how the syntactic category  $\mathcal{C}[\mathbb{T}]$  of every GAT  $\mathbb{T}$  can be viewed as a clan. The present section elaborates this for the specific case of the GAT  $\mathbb{T}_{\text{Cat}}$  of categories (10.1). We will use this clan and variations as a running example throughout the article.

Recall from Definition 10.1 that the objects of  $\mathcal{T}_{\text{Cat}} := \mathcal{C}[\mathbb{T}_{\text{Cat}}]$  are equivalence classes of contexts, and the arrows are equivalence classes of substitutions. By inspection of the axioms we see that sorts in  $\mathbb{T}_{\text{Cat}}$  cannot depend on non-variable terms, since the only non-constant sort symbol is  $x y : O \vdash A(x, y)$  and there are no function symbols of type  $O$ . This means that up to reordering, all contexts are of the form

$$(x_1 \dots x_n : O, y_1 : A(x_{s_1}, x_{t_1}), \dots, y_k : A(x_{s_k}, x_{t_k})) \quad (2.2)$$

where  $n, k \geq 0$  (such that  $n > 0$  whenever  $k > 0$ ) and  $1 \leq s_l, t_l \leq n$  for  $1 \leq l \leq k$ ; declaring first a list of object variables and then a list of arrow variables, each depending on a pair of the object variables. Given another context  $(u_1 \dots u_m, v_1 \dots v_h)$ , a substitution

$$u_1 \dots u_m, v_1 \dots v_h \vdash \sigma : x_1 \dots x_n, y_1 \dots y_k$$

is a tuple  $\sigma = (u_{i_1} \dots u_{i_n}, f_1 \dots f_k)$  where  $1 \leq i_1, \dots, i_n \leq m$  and the  $f_l$  are terms

$$u_1 \dots u_m, v_1 \dots v_h \vdash f_l : A(u_{i_{s_l}}, u_{i_{t_l}}).$$

Some reflection shows that  $\mathcal{C}[\mathbb{T}_{\text{Cat}}]$  is dual to the full subcategory of  $\mathbf{Cat}$  on free categories on finite graphs: the data of a context (2.2) is that of finite sets  $V = \{x_1, \dots, x_n\}$ ,  $E = \{y_1, \dots, y_k\}$  of *vertices* and *edges*, and source and target functions  $s, t : E \rightarrow V$ , and a substitution  $\sigma$  as above consists of a mapping from  $\{x_1, \dots, x_n\}$  to  $\{u_1, \dots, u_m\}$  and a mapping from  $\{y_1, \dots, y_k\}$  to suitable paths in the graph represented by the domain. This is not surprising, since every clan embeds contravariantly into its category of models by Remark 2.9(b). Finally, the

<sup>3</sup>Following Lurie [29] we use the shorter term “compact” instead of the more traditional “finitely presented” for objects whose covariant representable functor preserves filtered colimits.

display maps in  $\mathcal{T}_{\text{Cat}}$ , which syntactically correspond to projections “from longer contexts to shorter ones”, correspond to functors  $G^* \hookrightarrow H^*$  between free categories induced by inclusions (i.e., monomorphisms)  $G \hookrightarrow H$  of finite graphs in the dual presentation.

**2.3. The weak factorization system on models.** Next we introduce the *extension–full weak factorization system* on the category of models of a clan. We refer to §9 for basic facts about lifting properties and weak factorization systems (w.f.s.) as well as pointers to the literature.

DEFINITION 2.10. Let  $\mathcal{T}$  be a clan.

- (i) We call a map  $f : A \rightarrow B$  in  $\mathcal{T}\text{-Mod}$  *full*, if it has the *right lifting property* (r.l.p., see Definition 9.4(i)) w.r.t. all maps  $H(p)$  for  $p$  a display map.
- (ii) We call  $f : A \rightarrow B$  an *extension*, if it has the *left lifting property* (l.l.p.) w.r.t. all full maps.
- (iii) We call  $A \in \mathcal{T}\text{-Mod}$  a *0-extension*, if  $0 \rightarrow A$  is an extension.

REMARKS 2.11. (a) We use the arrow symbols “ $\twoheadrightarrow$ ” for extensions (just as for codisplay maps), and “ $\rightarrow$ ” for full maps. We write  $\mathbf{E}$  and  $\mathbf{F}$  for the classes of extensions and full maps in  $\mathcal{T}\text{-Mod}$ , respectively. By the *small object argument* (Theorem 9.5), extensions and full maps form a weak factorization system  $(\mathbf{E}, \mathbf{F})$ .

- (b) A map  $f : A \rightarrow B$  in  $\mathcal{T}\text{-Mod}$  is full if and only if the naturality square

$$\begin{array}{ccc} A(\Delta) & \xrightarrow{A(p)} & A(\Gamma) \\ f_\Gamma \downarrow & & \downarrow f_\Delta \\ B(\Delta) & \xrightarrow{B(p)} & B(\Gamma) \end{array}$$

is a *weak pullback*<sup>4</sup> in  $\mathbf{Set}$  for all display maps  $p : \Delta \rightarrow \Gamma$ . Setting  $\Gamma = 1$  we see that full maps are pointwise surjective and therefore regular epimorphisms (the pointwise kernel pair  $p, q : R \rightarrow A$  of  $f$  is in  $\mathcal{T}\text{-Mod}$  since  $\mathcal{T}\text{-Mod} \hookrightarrow [\mathcal{T}, \mathbf{Set}]$  creates limits, and pointwise surjective maps are coequalizers of their kernel pairs in  $[\mathcal{T}, \mathbf{Set}]$ , hence all the more so in  $\mathcal{T}\text{-Mod}$ ).

- (c) For every display map  $p : \Delta \rightarrow \Gamma$  in  $\mathcal{T}$ , the arrow  $H(p) : H(\Gamma) \twoheadrightarrow H(\Delta)$  is an extension—these are precisely the generators of the w.f.s. In particular, all representable models  $H(\Gamma)$  are 0-extensions, since all terminal projections  $\Gamma \rightarrow 1$  are display maps in  $\mathcal{T}$ .
- (d) The same w.f.s. was already defined by Simon Henry in [19, Definition 2.4.2], using the terminology of “cofibration categories” mentioned in Remark 2.5. There, extensions are called cofibrations, and full maps trivial fibrations. We have not used this homotopical terminology here since we don’t want to think about full maps as being “trivial” in any way.

EXAMPLES 2.12. (a) If  $\mathcal{T}$  is a finite-product clan, then  $(\mathbf{E}, \mathbf{F})$  is cofibrantly generated by initial injections  $0 \cong H(1) \twoheadrightarrow H(\Gamma)$ , since for every display map

<sup>4</sup>Meaning that the comparison map to the actual pullback is a surjection.



$p : \Delta \times \Gamma \rightarrow \Delta$  the generator  $H(p)$  is a pushout

$$\begin{array}{ccc} H(1) & \triangleright \longrightarrow & H(\Delta) \\ \downarrow & & \downarrow H(p) \\ H(\Gamma) & \triangleright \longrightarrow & H(\Gamma \times \Delta) \end{array}$$

in  $\mathcal{T}\text{-Mod}$  of an initial inclusion, and left classes of w.f.s.s are closed under pushout. It follows that the full maps are *precisely* the pointwise surjective maps, which in this case also coincide with the regular epis, since finite-product preserving functors are closed under image factorization in  $[\mathcal{T}, \text{Set}]$  (and thus every non-surjective arrow factors through a strict subobject). Thus, the 0-extensions are precisely the regular-projective objects in the finite-product case, which also play a central role in [3].

- (b) If  $\mathcal{T}$  is a finite-limit clan then *all* naturality squares of full maps  $f : A \rightarrow B$  are weak pullbacks, including the naturality squares

$$\begin{array}{ccc} A(\Gamma) & \longrightarrow & A(\Gamma \times \Gamma) \cong A(\Gamma) \times A(\Gamma) \\ \downarrow & & \downarrow \\ B(\Gamma) & \longrightarrow & B(\Gamma \times \Gamma) \cong B(\Gamma) \times B(\Gamma) \end{array}$$

of diagonals  $\Gamma \rightarrow \Gamma \times \Gamma$ . From this it follows easily that  $f_\Gamma$  is injective, and since we have shown that it is surjective above, we conclude that only isomorphisms are full in the finite-limit case.

- (c) The w.f.s. on  $\text{Cat}$  induced by the presentation  $\text{Cat} = \mathcal{T}_{\text{Cat}}\text{-Mod}$  (see §2.2) has as full maps functors  $F : \mathbb{C} \rightarrow \mathbb{D}$  which have the r.l.p. w.r.t. all functors  $G^* \hookrightarrow H^*$  for inclusions  $G \hookrightarrow H$  of finite graphs. It is not difficult to see that these are precisely the functors which are full in the classical sense and moreover surjective on objects, and that the w.f.s. is already generated by the functors  $(0 \hookrightarrow 1)$  and  $(2 \hookrightarrow 2)$ , where  $2$  is the discrete category with two objects and  $\mathbb{2}$  is the interval category.

### §3. Comodels and the universal property of $\mathcal{T}\text{-Mod}$ .

**3.1. Nerve–realization adjunctions.** We recall basic facts about *nerve–realization* adjunctions, to establish notation and conventions. Recall that for small  $\mathbb{C}$  the presheaf category  $\widehat{\mathbb{C}} = [\mathbb{C}^{\text{op}}, \text{Set}]$  is the *small-colimit completion* of  $\mathbb{C}$ , in the sense that for every cocomplete category  $\mathfrak{X}$ , precomposition with the Yoneda embedding  $\mathfrak{y} : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  induces an equivalence

$$\text{CoCont}(\widehat{\mathbb{C}}, \mathfrak{X}) \xrightarrow{\simeq} [\mathbb{C}, \mathfrak{X}] \quad (3.1)$$

between the categories of cocontinuous functors  $\widehat{\mathbb{C}} \rightarrow \mathfrak{X}$ , and of functors  $\mathbb{C} \rightarrow \mathfrak{X}$ . Specifically, the cocontinuous functor  $F_\otimes : \widehat{\mathbb{C}} \rightarrow \mathfrak{X}$  corresponding to  $F : \mathbb{C} \rightarrow \mathfrak{X}$  is the left Kan extension of  $F$  along  $\mathfrak{y} : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ , whose value at  $A \in \widehat{\mathbb{C}}$  admits alternative representations

$$\begin{aligned} F_\otimes(A) &= F \otimes A = \int^{C \in \mathbb{C}} F(C) \times A(C) \\ &= \text{colim}(\text{El}(A) \rightarrow \mathbb{C} \xrightarrow{F} \mathfrak{X}) \end{aligned}$$



as a coend and as a colimit indexed by the category  $\text{El}(A)$  of elements of  $A$ . If  $\mathfrak{X}$  is locally small then  $F_{\otimes}$  has a right adjoint  $F_N : \mathfrak{X} \rightarrow \widehat{\mathbb{C}}$  given by  $F_N(X) = \mathfrak{X}(F(-), X)$ . We call  $F_N$  and  $F_{\otimes}$  the *nerve* and *realization* functors of  $F$ , respectively, and  $F_{\otimes} \dashv F_N$  the *nerve–realization adjunction* of  $F$ .

**3.2. Comodels and the universal property of  $\mathcal{T}\text{-Mod}$ .** The universal property of  $\mathcal{T}\text{-Mod}$  is an equivalence between cocontinuous functors out of  $\mathcal{T}\text{-Mod}$  and coclan morphisms out of  $\mathcal{T}^{\text{op}}$ . Following a suggestion by Mathieu Anel, we refer to the latter as *comodels* of the clan. We will only use this term for coclan morphisms with cocomplete codomain.

**DEFINITION 3.1.** A *comodel* of a clan  $\mathcal{T}$  in a cocomplete category  $\mathfrak{X}$  is a functor  $F : \mathcal{T}^{\text{op}} \rightarrow \mathfrak{X}$  which sends 1 to 0, and display-pullbacks to pushouts. We write  $\mathcal{T}\text{-CoMod}(\mathfrak{X})$  for the category of comodels of  $\mathcal{T}$  in  $\mathfrak{X}$ , as a full subcategory of the functor category.

**REMARK 3.2.** In other words, a comodel of  $\mathcal{T}$  in  $\mathfrak{X}$  is a coclan morphism from  $\mathcal{T}^{\text{op}}$  to the large coclan with underlying category  $\mathfrak{X}$  and the maximal coclan structure.

**THEOREM 3.3** (The universal property of  $\mathcal{T}\text{-Mod}$ ). *Let  $\mathcal{T}$  be a clan.*

- (i) *The functor  $H : \mathcal{T}^{\text{op}} \rightarrow \mathcal{T}\text{-Mod}$  from Remark 2.9(b) is a comodel.*
- (ii) *For every cocomplete  $\mathfrak{X}$  and comodel  $F : \mathcal{T}^{\text{op}} \rightarrow \mathfrak{X}$ , the restriction of  $F_{\otimes} : [\mathcal{T}, \text{Set}] \rightarrow \mathfrak{X}$  to  $\mathcal{T}\text{-Mod}$  is cocontinuous. Thus, precomposition with  $H$  gives rise to an equivalence*

$$\text{CoCont}(\mathcal{T}\text{-Mod}, \mathfrak{X}) \xrightarrow{\simeq} \mathcal{T}\text{-CoMod}(\mathfrak{X}) \quad (3.2)$$

*between categories of continuous functors and of comodels.*

- (iii) *If  $F : \mathcal{T}^{\text{op}} \rightarrow \mathfrak{X}$  is a comodel and  $\mathfrak{X}$  is locally small, then the nerve functor  $F_N : \mathfrak{X} \rightarrow [\mathcal{T}, \text{Set}]$  factors through the inclusion  $\mathcal{T}\text{-Mod} \hookrightarrow [\mathcal{T}, \text{Set}]$ , giving rise to a restricted nerve realization adjunction  $F_{\otimes} : \mathcal{T}\text{-Mod} \rightleftarrows \mathfrak{X} : F_N$ .*

$$\begin{array}{ccccc} \mathcal{T}^{\text{op}} & \xrightarrow{H} & \mathcal{T}\text{-Mod} & \hookrightarrow & [\mathcal{T}, \text{Set}] \\ & \searrow F & \downarrow F_{\otimes} & \uparrow F_N & \nearrow F_{\otimes} \\ & & \mathfrak{X} & & \end{array}$$

$\dashv$

**PROOF.** Analogous statements to (i) and (ii) hold more generally for arbitrary small realized<sup>5</sup> limit sketches. As Brandenburg points out on MathOverflow<sup>6</sup>, the earliest reference for this seems to be [37, Theorem 2.5]. See also [10] which gives a careful account of an even more general statement for non-small sketches.

For claim (iii), it's easy to see that for  $X \in \mathfrak{X}$ , the functor  $F_N(X) = \mathfrak{X}(F(-), X)$  is a model since  $F$  is a comodel.  $\dashv$

**3.3. Slicing and coslicing.** As an application of Theorem 3.3, this subsection gives statements about clan presentations of slice categories  $\mathcal{T}\text{-Mod}/A$  of categories

<sup>5</sup>A sketch is called “realized” if all its designated cones are limiting.

<sup>6</sup><https://mathoverflow.net/q/403653>

of models (Proposition 3.6), and of coslice categories  $H(A)/\mathcal{T}\text{-Mod}$  under representable models (Proposition 3.5).

DEFINITION 3.4. For  $\mathcal{T}$  a clan and  $\Gamma \in \mathcal{T}$ , we write  $\mathcal{T}_\Gamma$  for the full subcategory of  $\mathcal{T}/\Gamma$  on display maps. Then  $\mathcal{T}_\Gamma$  is a clan where an arrow in  $\mathcal{T}_\Gamma$  is a display map if its underlying map is one in  $\mathcal{T}$ . Compare [26, Proposition 1.1.6].

PROPOSITION 3.5. *Let  $\Gamma$  be an object of a clan  $\mathcal{T}$ . Then the functor*

$$\Gamma/H : (\mathcal{T}_\Gamma)^\text{op} \rightarrow H(\Gamma)/\mathcal{T}\text{-Mod} \quad (3.3)$$

*which sends  $d : \Delta \rightarrow \Gamma$  to  $H(d) : H(\Gamma) \triangleright H(\Delta)$  is a comodel. Moreover, its restricted nerve–realization adjunction (in the sense of Theorem 3.3(iii))*

$$\begin{array}{ccc} (\mathcal{T}_\Gamma)^\text{op} & \xleftarrow{H} & \mathcal{T}_\Gamma\text{-Mod} \\ & \searrow \Gamma/H & \downarrow \scriptstyle (\Gamma/H)_\otimes \quad \uparrow \scriptstyle (\Gamma/H)_N \\ & & H(\Gamma)/\mathcal{T}\text{-Mod} \end{array} \quad (3.4)$$

*is an equivalence and identifies the extension–full w.f.s. on  $\mathcal{T}_\Gamma\text{-Mod}$  with the coslice w.f.s. on  $H(\Gamma)/\mathcal{T}\text{-Mod}$ .*

PROOF. It is easy to see that  $\Gamma/H$  is a comodel. For the second claim, since arrows  $H(\Gamma) \rightarrow A$  correspond to elements of  $A(\Gamma)$ , we can identify the coslice category  $H(\Gamma)/\mathcal{T}\text{-Mod}$  with the category of “ $\Gamma$ -pointed models of  $\mathcal{T}$ ”, i.e., pairs  $(A, x)$  of a model  $A$  and an element  $x \in A(\Gamma)$ , and morphisms preserving chosen elements.

Under this identification, we first verify that the functor  $(\Gamma/H)_N$  is given by

$$(\Gamma/H)_N(A, x)(\Delta \xrightarrow{d} \Gamma) = \{y \in A(\Delta) \mid d \cdot y = x\},$$

and then that it is an equivalence with inverse  $\Phi : \mathcal{T}_\Gamma\text{-Mod} \rightarrow H(\Gamma)/\mathcal{T}\text{-Mod}$  given by

$$\Phi(B) = (B(- \times \Gamma \xrightarrow{\pi_2} \Gamma), \delta \cdot \star)$$

where  $\star$  is the unique element of  $B(\text{id}_\Gamma)$  and  $\delta : \Gamma \rightarrow \Gamma \times \Gamma$  is the diagonal map viewed as global element of  $\pi_2 : \Gamma \times \Gamma \rightarrow \Gamma$  in  $\mathcal{T}_\Gamma$ . Thus,  $(\Gamma/H)_\otimes = \Phi$ .

Finally we note that the w.f.s. on  $H(\Gamma)/\mathcal{T}\text{-Mod}$  is cofibrantly generated by commutative triangles

$$\begin{array}{ccc} H(\Gamma) & & \\ \downarrow \scriptstyle H(\pi_2) & \searrow \scriptstyle H(\pi_2) & \\ H(\Delta \times \Gamma) & \xrightarrow{\scriptstyle H(d \times \Gamma)} & H(\Theta \times \Gamma) \end{array} \quad (3.5)$$

for display maps  $d : \Theta \rightarrow \Gamma$  [21, Theorem 2.7]. On the other hand, since  $(\Gamma/H)_\otimes \circ H = \Gamma/H$  (see (3.4)), the functor  $(\Gamma/H)_\otimes$  sends the generators of the extension–full

w.f.s. on  $\mathcal{T}_\Gamma\text{-Mod}$  to triangles

$$\begin{array}{ccc} H(\Gamma) & & \\ \downarrow H(e) & \nearrow H(f) & \\ H(\Delta) & \xrightarrow{H(d)} & H(\Theta) \end{array} \quad (3.6)$$

for arbitrary display maps  $d, e, f$  in  $\mathcal{T}$ . Now the triangles of shape (3.6) contain the triangles of shape (3.5), but are contained in their saturation, which is the left class of the coslice w.f.s. Thus, the two w.f.s.s are equal.  $\dashv$

**PROPOSITION 3.6.** *Let  $A$  be a model of a clan  $\mathcal{T}$ . Then the projection functor  $\text{El}(A) \rightarrow \mathcal{T}^{\text{op}}$  creates a coclan structure on  $\text{El}(A)$ , i.e.,  $\text{El}(A)$  is a coclan with codisplay maps those arrows that are mapped to display maps in  $\mathcal{T}$ . Moreover, the canonical functor*

$$H/A : \text{El}(A) \cong \mathcal{T}^{\text{op}}/A \rightarrow \mathcal{T}\text{-Mod}/A$$

*is a comodel, and its restricted nerve–realization adjunction*

$$\begin{array}{ccc} \text{El}(A) & \xleftarrow{H} & \text{El}(A)^{\text{op}}\text{-Mod} \\ & \searrow H/A & \downarrow (H/A)_\otimes \uparrow (H/A)_N \\ & & \mathcal{T}\text{-Mod}/A \end{array}$$

*is an equivalence which identifies the extension–full w.f.s. on  $\text{El}(A)^{\text{op}}\text{-Mod}$  and the slice w.f.s. on  $\mathcal{T}\text{-Mod}/A$ .*

**PROOF.** The verification that  $\text{El}(A)^{\text{op}}$  is a clan and  $H/A$  is a coclan morphism is straightforward. The equivalence is a restriction of the well-known equivalence  $\widehat{\mathcal{T}^{\text{op}}/A} \simeq \widehat{\mathcal{T}^{\text{op}}/A}$ . The w.f.s.s coincide since—again by  $(H/A)_\otimes \circ H = H/A$ —the functor  $(H/A)_\otimes$  sends the generators of the w.f.s. on  $\text{El}(A)^{\text{op}}\text{-Mod}$  to commutative triangles

$$\begin{array}{ccc} H(\Gamma) & \xrightarrow{d} & H(\Delta) \\ & \searrow \hat{x} & \downarrow \hat{y} \\ & & A \end{array}$$

in  $\mathcal{T}\text{-Mod}/A$ , where  $d : \Delta \rightarrow \Gamma$  is a display map in  $\mathcal{T}$  and  $x \in A(\Gamma)$  and  $y \in A(\Delta)$  are elements with  $d \cdot y = x$ . By [21, Theorem 1.5], these form a set of generators for the slice w.f.s. on  $\mathcal{T}\text{-Mod}/A$ .  $\dashv$

#### §4. (E, F)-categories and the biadjunction.

**DEFINITION 4.1.** An (E, F)-category is a l.f.p. category  $\mathcal{L}$  with a w.f.s. (E, F) whose maps we call *extensions* and *full maps*. A *morphism of (E, F)-categories* is a functor  $F : \mathcal{L} \rightarrow \mathcal{M}$  preserving small limits, filtered colimits, and full maps. We write  $\text{EFCat}$  for the 2-category of (E, F)-categories, morphisms of (E, F)-categories, and natural transformations.

LEMMA 4.2. *If  $F : \mathcal{L} \rightarrow \mathcal{M}$  is a morphism of  $(E, F)$ -categories, then it has a left adjoint  $L : \mathcal{M} \rightarrow \mathcal{L}$  which preserves compact objects and extensions. Conversely, if  $L : \mathcal{M} \rightarrow \mathcal{L}$  is a cocontinuous functor preserving compact objects and extensions, then it has a right adjoint  $F : \mathcal{L} \rightarrow \mathcal{M}$  which is a morphism of  $(E, F)$ -categories. Writing  $\text{EFCat}_L(\mathcal{M}, \mathcal{L})$  for the category of cocontinuous functors  $\mathcal{M} \rightarrow \mathcal{L}$  preserving extensions and compact objects, we thus have  $\text{EFCat}_L(\mathcal{M}, \mathcal{L}) \simeq \text{EFCat}(\mathcal{L}, \mathcal{M})^{\text{op}}$ .*

PROOF. That morphisms of  $(E, F)$ -categories have left adjoints follows from the adjoint functor theorem for presentable categories [2, Theorem 1.66], and conversely the special adjoint functor theorem [31, Section V-8] implies that cocontinuous functors between l.f.p. categories have right adjoints. It follows from standard arguments that the left adjoint preserves compact objects iff the right adjoint preserves filtered colimits, and that the left adjoint preserves extensions iff the right adjoint preserves full maps.  $\dashv$

LEMMA 4.3. *For any morphism  $F : \mathcal{S} \rightarrow \mathcal{T}$  of clans, the precomposition functor*

$$(-) \circ F : \mathcal{T}\text{-Mod} \rightarrow \mathcal{S}\text{-Mod}$$

*is a morphism of  $(E, F)$ -categories. Thus the assignment  $\mathcal{T} \mapsto \mathcal{T}\text{-Mod}$  extends to a 2-functor*

$$(-)\text{-Mod} : \text{Clan}^{\text{op}} \rightarrow \text{EFCat}$$

*from clans to  $(E, F)$ -categories.*

PROOF. The preservation of small limits and filtered colimits is obvious since they are computed pointwise (Remark 2.9(a)). To show that  $(- \circ F)$  preserves full maps, let  $f : A \rightarrow B$  be full in  $\mathcal{T}\text{-Mod}$ . It is sufficient to show that the  $(f \circ F)$ -naturality squares are weak pullbacks at all display maps  $p : \text{in } \mathcal{S}\text{-Mod}$ . But the  $(f \circ F)$ -naturality square at  $p$  is the same as the  $f$ -naturality square at  $F(p)$  so the claim follows since  $f$  is full and  $F$  preserves display maps.  $\dashv$

DEFINITION 4.4. Given an  $(E, F)$ -category  $\mathcal{L}$ , write  $\mathfrak{C}(\mathcal{L}) \subseteq \mathcal{L}$  for the full subcategory on compact 0-extensions.

LEMMA 4.5.  $\mathfrak{C}(\mathcal{L})$  is a coclan with extensions as codisplay maps.

PROPOSITION 4.6. The assignment  $\mathcal{L} \mapsto \mathfrak{C}(\mathcal{L})^{\text{op}}$  extends to a pseudofunctor

$$\mathfrak{C}(-)^{\text{op}} : \text{EFCat} \rightarrow \text{Clan}^{\text{op}}$$

which is left biadjoint to  $(-)\text{-Mod} : \text{Clan}^{\text{op}} \rightarrow \text{EFCat}$ .

PROOF. We show that for every  $(E, F)$ -category  $\mathcal{L}$ , the 2-functor

$$\text{EFCat}(\mathcal{L}, (-)\text{-Mod}) : \text{Clan}^{\text{op}} \rightarrow \text{Cat}$$

is birepresented by  $\mathfrak{C}(\mathcal{L})^{\text{op}}$ . Given a clan  $\mathcal{T}$  it is easy to see that the equivalence

$$\text{CoCont}(\mathcal{T}\text{-Mod}, \mathcal{L}) \simeq \mathcal{T}\text{-CoMod}(\mathcal{L})$$

from Theorem 3.3 restricts to an equivalence

$$\text{EFCat}_L(\mathcal{T}\text{-Mod}, \mathcal{L}) \simeq \text{CoClan}(\mathcal{T}^{\text{op}}, \mathfrak{C}(\mathcal{L})).$$

Taking opposite categories on both sides we get

$$\mathrm{EFCat}(\mathcal{L}, \mathcal{T}\text{-Mod}) \simeq \mathrm{Clan}(\mathcal{T}, \mathfrak{C}(\mathcal{L})^{\mathrm{op}}) \quad (4.1)$$

as required.  $\dashv$

REMARK 4.7. From the construction of the natural equivalence (4.1) we can extract explicit descriptions of the components

$$\Theta_{\mathcal{L}} : \mathcal{L} \rightarrow \mathfrak{C}(\mathcal{L})^{\mathrm{op}}\text{-Mod} \quad \text{and} \quad E_{\mathcal{T}} : \mathcal{T} \rightarrow \mathfrak{C}(\mathcal{T}\text{-Mod})^{\mathrm{op}}$$

of the unit  $\Theta$  and the counit  $E$  of the biadjunction

$$\mathfrak{C}(-)^{\mathrm{op}} : \mathrm{EFCat} \rightleftarrows \mathrm{Clan}^{\mathrm{op}} : (-)\text{-Mod} \quad (4.2)$$

at an  $(\mathbf{E}, \mathbf{F})$ -category  $\mathcal{L}$  and a clan  $\mathcal{T}$ , respectively. Specifically,  $\Theta_{\mathcal{L}}$  is the nerve of the inclusion  $J : \mathfrak{C}(\mathcal{L}) \hookrightarrow \mathcal{L}$  (which is obviously a comodel), and  $E_{\mathcal{T}}$  is  $(-)^{\mathrm{op}}$  of the evident corestriction of  $H : \mathcal{T}^{\mathrm{op}} \rightarrow \mathcal{T}\text{-Mod}$ .

In §5 and §6 we show that the biadjunction (4.2) is *idempotent* (in the sense that the associated monad and comonad are), and characterize the fixed points on both sides (Theorems 5.9 and 6.18).

### §5. Cauchy complete clans and the fat small object argument.

DEFINITION 5.1. A clan  $\mathcal{T}$  is called *Cauchy complete* if its underlying category is Cauchy complete (i.e., idempotents split), and retracts of display maps are display maps.

EXAMPLES 5.2. (a) Finite-limit clans are always Cauchy complete, since finite-limit categories are and all arrows are display maps in finite-limit clans.

(b) A finite-product clan is Cauchy complete if and only if idempotents split in the underlying finite-product category, which may or may not be the case for the presentation of a single-sorted algebraic theory  $\mathbb{T}$  as *Lawvere theory* (i.e., the opposite of the full subcategory of  $\mathrm{Mod}(\mathbb{T})$  on *finitely generated free models*). For example the Lawvere theory of abelian groups is Cauchy complete since all finitely presented projective abelian groups are free, whereas the Lawvere theory of distributive lattices is *not* Cauchy complete. A non-free retract of a finitely generated free distributive lattice may be obtained by starting with a section–retraction pair  $s : \{0 < 1 < 2\} \rightleftarrows \{0 < 1\}^2 : r$  in posets, and then taking the distributive lattice of *upper* sets on both sides, i.e., applying the functor  $\mathrm{Pos}(-, \{0 < 1\}) : \mathrm{Pos}^{\mathrm{op}} \rightarrow \mathrm{DLat}$ . Then  $\mathrm{Pos}(\{0 < 1\}^2, \{0 < 1\})$  is the free distributive lattice on 2 generators, but  $\mathrm{Pos}(\{0 < 1 < 2\}, \{0 < 1\})$  is not free.

Further details on the question of Cauchy-completeness of finite-limit theories, including a discussion of how the classical theory of *Morita equivalence of rings* fits into the picture, can be found in [3, Sections 8, 15].

(c) The clan  $\mathcal{T}_{\mathrm{Cat}}$  of categories is Cauchy complete. To see this assume that  $G$  is a finite graph and that  $\mathbb{D}$  is a retract of the free category  $G^*$  on  $G$ . Then we know that  $\mathbb{D}$  is a compact 0-extension and we have to show that  $\mathbb{D}$  is free on a finite graph. Call an arrow  $f$  in  $\mathbb{D}$  *irreducible* if it is not an identity and in any decomposition  $f = gh$ , either  $g$  or  $h$  is an identity. Since the factors

of every non-trivial decomposition have shorter length in  $G^*$ , every arrow in  $\mathbb{D}$  admits a decomposition into irreducible factors. Let  $H$  be the graph of irreducible arrows in  $\mathbb{D}$ , and let  $F : H^* \rightarrow \mathbb{D}$  be the canonical functor. Then  $F$  is full since all arrows in  $\mathbb{D}$  are composites of irreducibles, and it admits a section  $K : \mathbb{D} \rightarrow H^*$  since  $\mathbb{D}$  is a 0-extension. As a section,  $K$  sends arrows in  $\mathbb{D}$  to decompositions into irreducibles, thus it sends irreducible arrows to themselves. It follows that  $K(F(j)) = j$  for generators  $j$  in  $H$ , and from this we can deduce that  $K \circ F = \text{id}_{H^*}$ . Thus,  $\mathbb{D} \cong H^*$ . Finiteness of  $H$  follows from compactness.

This argument is an adaption of a similar argument for monoids [18].

- (d) For every  $(E, F)$ -category  $\mathcal{L}$ , the clan  $\mathfrak{C}(\mathcal{L})^{\text{op}}$  (Definition 4.4) is Cauchy complete, since compact objects and extensions are closed under retracts.

By Example 5.2(d), Cauchy completeness is a necessary condition for the counit  $E_{\mathcal{T}} : \mathcal{T} \rightarrow \mathfrak{C}(\mathcal{T}\text{-Mod})^{\text{op}}$  of the biadjunction (4.2) to be an equivalence. We will show that it is also sufficient, but for this we need the notion of *flat model*, and the *fat small object argument*.

Recall that for small  $\mathbb{C}$ , a functor  $F : \mathbb{C} \rightarrow \text{Set}$  is called *flat* if  $\text{El}(F)$  is filtered, or equivalently if  $F_{\otimes} : [\mathbb{C}^{\text{op}}, \text{Set}] \rightarrow \text{Set}$  preserves finite limits [8, Definition 6.3.1 and Proposition 6.3.8]. From the second characterization it follows that flat functors preserve all finite limits that exist in  $\mathbb{C}$ , thus for the case of a clan  $\mathcal{T}$ , flat functors  $F : \mathcal{T} \rightarrow \text{Set}$  are always models. We refer to them as *flat models*.

DEFINITION 5.3. A model  $A : \mathcal{T} \rightarrow \text{Set}$  of a clan  $\mathcal{T}$  is called *flat*, if  $\text{El}(F)$  is filtered.

LEMMA 5.4. A  $\mathcal{T}$ -model  $A$  is flat iff it is a filtered colimit of representable models.

PROOF. We always have  $A = \text{colim}(\text{El}(A) \rightarrow \mathcal{T}^{\text{op}} \xrightarrow{H} \mathcal{T}\text{-Mod})$ , thus if  $A$  is flat then it is a filtered colimit of representable models. The other direction follows since representable models are flat, and flat functors are closed under filtered colimits in  $[\mathcal{T}, \text{Set}]$  [8, Proposition 6.3.6].  $\dashv$

COROLLARY 5.5. For any clan  $\mathcal{T}$ , the 0-extensions in  $\mathcal{T}\text{-Mod}$  are flat.

PROOF. This follows from the *fat small object argument* and can be seen as a special case of [33, Corollary 5.1], but we give a direct proof in Appendix 11 (Corollary 11.9).  $\dashv$

DEFINITION 5.6. Let  $\mathfrak{X}$  be a cocomplete locally small category.

- (i) We say that an arrow  $f : A \rightarrow B$  is *orthogonal* to a small diagram  $D : \mathbb{J} \rightarrow \mathfrak{X}$ , and write  $f \perp D$ , if the following square is a pullback in  $\text{Set}$ .

$$\begin{array}{ccc} \text{colim}_{j \in \mathbb{J}} \mathfrak{X}(B, D_j) & \longrightarrow & \mathfrak{X}(B, \text{colim}(D)) \\ \downarrow & & \downarrow \\ \text{colim}_{j \in \mathbb{J}} \mathfrak{X}(A, D_j) & \longrightarrow & \mathfrak{X}(A, \text{colim}(D)). \end{array}$$

- (ii) We call  $f$  *compact* if it is orthogonal to all small filtered diagrams.

LEMMA 5.7. Let  $\mathfrak{X}$  be a locally small cocomplete category.

- (i) An object  $A \in \mathfrak{X}$  is compact in the usual sense that  $\mathfrak{X}(A, -)$  preserves filtered colimits, if and only if the arrow  $0 \rightarrow A$  is compact in the sense of Definition 5.6.

- (ii) If the arrow  $g$  in a commutative triangle 
$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ & \searrow f & \downarrow h \\ & & C \end{array}$$
 is compact, then  $f$  is

compact if and only if  $h$  is compact. In other words, compact arrow are closed under composition and have the right cancellation property.

- (iii) If  $f : A \rightarrow B$  is compact as an arrow in  $\mathfrak{X}$ , then it is compact as an object in  $A/\mathfrak{X}$ .  
 (iv) If  $h : B \rightarrow C$  is an arrow between compact objects in  $\mathfrak{X}$ , then  $h$  is compact as an object in  $B/\mathfrak{X}$ .

PROOF. (i) is obvious, and (ii) follows from the pullback lemma.

For (iii) assume that  $f$  is compact as an arrow in  $\mathfrak{X}$  and consider a filtered diagram in  $A/\mathfrak{X}$ , given by a filtered diagram  $D : \mathbb{I} \rightarrow \mathfrak{X}$  and a cocone  $\gamma = (\gamma_i : A \rightarrow D_i)_{i \in \mathbb{I}}$ . Note that since the forgetful functor  $A/\mathfrak{X} \rightarrow \mathfrak{X}$  creates connected colimits, we have  $\text{colim}(\gamma) : A \rightarrow \text{colim}(D)$ . Also because  $\mathbb{I}$  is connected, all  $\gamma_i$  are in the same equivalence class in  $\text{colim}_{i \in \mathbb{I}} \mathfrak{X}(A, D_i)$ , which we denote by  $\bar{\gamma} : 1 \rightarrow \text{colim}_{i \in \mathbb{I}} \mathfrak{X}(A, D_i)$ . We have to show that the canonical map

$$\text{colim}_i (A/\mathfrak{X})(f, \gamma_i) \longrightarrow (A/\mathfrak{X})(f, \text{colim}(\gamma))$$

is a bijection. This follows because this function can be presented by a pullback in  $\text{Set}^2$  as in the following diagram.

$$\begin{array}{ccccc} & & \text{colim}_i \mathfrak{X}(B, D_i) & \longrightarrow & \mathfrak{X}(B, \text{colim}(D)) \\ & \nearrow & \downarrow & & \downarrow \\ \text{colim}_i (A/\mathfrak{X})(f, \gamma_i) & \longrightarrow & (A/\mathfrak{X})(f, \text{colim}(\gamma)) & & \\ \downarrow & & \downarrow & & \downarrow \\ & \nearrow \bar{\gamma} & \text{colim}_i \mathfrak{X}(A, D_i) & \longrightarrow & \mathfrak{X}(A, \text{colim}(D)) \\ & & \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 & \xrightarrow{\text{colim}(\gamma)} & \end{array}$$

The front square is a pullback since the back one is by compactness of  $f$  as an arrow, and since the side ones are pullbacks by construction. Thus the gray horizontal arrow is a bijection since  $1 \rightarrow 1$  is.

Finally, claim (iv) now follows directly from (i), (ii), and (iii).  $\dashv$

REMARK 5.8. One can show the implication of Lemma 5.7(iii) is actually an equivalence, i.e.,  $f : A \rightarrow B$  is compact as an arrow if and only if it is so as an object of the coslice category, but the other direction is more awkward to write down and we don't need it.

THEOREM 5.9. If  $\mathcal{T}$  is a Cauchy complete clan, then  $E_{\mathcal{T}} : \mathcal{T} \rightarrow \mathfrak{C}(\mathcal{T}\text{-Mod})^{\text{op}}$  is an equivalence of clans.

PROOF. Let  $C \in \mathcal{T}\text{-Mod}$  be a compact 0-extension. Then by Corollary 5.5,  $C$  is a filtered colimit of representable models, and since  $C$  is compact the identity  $\text{id}_C$  factors through one of the colimit inclusions, whence  $C$  is a retract of a representable model. By Cauchy completeness,  $C$  is thus itself representable, i.e., we have an equivalence of categories.

It remains to show that  $E_{\mathcal{T}}$  reflects extensions to display maps. Assume  $f : \Delta \rightarrow \Gamma$  in  $\mathcal{T}$  such that  $H(f) : H(\Gamma) \rightarrow H(\Delta)$  is an extension. Then  $H(f)$  is compact in



$H(\Gamma)/\mathcal{T}\text{-Mod}$  by Lemma 5.7(iv) and  $H(\Gamma)/\mathcal{T}\text{-Mod} \simeq \mathcal{T}_\Gamma\text{-Mod}$  by Proposition 3.5. This means that the object corresponding to  $H(f)$  in  $\mathcal{T}_\Gamma\text{-Mod}$  is a compact 0-extension, and thus it is isomorphic to a representable model  $\mathcal{T}_\Gamma(d, -)$  for a display map  $d : \Theta \rightarrow \Gamma$  by the argument in the first part of the proof. This means that  $f$  is isomorphic to  $d$  over  $\Gamma$ , and therefore a display map.  $\dashv$

The preceding proposition together with Example 5.2(d) shows that the pseudomonad on  $\text{Clan}$  induced by the biadjunction (4.2) is *idempotent*: applying the pseudomonad once produces a Cauchy complete clan, and applying it again gives something equivalent. By general facts about (bi)adjunctions, the induced pseudomonad on  $\text{EFCat}$  is also idempotent. In the next section we characterize its fixed points as being *clan-algebraic categories*.

## §6. Clan-algebraic categories.

DEFINITION 6.1. An  $(E, F)$ -category  $\mathcal{L}$  is called *clan-algebraic* if

- (D) the inclusion  $J : \mathfrak{C}(\mathcal{L}) \hookrightarrow \mathcal{L}$  is dense,
- (CG) the w.f.s.  $(E, F)$  is cofibrantly generated by  $E \cap \mathfrak{C}(\mathcal{L})$ , and
- (FQ) equivalence relations  $\langle p, q \rangle : R \rightrightarrows A \times A$  in  $\mathcal{L}$  with *full components*  $p, q$  are effective, and have full coequalizers.

A *clan-algebraic weak factorization system* is a w.f.s. on a l.f.p. category  $\mathcal{L}$  making  $\mathcal{L}$  into a clan-algebraic category.

THEOREM 6.2. *The category  $\mathcal{T}\text{-Mod}$  is clan-algebraic for every clan  $\mathcal{T}$ .*

PROOF. Conditions (D) and (CG) are straightforward. For (FQ) let  $\langle p, q \rangle : R \rightrightarrows A \times A$  be an equivalence relation with full components. This means that we have an equivalence relation  $\sim$  on each  $A(\Gamma)$ , such that

- for all arrows  $s : \Delta \rightarrow \Gamma$ , the function  $A(s) = s \cdot (-) : A(\Delta) \rightarrow A(\Gamma)$  preserves this relation, and
- for every display map  $p : \Gamma^+ \rightarrow \Gamma$  and all  $a, b \in A(\Gamma)$  and  $c \in A(\Gamma^+)$  such that  $a \sim b$  and  $p \cdot c = a$ , there exists a  $d \in A(\Gamma^+)$  with  $c \sim d$  and  $p \cdot d = b$ .

We show first that the pointwise quotient  $A/R$  is a model. Clearly  $(A/R)(1) = 1$ , and it remains to show that given a pullback

$$\begin{array}{ccc} \Delta^+ & \xrightarrow{t} & \Gamma^+ \\ \downarrow g & & \downarrow f \\ \Delta & \xrightarrow{s} & \Gamma \end{array}$$

with  $f$  and  $g$  display maps, and elements  $a \in A(\Delta)$ ,  $b \in A(\Gamma^+)$  with  $s \cdot a \sim f \cdot b$ , there exists a unique-up-to- $\sim$  element  $c \in A(\Delta^+)$  with  $g \cdot c \sim a$  and  $t \cdot c \sim b$ . Since  $f$  is a display map, there exists a  $b'$  with  $b \sim b'$  and  $v \cdot b' = s \cdot a$ , and since  $A$  is a model there exists therefore a  $c$  with  $q \cdot c = a$  and  $t \cdot c = b'$ . For uniqueness assume that  $c, c' \in A(\Delta^+)$  with  $q \cdot c \sim q \cdot c'$  and  $t \cdot c \sim t \cdot c'$ . Then  $c \sim c'$  follows from the fact that  $R$  is a model. This shows that  $A/R$  is a model, and also that the quotient is effective, since the kernel pair is computed pointwise. The fact that  $A \rightarrow A/R$  is full is similarly easy to see.  $\dashv$

The following counterexample shows that conditions (D) and (CG) alone are not sufficient to characterize categories  $\mathcal{T}\text{-Mod}$ .

EXAMPLE 6.3. Let  $\text{Inj}$  be the full subcategory of  $\widehat{\mathcal{A}}$  on injections, and let  $(E, F)$  be the w.f.s. generated by  $0 \rightarrow \mathcal{A}(0)$  and  $0 \rightarrow \mathcal{A}(1)$ . Then  $(E, F)$  satisfies (D) and (CG), and  $F$  consists precisely of the pointwise surjective maps, in particular it contains all split epis. However, the equivalence relation on  $\text{id}_2$  which is discrete on the domain and codiscrete on the codomain is not effective.

The following is a restatement of Remark 2.11(b) for clan-algebraic categories.

LEMMA 6.4. *Full maps in clan-algebraic categories are regular epimorphisms.*

PROOF. Given a full map in a clan-algebraic category  $\mathcal{L}$ , the lifting property against 0-extensions implies that  $\Theta_{\mathcal{L}}(f) = J_N(f)$  is componentwise surjective in  $\mathcal{C}(\mathcal{L})^{\text{op}}\text{-Mod}$ , and therefore the coequalizer of its kernel pair. Since left adjoints preserve regular epis, we deduce that  $J_{\otimes}(J_N(f))$  is regular epic in  $\mathcal{L}$  and the claim follows since  $J_{\otimes} \circ J_N \cong \text{id}$  by (D).  $\dashv$

REMARK 6.5. Observe that we only used property (D) in the proof, no exactness.

LEMMA 6.6. *The class  $F$  of full maps in a clan-algebraic category  $\mathcal{L}$  has the right cancellation property, i.e., we have  $g \in F$  whenever  $gf \in F$  and  $f \in F$  for composable pairs  $f : A \rightarrow B$ ,  $g : B \rightarrow C$ .*

PROOF. By (CG) it suffices to show that  $g$  has the r.l.p. w.r.t. extensions  $e : I \twoheadrightarrow J$  between compact 0-extensions  $I, J$ . Let

$$\begin{array}{ccc} I & \xrightarrow{h} & B \\ \downarrow e & & \downarrow g \\ J & \xrightarrow{k} & C \end{array}$$

be a filling problem. Since  $I$  is a 0-extension and  $f$  is full, there exists a map  $h' : I \rightarrow A$  with  $fh' = h$ . We obtain a new filling problem

$$\begin{array}{ccc} I & \xrightarrow{h'} & A \\ \downarrow e & & \downarrow gf \\ J & \xrightarrow{k} & C \end{array}$$

which can be filled by a map  $m : J \rightarrow A$  since  $gf$  is full. Then  $m' = fm$  is a filler for the original problem (the upper triangle commutes since  $m'e = fme = fh' = h$ ).  $\dashv$

LEMMA 6.7. *Let  $\mathcal{L}$  be a clan-algebraic category, let  $f : A \rightarrow B$  be an arrow in  $\mathcal{L}$  with componentwise full kernel pair  $p, q : R \rightrightarrows A$ , and let  $e : A \twoheadrightarrow C$  be the coequalizer of  $p$  and  $q$ . Then the unique  $m : C \rightarrow B$  with  $me = f$  is monic.*

PROOF. By (D) it is sufficient to test monicity of  $m$  on maps out of compact 0-extensions  $E$ . Let  $h, k : E \rightarrow C$  such that  $mh = mk$ . Since  $e$  is full by (FQ), there exist  $h', k' : E \rightarrow A$  with  $eh' = h$  and  $ek' = k$ . In particular we have  $fh' = fk'$  and therefore there is a unique  $u : E \rightarrow R$  with  $pu = h'$  and  $qu = k'$ . Thus we can argue

$$h = eh' = epu = equ = ek' = k$$

which shows that  $m$  is monic.  $\dashv$

LEMMA 6.8. *If  $A \in \mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Mod}$  is flat then  $A \rightarrow J_N(J_{\otimes}(A))$  is an isomorphism. Thus,  $J_{\otimes}$  restricted to flat models is fully faithful.*

PROOF. For the first claim we have

$$\begin{aligned} J_N(J_{\otimes}(A))(C) &= \mathcal{L}(C, \text{colim}(\text{El}(A) \rightarrow \mathfrak{C}(\mathcal{L}) \hookrightarrow \mathcal{L})) \\ &\cong \text{colim}(\text{El}(A) \rightarrow \mathfrak{C}(\mathcal{L}) \xrightarrow{\mathfrak{J}(C)} \text{Set}) \quad \text{since El}(A) \text{ is filtered} \\ &\cong \mathfrak{J}(C) \otimes A \cong A(C). \end{aligned}$$

The second claim follows since for flat  $B$ , the mapping

$$(\mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Mod})(A, B) \rightarrow \mathcal{L}(J_{\otimes}(A), J_{\otimes}(B))$$

can be decomposed as

$$(\mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Mod})(A, B) \rightarrow (\mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Mod})(A, J_N(J_{\otimes}(B))) \rightarrow \mathcal{L}(J_{\otimes}(A), J_{\otimes}(B)).$$

⊥

LEMMA 6.9. *The following are equivalent for a cone  $\phi : \Delta C \rightarrow D$  on a diagram  $D : \mathbb{J} \rightarrow \mathcal{L}$  in an  $(\mathbf{E}, \mathbf{F})$ -category  $\mathcal{L}$ .*

- (i) *Given an extension  $e : A \rightarrow B$ , an arrow  $h : A \rightarrow C$ , and a cone  $\kappa : \Delta B \rightarrow D$  such that  $\phi_j h = \kappa_j e$  for all  $j \in \mathbb{J}$ , there exists  $l : B \rightarrow C$  such that  $le = h$  and  $\phi_j l = \kappa_j$  for all  $j \in \mathbb{J}$ .*

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ e \downarrow & \nearrow l & \downarrow \phi_j \\ B & \xrightarrow{\kappa_j} & D_j \end{array}$$

- (ii) *The mediating arrow  $C \rightarrow \lim(D)$  is full.*

PROOF. The data of  $e, h, \kappa$  is equivalent to  $e, h$ , and  $k : B \rightarrow \lim(D)$  such that

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ e \downarrow & & \downarrow f \\ B & \xrightarrow{k} & \lim(D) \end{array}$$

commutes, and  $l : B \rightarrow C$  fills the latter square iff it fills all the squares with the  $D_j$ . ⊥

DEFINITION 6.10. A cone  $\phi$  satisfying the conditions of the lemma is called *jointly full*.

REMARK 6.11. The interest of this is that it allows us to talk about full “covers” of limits without actually computing the limits, which is useful when talking about cones and diagrams in the full subcategory of a clan-algebraic category on 0-extensions, which does not admit limits.

DEFINITION 6.12. A *nice diagram* in an  $(E, F)$ -category  $\mathcal{L}$  is a 2-truncated semi-simplicial diagram

$$A_{\bullet} = \left( \begin{array}{ccc} & \xrightarrow{d_0} & \\ A_2 & \xrightarrow{d_1} & A_1 \\ & \xrightarrow{d_2} & \\ & & \xrightarrow{d_1} & A_0 \end{array} \right)$$

where

- (i)  $A_0$ ,  $A_1$ , and  $A_2$  are 0-extensions,
- (ii) the maps  $d_0, d_1 : A_1 \rightarrow A_0$  are full,
- (iii) in the commutative square  $\begin{array}{ccc} A_2 & \xrightarrow{d_0} & A_1 \\ d_2 \downarrow & & \downarrow d_1 \\ A_1 & \xrightarrow{d_0} & A_0 \end{array}$  the span constitutes a jointly full cone over the cospan,
- (iv) there exists a “symmetry” map  $\begin{array}{ccc} A_1 & \xrightarrow{d_1} & A_0 \\ d_0 \downarrow & \searrow \sigma & \uparrow d_0 \\ A_0 & \xleftarrow{d_1} & A_1 \end{array}$  making the triangles commute, and
- (v) there exists a 0-extension  $\tilde{A}$  and full maps  $f, g : \tilde{A} \rightarrow A_1$  constituting a jointly full cone over the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{d_1} & A_1 \\ d_0 \downarrow & \searrow & \downarrow d_1 \\ A_0 & \xrightarrow{d_0} & A_0 \end{array}.$$

LEMMA 6.13. If  $A_{\bullet}$  is a nice diagram in a clan-algebraic category  $\mathcal{L}$ , the pairing  $\langle d_0, d_1 \rangle : A_1 \rightarrow A_0 \times A_0$  factors as  $A_1 \xrightarrow{f} R \xrightarrow{\langle r_0, r_1 \rangle} A_0 \times A_0$ , where  $f$  is full and  $r = \langle r_0, r_1 \rangle$  is monic and a componentwise full equivalence relation.

PROOF. Condition (v) of the preceding definition gives us the following diagram

$$\begin{array}{ccccc} \tilde{A} & & & & \\ \downarrow f & \searrow h & \xrightarrow{g} & & \\ & S & \xrightarrow{p} & A_1 & \\ & \downarrow q & \lrcorner & \downarrow \langle d_0, d_1 \rangle & \\ & A_1 & \xrightarrow{\langle d_0, d_1 \rangle} & A_0 \times A_0 & \end{array},$$

i.e.,  $S$  is the kernel of  $\langle d_0, d_1 \rangle$  with projections  $p, q$ ,  $\tilde{A}$  is a 0-extension, and  $f, g, h$  are full. By right cancellation we deduce that  $p$  and  $q$  are full, and the existence of the factorization follows from Lemma 6.7. Fullness of  $r_0, r_1$  follows again from right cancellation because  $f, d_0$ , and  $d_1$  are full.

It remains to show that  $r$  is an equivalence relation. This is easy: condition 4 gives symmetry, and condition 3 gives transitivity, and reflexivity follows from the fact that  $r_0$  admits a section as a full map into a 0-extension, together with symmetry (we internalize the argument that if in a symmetric and transitive relation everything is related to *something*, then it is reflexive.)  $\dashv$

DEFINITION 6.14. A 0-extension replacement of an object  $A$  in an  $(E, F)$ -category  $\mathcal{L}$  is a full map  $f : \overline{A} \twoheadrightarrow A$  from a 0-extension  $\overline{A}$  to  $A$ .

0-extension replacements can always be obtained as  $(E, F)$ -factorizations of  $0 \rightarrow A$ .

LEMMA 6.15. For every object  $A$  in an  $(E, F)$ -category  $\mathcal{L}$  there exists a nice diagram  $A_\bullet$  with colimit  $A$ .

PROOF.  $A_0$  is constructed as a 0-extension replacement  $f : A_0 \twoheadrightarrow A$  of  $A$ . Similarly,  $A_1$  is given by a 0-extension replacement  $f_1 : A_1 \twoheadrightarrow A_0 \times_A A_0$  of  $A_0 \times_A A_0$ , and  $A_2$  is a 0-extension replacement  $f_2 : A_2 \twoheadrightarrow P$  of the pullback

$$\begin{array}{ccc} P & \xrightarrow{p_0} & A_1 \\ p_1 \downarrow & \lrcorner & \downarrow d_0 \\ A_1 & \xrightarrow{d_1} & A_0 \end{array},$$

with  $d_0, d_1, d_2 : A_2 \rightarrow A_1$  given by  $d_0 = p_0 \circ f$ ,  $d_2 = p_1 \circ f$ , and  $d_1$  a lifting of  $\langle d_0 \circ d_0, d_1 \circ d_2 \rangle$  along  $f_1$ . The map  $\sigma$  is constructed as a lifting of the symmetry of  $A_0 \times_A A_0$  along  $f_1$ . The object  $\tilde{A}$  is a 0-extension replacement of the kernel of  $f_1$ .  $\dashv$

LEMMA 6.16. For any clan-algebraic category  $\mathcal{L}$ , the realization functor  $J_\otimes$  preserves jointly full cones in flat models, and nice diagrams.

PROOF. The first claim follows since  $J_\otimes$  is fully faithful on 0-extensions by Lemma 6.8 and in both sides the weak factorization system determined by the same generators. Thus there's a one-to-one correspondence between lifting problems. The second claim follows since  $J_\otimes$  preserves 0-extensions and 0-extensions are flat by the fat small object argument.  $\dashv$

LEMMA 6.17. For any clan-algebraic category  $\mathcal{L}$ , the functor  $J_N : \mathcal{L} \rightarrow \mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Mod}$  preserves quotients of nice diagrams.

PROOF. Given a nice diagram  $A_\bullet$  in  $\mathcal{L}$ , its colimit is the coequalizer of  $d_0, d_1 : A_1 \rightarrow A_0$ . By Lemma 6.13,  $\langle d_0, d_1 \rangle$  factors as  $\langle r_0, r_1 \rangle \circ f$  with  $f$  full and  $r = \langle r_0, r_1 \rangle$  an equivalence relation. The pairs  $d_0, d_1$  and  $r_0, r_1$  have the same coequalizer (since  $f$  is epic), and  $J_N$  preserves the coequalizer of  $r_0, r_1$  since it preserves full maps and kernel pairs. Finally, the coequalizer of  $J_N(r_0), J_N(r_1)$  is also the coequalizer of  $J_N(d_0), J_N(d_1)$  since  $J_N(f)$  is full and therefore epic.  $\dashv$

THEOREM 6.18. If  $\mathcal{L}$  is clan-algebraic, then  $J_N : \mathcal{L} \rightarrow \mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Mod}$  is an equivalence.

PROOF. By density,  $J_N$  is fully faithful. It remains to verify that it is essentially surjective, and to this end we show that the unit map  $\eta_A : A \rightarrow J_N(J_\otimes(A))$  is an

isomorphism for all  $A \in \mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Mod}$ . Let  $A_\bullet$  be a nice diagram with colimit  $A$ . We have:

$$\begin{aligned} J_N(J_\otimes(A)) &= J_N(J_\otimes(\text{colim}(A_\bullet))) \\ &= \text{colim}(J_N(J_\otimes(A_\bullet))) && \text{by Lemmas 6.16 and 6.17} \\ &= \text{colim}(A_\bullet) && \text{by Lemma 6.8} \\ &= A. \end{aligned}$$

⊣

We are now ready to prove our main theorem.

**THEOREM 6.19.** *The contravariant biadjunction between clans and  $(E, F)$ -categories from Proposition 4.6 restricts to a contravariant biequivalence between Cauchy-complete clans and clan-algebraic categories.*

$$\begin{array}{ccc} \text{Clan}_{\text{cc}}^{\text{op}} & \xrightarrow{\quad} & \text{Clan}^{\text{op}} \\ \simeq \updownarrow & \mathfrak{C}(-)^{\text{op}} \uparrow \dashv \downarrow (-)\text{-Mod} & \\ \text{ClanAlg} & \xrightarrow{\quad} & \text{EFCat} \end{array}$$

**PROOF.** By Theorem 5.9 the biadjunction is idempotent, and its fixed points in  $\text{Clan}$  are Cauchy-complete clans. By Theorems 6.2 and 6.18, the fixed points in  $\text{EFCat}$  are precisely the clan-algebraic categories. ⊣

**REMARKS 6.20.** (a) The characterization of  $(E, F)$ -categories of the form  $\mathcal{T}\text{-Mod}$  in terms of conditions (D), (CG), and (FQ) generalizes, and is in fact inspired by, Adámek–Rosický–Vitale’s characterization of *algebraic categories* (i.e., categories of models of algebraic theories) as *Barr-exact cocomplete locally small categories with a strong generator of compact regular projectives* [3, Corollary 18.4]<sup>7</sup>. In particular, if  $\mathcal{T}$  is a finite-product clan then the regular projectives in  $\mathcal{T}\text{-Mod}$  are precisely the 0-extensions, thus the strong generation requirement corresponds to (D) (which we have elected to state in terms of density). The Barr-exactness requirement refines to the existence of *full coequalizers of componentwise full equivalence relations* in the clanic case. The fullness requirements are void in the finite-product case, since coequalizers as well as split epimorphisms are always regular epimorphisms.

(b) As emphasized by the referee, Definition 6.12 of “nice diagrams” resembles Carboni–Vitale’s definition of *pseudo-equivalence relations* [11, Definition 6]. Indeed, the resolution by nice diagrams given in Lemma 6.15 reduces to pseudo-equivalence relations in projective objects in the case of finite-product clans, which are related to the fact that algebraic categories are *ex/wlex-completions* in the sense of Carboni and Vitale. An analogous presentation of clan-algebraic categories as a completion of categories of 0-extensions (suitably axiomatized) is subject of future work.

**6.1. Clan-algebraic weak factorization systems on  $\text{Cat}$ .** The characterization of  $(E, F)$ -categories of models of clans as clan-algebraic categories allows to exhibit

<sup>7</sup>See also [1, Theorem 5.5] for an earlier version of this using slightly different notions

new clans by defining suitable w.f.s.s on l.f.p. categories. In this subsection we demonstrate this by defining three more clan-algebraic w.f.s.s on  $\text{Cat}$ .

We start with the clan-algebraic w.f.s. from Example 2.12(c), corresponding to the “standard” clan-presentation  $\mathcal{T}_{\text{Cat}}$  from §2.2. We observed that this w.f.s. is cofibrantly generated by the functors  $0 \hookrightarrow 1$  and  $2 \hookrightarrow 2$ . Our strategy to define new clan-algebraic w.f.s.s is to add additional generators. If we make sure that the domain and codomain of these are compact 0-extensions, we only have to verify condition (FQ) when verifying that the new w.f.s. is still clan-algebraic. The additional generators we consider are the arrows  $\mathbb{P} \rightarrow 2$  and  $2 \rightarrow 1$ , where  $\mathbb{P}$  is the “parallel pair category”  $\bullet \rightrightarrows \bullet$ . By adding either one or both of the additional generators we obtain three additional w.f.s.s  $(E_O, F_O)$ ,  $(E_A, F_A)$ , and  $(E_{OA}, F_{OA})$ , where:

$$\begin{aligned} F &= \{(0 \rightarrow 1), (2 \rightarrow 2)\}^{\text{cl}} \\ F_O &= \{(0 \rightarrow 1), (2 \rightarrow 2), (2 \rightarrow 1)\}^{\text{cl}} \\ F_A &= \{(0 \rightarrow 1), (2 \rightarrow 2), (\mathbb{P} \rightarrow 2)\}^{\text{cl}} \\ F_{OA} &= \{(0 \rightarrow 1), (2 \rightarrow 2), (2 \rightarrow 1), (\mathbb{P} \rightarrow 2)\}^{\text{cl}}. \end{aligned}$$

We have already observed that  $F$  consists of the functors that are full and surjective on objects, and it is easy to see that  $F_O$  contains only those functors which are full and *bijective* on objects, whereas  $F_A$  consists of functors which are *fully faithful* and surjective on objects. Finally,  $F_{OA}$  only contains functors which are fully faithful and bijective on objects, i.e., isomorphisms of categories.

To convince ourselves that the new w.f.s.s are indeed clan-algebraic we only have to verify that for every equivalence relation  $\langle p, q \rangle : \mathbb{R} \rightarrow \mathbb{A} \times \mathbb{A}$  in  $\text{Cat}$ , the coequalizer is in either of  $F_O, F_A, F_{OA}$  whenever  $p$  and  $q$  are, since effectivity has already been established for equivalence relations with components in  $F$ . This is not difficult to see for  $F_O$  and  $F_A$ , and trivial for  $F_{OA}$ .

The coclans corresponding to the new w.f.s.s are:

- $\mathcal{T}_{\text{Cat}_O}^{\text{op}} = \{\text{categories free on finite graphs}\}$ , with functors  $G^* \rightarrow H^*$  arising from faithful graph morphisms as codisplay maps,
- $\mathcal{T}_{\text{Cat}_A}^{\text{op}} = \{\text{finitely presented categories}\}$ , with injective-on-objects functors as codisplay maps, and
- $\mathcal{T}_{\text{Cat}_{OA}}^{\text{op}} = \{\text{finitely presented categories}\}$ , with arbitrary functors as codisplay maps.

We note the clan  $\mathcal{T}_{\text{Cat}_{OA}}$  is simply the finite-limit theory of categories. One may ask whether the clans  $\mathcal{T}_{\text{Cat}_O}$ ,  $\mathcal{T}_{\text{Cat}_A}$ , and  $\mathcal{T}_{\text{Cat}_{OA}}$  admit simple syntactic presentations by GATs, and indeed they do. To obtain such a presentation e.g., for  $\mathcal{T}_{\text{Cat}_O}$ , we have to modify the GAT  $\mathbb{T}_{\text{Cat}}$  in such a way that the syntactic category stays the same, but acquires additional display maps, such as the diagonal  $(x : O) \rightarrow (x y : O)$  corresponding to the new generator  $2 \rightarrow 1$ . Display maps in the syntactic category of a GAT are always of the form  $p \circ i$  where  $p$  is a projection omitting a finite number of variables and  $i$  is an isomorphism (Proposition 10.3), so to turn  $(x : O) \rightarrow (x y : O)$  into a display map we have to make  $(x : O)$  isomorphic to an *extension* of  $(x y : O)$ . To achieve this we postulate a new type family  $x y : O \vdash E_O(x, y)$  and add axioms forcing the projection  $(x y : A, z : E_O(x, y)) \rightarrow (x : A)$  to



become an isomorphism:

$$\begin{aligned}
 & x y : O \vdash E_O(x, y) \\
 & x : O \vdash r_O(x) : E_O(x, x) \\
 & x y : O, p : E_O(x, y) \vdash x = y \\
 & x : O, p : E_O(x, x) \vdash r_O(x) = p.
 \end{aligned} \tag{6.1}$$

The function symbol  $r_O$  gives a section for the projection, and the two last axioms force the retract to be an isomorphism. We recognize at once that these axioms make  $E_O$  an *extensional identity type* of  $O$  [23, Section 3.2]: the term  $r_O$  is reflexivity, and the third and fourth rule give equality reflection and uniqueness of identity proofs. We write  $\mathbb{T}_{\text{Cat}_O}$  for the extension of the GAT  $\mathbb{T}_{\text{Cat}}$  by the axioms (6.1), and  $\text{Cat}_O$  for the corresponding clan-algebraic category.

Similarly, we obtain a GAT-representation  $\mathbb{T}_{\text{Cat}_A}$  of the clan  $\mathcal{T}_{\text{Cat}_A}$  by augmenting  $\mathbb{T}_{\text{Cat}}$  by a type family  $E_A$  with the following rules:

$$\begin{aligned}
 & x y : O, f g : A(x, y) \vdash E_A(f, g) \\
 & x y : O, f : A(x, y) \vdash r_A(f) : E_A(f, f) \\
 & x y : O, f g : A(x, y), p : E_A(f, g) \vdash f = g \\
 & x y : O, f : A(x, y), p : E_A(f, f) \vdash r_A(f) = p.
 \end{aligned} \tag{6.2}$$

Adding both sets of axioms (6.1) and (6.2) to  $\mathbb{T}_{\text{Cat}}$  yields a GAT for the clan  $\mathcal{T}_{\text{Cat}_{OA}}$ , i.e., the finite-limit theory of categories.

**REMARK 6.21.** As pointed out by the referee, an enriched version of the  $(E_A, F_A)$ -instance of the exactness condition (FQ) appears under the name *ff-exactness* in Bourke and Garner’s [9]. Further investigation may be needed.

**§7. A counterexample.** This section gives a common counterexample to two related natural questions about the extension–full w.f.s. on a clan-algebraic category  $\mathcal{L}$ :

- (1) Does every compact object admit a full map from a compact 0-extension?
- (2) Does the weak factorization system always restrict to compact 0-extensions?

The counterexample to both question is given by the category of models of the following GAT with infinitely many sorts and operations:

$$\begin{aligned}
 & \vdash X \\
 & \vdash Y \\
 & y : Y \vdash Z_n(y) & n \in \mathbb{N} \\
 & x : X \vdash f(x) : Y \\
 & x : X \vdash g_n(x) : Z_n(f(x)). & n \in \mathbb{N}
 \end{aligned}$$

Its category of models is equivalent to the set-valued functors on the posetal category

$$\mathbb{C} = \left( \begin{array}{c} X \\ \downarrow g_0 \quad \searrow g_1 \quad \searrow g_n \\ Z_0 \quad Z_1 \quad \dots \quad Z_n \quad \dots \\ \downarrow z_0 \quad \swarrow z_1 \quad \swarrow z_n \\ Y \end{array} \right)$$

and the w.f.s. on  $[\mathbb{C}, \text{Set}]$  is generated by the arrows  $(\emptyset \rightarrow \mathfrak{J}(X))$ ,  $(\emptyset \rightarrow \mathfrak{J}(Y))$ , and  $(\mathfrak{J}(Y) \rightarrow \mathfrak{J}(Z_n))$  for  $n \in \mathbb{N}$ , reflecting the idea that models  $A : \mathbb{C} \rightarrow \text{Set}$  can be built up by successively adding elements to  $A(X)$  or  $A(Y)$ , and to  $A(Z_n)$  over a given element  $x$  of  $A(Y)$ , as in the following pushouts.

$$\begin{array}{ccc} \emptyset \longrightarrow A & \emptyset \longrightarrow A & \mathfrak{J}(Y) \xrightarrow{\hat{x}} A \\ \Downarrow \quad \sqsubset \quad \Downarrow & \Downarrow \quad \sqsubset \quad \Downarrow & \Downarrow \quad \sqsubset \quad \Downarrow \\ 1 = \mathfrak{J}(X) \longrightarrow 1 + A & \mathfrak{J}(Y) \longrightarrow \mathfrak{J}(Y) + A & \mathfrak{J}(Z_n) \longrightarrow B \end{array}$$

The following lemma gives explicit descriptions of the w.f.s. and the compact objects in  $[\mathbb{C}, \text{Set}]$ .

LEMMA 7.1. *Let  $f : A \rightarrow B$  in  $[\mathbb{C}, \text{Set}]$ .*

- (i)  *$f$  is full if and only if it is componentwise surjective and the naturality squares for  $z_n$  are weak pullbacks for all  $n \in \mathbb{N}$ .*
- (ii)  *$f$  is an extension if and only if  $f_X : A(X) \rightarrow B(X)$  is injective, and the squares*

$$\begin{array}{ccc} A(X) \longrightarrow B(X) & A(X) \longrightarrow B(X) \\ \downarrow \quad \quad \downarrow & \downarrow \quad \quad \downarrow \\ A(Y) \longrightarrow B(Y) & A(Z_n) \longrightarrow B(Z_n) \end{array}$$

*are quasi-pushouts, in the sense that the gap maps  $A(Y) +_{A(X)} B(X) \rightarrow B(Y)$  and  $A(Z_n) +_{A(X)} B(X) \rightarrow B(Z_n)$  are injective. (This implies that the components  $f_Y$  and  $f_{Z_n}$  are also injective).*

- (iii)  *$A$  is a 0-extension if and only if  $A(f)$  and all  $A(g_n)$  are injective.*
- (iv)  *$A$  is compact if and only if (a) it is componentwise finite, and (b)  $A(f_n) : A(X) \rightarrow A(Z_n)$  is a bijection for all but finitely many  $n \in \mathbb{N}$ .*

Using this lemma, we can give negative answers to the two question at the beginning of the section.

PROPOSITION 7.2. (i) *The object  $P$  in the pushout*

$$\begin{array}{ccc} \mathfrak{J}(Y) + \mathfrak{J}(Y) \longrightarrow \mathfrak{J}(X) + \mathfrak{J}(X) \\ \downarrow \quad \quad \quad \sqsubset \quad \downarrow \\ \mathfrak{J}(Y) \longrightarrow P \end{array}$$

*is compact, but does not admit a full map from a compact 0-extension.*

- (ii) *The map  $\mathfrak{J}(Y) \rightarrow \mathfrak{J}(X)$  does not admit an extension–full factorization through a compact object.*

PROOF. For the first claim,  $P$  is compact as a finite colimit of representables. Let  $f : E \rightarrow P$  be a full map with  $E$  a 0-extension. For each  $n \in \mathbb{N}$  we get a diagram

$$\begin{array}{ccc}
 E(X) & \xrightarrow{f_X} & 2 \\
 \downarrow E(f) & \searrow E(g_n) & \downarrow \\
 & E(Z_n) & \xrightarrow{f_{Z_n}} 2 \\
 & \downarrow E(z_n) & \downarrow \\
 E(Y) & \xrightarrow{f_Y} & 1
 \end{array}$$

where  $E(f)$  and  $E(g_n)$  are injective because  $E$  is a 0-extension, and the components of  $f$  are surjective and the  $z_n$ -naturality square is a weak pullback since because  $f$  is full. In particular  $E(Y)$  is inhabited and the fibers of  $E(z_n)$  have at least two elements. Since the fibers of  $E(f)$  have at most one element, this means that  $E(g_n)$  can't be surjective for any  $n$ , and it follows from Lemma 7.1(iv) that  $E$  is not compact.

For the second claim consider an extension  $e : \mathcal{K}(Y) \rightarrow A$  such that  $A \rightarrow \mathcal{K}(X) = 1$  is full. Then  $A(Y)$  is inhabited, all  $A(Z_n) \rightarrow A(Y)$  are surjective, and  $1 + A(X) \rightarrow A(Y)$  is injective. From this we can again deduce that none of the  $A(g_n)$  are surjective and thus  $A$  is not compact.  $\dashv$

**§8. Models in higher types.** One practical use of having inequivalent clans with equivalent categories of Set-models is that they can have inequivalent  $\infty$ -categories of models in the  $\infty$ -category  $\mathcal{S}$  of *homotopy types* (a.k.a. “spaces”). We leave this issue for future work and content ourselves here with outlining some main ideas.

Reasoning informally, in the following we identify sets with discrete (0-truncated) homotopy types, and 1-categories with  $\infty$ -categories all of whose hom-types are 0-truncated. Given a clan  $\mathcal{T}$  it then makes sense to define an  $\infty$ -model to be an  $\infty$ -functor  $\mathcal{T} \rightarrow \mathcal{S}$  into spaces which preserves the terminal object and pullbacks of display maps in the suitable  $\infty$ -categorical sense, and to write  $\mathcal{T}\text{-Mod}_\infty \subseteq [\mathcal{T}, \mathcal{S}]$  for the  $\infty$ -category of models, as a full  $\infty$ -subcategory of the functor- $\infty$ -category  $[\mathcal{T}, \mathcal{S}]$ .

The first observation is that for  $\mathcal{T}$  a *finite-limit clan*, the  $\infty$ -category  $\mathcal{T}\text{-Mod}_\infty$  of  $\infty$ -models is in fact equivalent to (the  $\infty$ -category corresponding to) the 1-category  $\mathcal{T}\text{-Mod}$  of 1-models! This is because finite  $\infty$ -limit preserving  $\infty$ -functors preserve truncation levels, and thus every finite  $\infty$ -limit preserving  $F : \mathcal{L} \rightarrow \mathcal{S}$  must factor through the inclusion of  $\text{Set} \hookrightarrow \mathcal{S}$ , since all objects of 1-categories are (representably) 0-truncated.

For finite-product clans, on the other hand, there is no such restriction. The  $\infty$ -models of the finite-product clan  $\mathcal{C}_{\text{Mon}}$  of monoids, for example, are the models of the *associative  $\infty$ -operad* [30, Section 4.11], whereas the  $\infty$ -models of the finite-product theory of abelian groups are related to the Dold–Kan correspondence. Variants of this phenomenon are discussed under the name “animation” in [14]; Rosický’s [40] contains an earlier account.

Now the nice thing about clans is that they admit finer graduations of “levels of strictness”. Among the clans  $\mathcal{T}$ ,  $\mathcal{T}_{\text{Cat}_0}$ ,  $\mathcal{T}_{\text{Cat}_A}$ , and  $\mathcal{T}_{\text{Cat}_{0A}}$  from §6.1, for example, we know that the  $\infty$ -models of the finite-limit clan  $\mathcal{T}_{\text{Cat}_{0A}}$  are precisely the strict

1-categories. The presence of the extensional identity type on  $O$  in  $\mathcal{T}_{\text{Cat}_O}$  behaves like a kind of “partial finite-limit completion”, and has the effect that the sort  $O$  is interpreted by a 0-type in every model  $\mathcal{T}_{\text{Cat}_O} \rightarrow \mathcal{S}$ , whereas the presence of extensional identities on  $A$  in  $\mathcal{T}_{\text{Cat}_A}$  has the effect that the projection  $(x\ y : O, f : A(x, y)) \rightarrow (x\ y : O)$  is mapped to a function with 0-truncated fibers by every  $\infty$ -model  $C : \mathcal{T}_{\text{Cat}_A} \rightarrow \mathcal{S}$ . This means that  $\infty$ -models of  $\mathcal{T}_{\text{Cat}_A}$  are *pre-categories* in the sense of *Homotopy Type Theory* [43, Definition 9.1.1], whereas  $\infty$ -models of  $\mathcal{T}_{\text{Cat}_O}$  seem to correspond to *Segal-categories* [22, Section 2], [7, Section 5]. Finally, the clan  $\mathcal{T}_{\text{Cat}}$  does not impose any truncation conditions, which makes its  $\infty$ -models resemble *Segal spaces* (not necessarily complete), in the sense of [38, Section 4].

REMARK 8.1. Notably absent from the list of higher algebraic structures represented by the variants of  $\mathcal{T}_{\text{Cat}}$  are univalent  $\infty$ -categories. In fact, the requirement on a clan  $\mathcal{T}$  that  $\mathcal{T}\text{-Mod}_\infty \simeq \infty\text{-Cat}$  is incompatible with the requirement that  $\mathcal{T}\text{-Mod} \simeq \text{Cat}$ , since  $\mathcal{T}\text{-Mod}$  is the full subcategory of  $\mathcal{T}\text{-Mod}_\infty$  on 0-truncated objects, and the only 0-truncated univalent 1-categories are the *rigid* ones, i.e., those whose with only trivial automorphisms. It seems unlikely to the author that  $\infty\text{-Cat}$  can be described by a “1-clan”. See [4, 5] for recent work addressing this univalence issue, using techniques building on Makkai’s *First order logic with dependent sorts (FOLDS)* [32]. Another recent work connecting Cartmell’s GATs and Makkai’s FOLDS is Chaitanya Leena Subramaniam’s thesis [28].

**§9. Appendix. Locally finitely presentable categories, weak factorization systems, and Quillen’s small object argument.** This appendix recalls basic definitions and facts about the concepts mentioned in the title.

DEFINITION 9.1. A category  $\mathcal{C}$  is called *filtered*, if every diagram  $D : \mathbb{J} \rightarrow \mathcal{C}$  with finite domain admits a cocone. A *filtered colimit* is a colimit of a diagram indexed by a filtered category.

DEFINITION 9.2. Let  $\mathfrak{X}$  be a cocomplete locally small category.

- (i) An object  $C \in \mathfrak{X}$  is called *compact*, if the covariant representable functor

$$\mathfrak{X}(C, -) : \mathfrak{X} \rightarrow \text{Set}$$

preserves small filtered colimits.

- (ii)  $\mathfrak{X}$  is called *locally finitely presentable* (l.f.p.) if it admits a *small dense family of compact objects*, i.e., a family  $(C_i)_{i \in I}$  of compact objects indexed by a small set  $I$ , such that the nerve functor

$$J_N : \mathfrak{X} \rightarrow \widehat{\mathbb{C}}$$

of the inclusion  $J : \mathbb{C} \hookrightarrow \mathfrak{X}$  of the full subcategory on the  $(C_i)_{i \in I}$  is fully faithful.

REMARKS 9.3. (a) Compact objects are also known as *finitely presentable objects*, e.g., in [16, 2]. We adopted the term compact from [29, Definition A.1.1.1] since it is more concise, and in particular since *compact 0-extension* sounds less awkward than *finitely presented 0-extension*. Moreover I think the fact that objects of algebraic categories (such as groups, rings, modules ...) are

compact if and only if they admit a presentation by finitely many generators and relations is an important theorem, which is difficult to state if one uses the same terminology for the syntactic and the categorical notion.

- (b) The density condition in the definition is equivalent to saying that the family  $(C_i)_{i \in I}$  is a *strong generator*, in the sense that the canonical arrow

$$\coprod_{i \in I, f: C_i \rightarrow A} C_i \rightarrow A$$

is an extremal epimorphism for all  $A \in \mathfrak{X}$ . We stated the definition in terms of density here, since nerve functors play a central role in this work, contrary to strong generation.

- (c) The notion of l.f.p. category is a special case of the notion of *locally  $\alpha$ -presentable category* for a regular cardinal  $\alpha$  [16, 2]. In this work, only the case  $\alpha = \omega$  plays a role.

DEFINITION 9.4. Let  $\mathcal{C}$  be a category.

- (i) Given two arrows  $f : A \rightarrow B$ ,  $g : X \rightarrow Y$  in  $\mathcal{C}$ , we say that  $f$  has the *left lifting property* (l.l.p.) w.r.t.  $g$  (or equivalently that  $g$  has the *right lifting property* (r.l.p.) w.r.t.  $f$ ), and write  $f \pitchfork g$ , if in each commutative square

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ f \downarrow & \nearrow m & \downarrow g \\ B & \xrightarrow{k} & Y \end{array}$$

there exists a diagonal arrow  $h$  making the two triangles commute.

- (ii) Given a class  $E \subseteq \text{mor}(\mathcal{C})$  of arrows in  $\mathcal{C}$ , we define:

$$\pitchfork E = \{f \in \text{mor}(\mathcal{C}) \mid \forall g \in E. f \pitchfork g\}$$

$$E^\pitchfork = \{g \in \text{mor}(\mathcal{C}) \mid \forall f \in E. f \pitchfork g\}.$$

- (iii) A *weak factorization system* (w.f.s.) on  $\mathcal{C}$  is a pair  $\mathcal{L}, \mathcal{R} \subseteq \text{mor}(\mathcal{C})$  of classes of morphisms such that  $\mathcal{L}^\pitchfork = \mathcal{R}$ ,  $\mathcal{R}^\pitchfork = \mathcal{L}$ , and every  $f : A \rightarrow B$  in  $\mathcal{C}$  admits a factorization  $f = r \circ l$  with  $l \in \mathcal{L}$  and  $r \in \mathcal{R}$ .

We call  $\mathcal{L}$  the *left class*, and  $\mathcal{R}$  the *right class* of the w.f.s. One can show that left classes of w.f.s. contain all isomorphisms, and are closed under composition and pushouts, i.e., if

$$\begin{array}{ccc} A & \longrightarrow & B \\ l \downarrow & \lrcorner & \downarrow m \\ C & \longrightarrow & D \end{array}$$

is a pushout in  $\mathcal{C}$  and is a left map, then so is  $m$ . Dually, right maps are closed under (isomorphisms, composition, and) pullbacks. With this, we have the prerequisites to state Quillen's *small object argument*.

THEOREM 9.5 (Small object argument for l.f.p. categories). *Let  $E \subseteq \text{mor}(\mathfrak{X})$  be a small set of morphisms in a l.f.p. category. Then  $(\pitchfork(E^\pitchfork), E^\pitchfork)$  is a w.f.s. on  $\mathfrak{X}$ .*

PROOF. Hovey [24, Theorem 2.1.14] and Riehl [39, Theorem 12.2.2] prove stronger statements in a more general setting.  $\dashv$

**§10. Appendix. Generalized algebraic theories.** Cartmell's *generalized algebraic theories* extend the notion of algebraic theory (which can be ‘single sorted’, such as the theories of *groups* or *rings*, or ‘many sorted’, such as the theories of *reflexive graphs*, *chain complexes of abelian groups*, or *modules over a non-fixed base ring*) by introducing *dependent sorts* (a.k.a. dependent ‘types’), which represent families of sets and can be used e.g., to axiomatize the notion of a (small) category  $\mathbb{C}$  as a structure with a set  $\mathbb{C}_0$  of objects, and a family  $(\mathbb{C}(A, B))_{A, B \in \mathbb{C}_0}$  of hom-sets (see (10.1) below).

Compared to ordinary algebraic theories, whose specification in terms of sorts, operations, and equations is fairly straightforward, the syntactic description of generalized algebraic theories is complicated by the fact that the domains of definition of operations and dependent sorts, and the codomains of operations, may themselves be compound expressions involving previously declared operations and sorts, whose well-formedness has to be ensured and may even depend on the equations of the theory. This means that we have to state the declarations of *sorts* and of *operations*, and the *equations* (which we collectively refer to as *axioms* of the theory) in an ordered way, where the later axioms have to be well-formed on the basis of the earlier axioms. This looks as follows in the case of the generalized algebraic theory  $\mathbb{T}_{\text{Cat}}$  of categories:

$$\begin{aligned}
 & \vdash O \\
 & x\ y : O \vdash A(x, y) \\
 & x : O \vdash \text{id}(x) : A(x, x) \\
 & x\ y\ z : O, f : A(x, y), g : A(y, z) \vdash g \circ f : A(x, z) \quad (10.1) \\
 & x\ y : O, f : A(x, y) \vdash \text{id}(y) \circ f = f \\
 & x\ y : O, f : A(x, y) \vdash f \circ \text{id}(x) = f \\
 & w\ x\ y\ z : O, e : A(w, x), f : A(x, y), g : A(y, z) \vdash (g \circ f) \circ e = g \circ (f \circ e)
 \end{aligned}$$

Each line contains one axiom, the first two declaring the sort  $O$  of objects and the dependent sort  $A(x, y)$  of arrows, the third and the fourth declaring the identity and composition operations, and the last three stating the identity and associativity axioms.

Each axiom is of the form  $\Gamma \vdash \mathcal{J}$ , where the  $\mathcal{J}$  on the right of the ‘turnstile’ symbol ‘ $\vdash$ ’ is the actual declaration or equation, and the part  $\Gamma$  on the left—called ‘context’—specifies the sorts of the variables occurring in  $\mathcal{J}$ . Note that the ordering of these ‘variable declarations’ is not arbitrary, since the sorts of variables may themselves contain variables which have to be declared further left in the context. An example is the context  $(x\ y\ z : O, f : A(x, y), g : A(y, z))$  of the composition operation, where the sorts of the ‘arrow’ variables  $f, g$  depend on the ‘object’ variables  $x, y, z$ . See Figure 1 for another example generalized algebraic theory: the generalized algebraic theory of *rings graded over monoids*.

The dependent structure of contexts and the well-formedness requirement of axioms on the basis of other axioms makes the formulation of a general notion of

---

	$\vdash M$
$u : M$	$\vdash R(u)$
	$\vdash e : M$
$u v : M$	$\vdash u \cdot v : M$
$u : M$	$\vdash 0(u) : R(u)$
$u : M, x y : R(u)$	$\vdash x + y : R(u)$
$u : M, x : R(u)$	$\vdash -x : R(u)$
	$\vdash 1 : R(e)$
$u v : M, x : R(u), y : R(v)$	$\vdash x \cdot y : R(u \cdot v)$
$u : M$	$\vdash e \cdot u = u = u \cdot e$
$u v w : M$	$\vdash (u \cdot v) \cdot w = u \cdot (v \cdot w)$
$u : M, x y : R(u)$	$\vdash x + y = y + x$
$u : M, x y : R(u)$	$\vdash x + 0(u) = x$
$u : M, x y : R(u)$	$\vdash x + (-x) = 0(u)$
$u : M, x : R(u)$	$\vdash 1 \cdot x = x = x \cdot 1$
$u v w : M, x : R(u), y : R(v), z : R(w)$	$\vdash (x \cdot y) \cdot z = x \cdot (y \cdot z)$
$u v : M, x : R(u), y z : R(v)$	$\vdash x \cdot (y + z) = x \cdot y + x \cdot z$
$u v : M, x y : R(u), z : R(v)$	$\vdash (x + y) \cdot z = x \cdot z + y \cdot z$

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FIGURE 1. The generalized algebraic theory of monoid-graded rings

generalized algebraic theory somewhat subtle and technical. We refer to [12, 13] for the authoritative account and to [36, Section 6] and [17, Section 2] for rigorous and concise summaries. The good news is that to understand specific examples of GATs, these technicalities may safely be ignored: all we have to know is that for every generalized algebraic theory  $\mathbb{T}$  there is a notion of ‘derivable judgment’ which includes the axioms and is closed under various *rules* expressing that the set of derivable judgments is closed under operations like substitutions and weakening, and that equality is reflexive, symmetric, and transitive.

Besides the forms of judgments

$\Gamma \vdash S$	‘ $S$ is a sort in context $\Gamma$ ’
$\Gamma \vdash t : S$	‘ $t$ is term of sort $S$ in context $\Gamma$ ’
$\Gamma \vdash s = t : S$	‘ $s$ and $t$ are equal terms in context $\Gamma$ ’

that we have already encountered, we consider the following additional forms of judgments:

$\Gamma \vdash S = T$	‘ $S$ and $T$ are equal sorts in context $\Gamma$ ’
$\Gamma \vdash$	‘ $\Gamma$ is a context’
$\Gamma = \Delta \vdash$	‘ $\Gamma$ and $\Delta$ are equal contexts’
$\Gamma \vdash \sigma : \Delta$	‘ $\sigma$ is a substitution from $\Gamma$ to $\Delta$ ’
$\Gamma \vdash \sigma = \tau : \Delta$	‘ $\sigma$ and $\tau$ are equal substitutions from $\Gamma$ to $\Delta$ ’

The last two of these introduce a novel kind of expression called *substitution*: a substitution  $\Gamma \vdash \sigma : \Delta$  is a list of terms that is suitable to be simultaneously



substituted for the variables in a judgment in context  $\Delta$  (in particular  $\sigma$  and  $\Delta$  must have the same length), to produce a new judgment in context  $\Gamma$ , as expressed by the following *substitution* rule.

$$\frac{\Gamma \vdash \sigma : \Delta \quad \Delta \vdash \mathcal{J}}{\Gamma \vdash \mathcal{J}[\sigma]} \quad (\text{Subst})$$

Here,  $\mathcal{J}[\sigma]$  is the result of simultaneous substitution of the terms in  $\sigma$  for the variables in  $\mathcal{J}$ , replacing each occurrence of the  $i$ th variable declared in  $\Delta$  with the  $i$ th term in  $\sigma$ . This operation of simultaneous substitution also appears in the derivation rules for substitutions themselves, which we present in the following table together with the rules for the formation of well-formed contexts:

$$\begin{array}{c} \frac{}{* \vdash} \quad \frac{}{\Gamma \vdash () : *} \\[10pt] \frac{\Gamma \vdash A}{\Gamma, y : A \vdash} \quad \frac{\Gamma \vdash \sigma : \Delta \quad \Delta \vdash A \quad \Gamma \vdash t : A[\sigma]}{\Gamma \vdash (\sigma, t) : (\Delta, x : A)} \end{array} \quad (10.2)$$

The two rules in the first line say respectively that the *empty context*  $\emptyset$  is a context, and that for any context  $\Gamma$ , the *empty substitution*  $()$  is a substitution to the empty context. The first rule in the second line is known as *context extension*, since it says that we can extend any context by a well-formed sort in this context (here  $y$  has to be a ‘fresh’ variable, i.e., a variable not appearing in  $\Gamma$ ). The last rule says that a substitution to an extended context is a pair of a substitution into the original context and a term whose sort is a *substitution instance* of the extending sort—it wouldn’t make sense to ask for  $t$  to be of sort  $A$  since  $A$  is only well-formed in context  $\Delta$ , and we want something in context  $\Gamma$ .

### 10.1. The syntactic category of a generalized algebraic theory.

DEFINITION 10.1. The *syntactic category*  $\mathcal{C}[\mathbb{T}]$  of a generalized algebraic theory  $\mathbb{T}$  is given as follows.

- The objects are the contexts of  $\Gamma$  *modulo derivable equality*, i.e., contexts  $\Gamma$  and  $\Delta$  are identified if the judgment  $\Gamma = \Delta \vdash$  is derivable.
- Similarly, morphisms  $[\Gamma] \rightarrow [\Delta]$  from the equivalence class of  $\Gamma$  to the equivalence class of  $\Delta$  are substitutions  $\Gamma \vdash \sigma : \Delta$  modulo derivable equality. (The closure conditions on the set of derivable judgments ensure independence of representatives, e.g., that  $\Gamma' \vdash \sigma : \Delta'$  whenever  $\Gamma \vdash \sigma : \Delta$  and  $\Gamma = \Gamma' \vdash$  and  $\Delta = \Delta' \vdash$ .)
- Composition is given by substitution of representatives, and identities are given by lists of variables:
  - $[\Delta \vdash \tau : \Theta] \circ [\Gamma \vdash \sigma : \Delta] = [\Gamma \vdash \tau[\sigma] : \Theta]$
  - $\text{id}_\Gamma = (\Gamma \vdash (\vec{x}) : \Gamma)$  where  $\vec{x}$  is the list of variables declared in  $\Gamma$ .

The syntactic category  $\mathcal{C}[\mathbb{T}]$  of a GAT  $\mathbb{T}$  has the structure of a *contextual category*:

DEFINITION 10.2. A *contextual category* consists of

- (1) a small category  $\mathcal{C}$  with a grading function  $\deg : \mathcal{C}_0 \rightarrow \mathbb{N}$  on its objects, and

- (2) a presheaf  $\text{Ty} : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ , together with
- an object  $\Gamma.A$  and an arrow  $p_A : \Gamma.A \rightarrow \Gamma$  for each  $\Gamma \in \mathcal{C}$  and  $A \in \text{Ty}(\Gamma)$ , and
  - an arrow  $\sigma.A : \Delta.A\sigma \rightarrow \Gamma.A$  for each  $\Gamma \in \mathcal{C}$ ,  $A \in \text{Ty}(\Gamma)$ , and  $\sigma : \Delta \rightarrow \Gamma$ ,

such that:

- (i) The square 
$$\begin{array}{ccc} \Delta.A\sigma & \xrightarrow{\sigma.A} & \Gamma.A \\ p_{A\sigma} \downarrow & \lrcorner & \downarrow p_A \\ \Delta & \xrightarrow{\sigma} & \Gamma \end{array}$$
 is a pullback for all  $A \in \text{Ty}(\Gamma)$  and  $\sigma : \Delta \rightarrow \Gamma$ .
- (ii) The mappings  $(\Gamma, A) \mapsto \Gamma.A$  and  $(\sigma, A) \mapsto \sigma.A$  constitute a functor  $\text{El}(\text{Ty}) \rightarrow \mathcal{C}$ .
- (iii) We have  $\deg(\Gamma.A) = \deg(\Gamma) + 1$  for all  $\Gamma \in \mathcal{C}$  and  $A \in \text{Ty}(\Gamma)$ .
- (iv) There is a unique object  $*$  of degree 0, and  $*$  is terminal.
- (v) For all  $\Gamma$  with  $\deg(\Gamma) > 0$  there is a unique  $(\Gamma_0, A) \in \text{El}(\text{Ty})$  with  $\Gamma = \Gamma_0.A$ .

In the case of the syntactic category  $\mathcal{C}[\mathbb{T}]$  of a GAT  $\mathbb{T}$ , the grading assigns to each context its length, and  $\text{Ty}(\Gamma)$  is the set of ‘types in context  $\Gamma$ ’, i.e., equivalence classes of type expressions  $A$  such that  $\Gamma \vdash A$  is derivable, modulo the equivalence relation of derivable equality. The presheaf action is given by substitution. Given a type  $A \in \text{Ty}(\Gamma)$ , the extended context  $\Gamma.A$  is given by  $\Gamma, y:A$  obtained via the context formation rule in (10.2), and  $p_A$  is the substitution

$$\Gamma, y:A \vdash (\vec{x}) : \Gamma$$

where  $\vec{x}$  is the list of variables declared in  $\Gamma$ . For  $\sigma : \Gamma \rightarrow \Delta$  and  $A \in \text{Ty}(\Delta)$ , the substitution  $\sigma.A$  is given by

$$\Gamma, y:A[\sigma] \vdash (\sigma, x) : \Delta, y:A.$$

Then the fact that the square in Definition 10.210.2 is a pullback follows from the substitution formation rule in (10.2) together with the equality rules for substitutions that can be found in the cited references.

The following describes the relationship between contextual categories and clans.

**PROPOSITION 10.3.** *Every contextual category  $\mathcal{C}$  admits a clan structure where the display maps are the composites  $p_{A_1} \circ \cdots \circ p_{A_n} \circ i$  of a finite sequence of projections and an isomorphism.*

## §11. Appendix. The fat small object argument for clans.

**11.1. Colimit decomposition formula and pushouts of sieves.** This subsection discusses two results that are needed in the proof of the fat small object argument.

**THEOREM 11.1** (Colimit decomposition formula (CDF)). *Let  $\mathbb{C} : \mathbb{J} \rightarrow \text{Cat}$  be a small diagram in the 1-category of small categories, and let  $D : \text{colim}(\mathbb{C}) \rightarrow \mathfrak{X}$  be a diagram in a category  $\mathfrak{X}$  such that*

- (i) *for all  $j \in \mathbb{J}$ , the colimit  $\text{colim}_{c \in \mathbb{C}_j} D_{\sigma_j c} = \text{colim}(\mathbb{C}_j \xrightarrow{\sigma_j} \text{colim}(\mathbb{C}) \xrightarrow{D} \mathfrak{X})$  exists, and*
- (ii) *the iterated colimit  $\text{colim}_{j \in \mathbb{J}} \text{colim}_{c \in \mathbb{C}_j} D_{\sigma_j c}$  exists.*

*Then  $\text{colim}_{j \in \mathbb{J}} \text{colim}_{c \in \mathbb{C}_j} D_{\sigma_j c}$  is a colimit of  $D$ .*

PROOF. Peschke and Tholen [35] give three proofs of this under the additional assumption that  $\mathfrak{X}$  is cocomplete. The third proof (§5.3, ‘via Fubini’) easily generalizes to the situation where only the necessary colimits are assumed to exist. We sketch a simplified argument here. Let  $\int \mathbb{C}$  be the covariant Grothendieck construction of  $\mathbb{C}$ , whose projection  $\int \mathbb{C} \rightarrow \mathbb{J}$  is a split opfibration. Then  $\text{colim}(\mathbb{C})$  is the ‘joint coidentifier’ of the splitting, i.e., there is a functor  $E : \int \mathbb{C} \rightarrow \text{colim}(\mathbb{C})$  such that for every category  $\mathfrak{X}$ , the precomposition functor

$$(- \circ E) : [\text{colim}(\mathbb{C}), \mathfrak{X}] \rightarrow [\int \mathbb{C}, \mathfrak{X}]$$

restricts to an isomorphism between the functor category  $[\text{colim}(\mathbb{C}), \mathfrak{X}]$  and the full subcategory of  $[\int \mathbb{C}, \mathfrak{X}]$  on functors which send the arrows of the splitting to identities. In particular,  $(- \circ E)$  is fully faithful and thus it induces an isomorphism

$$(\text{colim}(\mathbb{C}))(D, \Delta -) \xrightarrow{\cong} (\int \mathbb{C})(D \circ E, \Delta -) : \mathfrak{X} \rightarrow \text{Set}$$

of co-presheaves of cocones for every diagram  $D : \text{colim}(\mathbb{C}) \rightarrow \mathfrak{X}$ . In other words,  $E$  is *final*, which is the crucial point of the argument, and for which Peschke and Tholen give a lengthier proof in [35, Theorem 5.8].

Finality of  $E$  implies that  $D$  has a colimit if and only if  $D \circ E$  has a colimit, and the existence of the latter follows if successive left Kan extensions along the composite  $\int \mathbb{C} \rightarrow \mathbb{J} \rightarrow 1$  exist. The first of these can be computed as fiberwise colimit since  $\int \mathbb{C} \rightarrow \mathbb{J}$  is a split cofibration [35, Theorem 4.6], which yields the inner term in the double colimit in the proposition.  $\dashv$

In the following we use the CDF specifically for pushouts of sieve inclusions of posets. Recall that a *sieve* (a.k.a. *downset* or *lower set*) in a poset  $P$  is a subset  $U \subseteq P$  satisfying

$$x \in U \wedge y \leq x \implies y \in U$$

for all  $x, y \in P$ . A monotone map  $f : P \rightarrow Q$  is called a *sieve inclusion* if it is order-reflecting and its image  $\text{im}(f) = f[P]$  is a sieve in  $Q$ . The proof of the following lemma is straightforward, but we state it explicitly since it will play a central role.

LEMMA 11.2. (i) If  $f : P \rightarrow Q$  and  $g : P \rightarrow R$  are sieve inclusions of posets, a pushout of  $f$  and  $g$  in the 1-category  $\text{Cat}$  of small categories is given by

$$\begin{array}{ccc} P & \xrightarrow{g} & R \\ f \downarrow & & \downarrow \sigma_2 \\ Q & \xrightarrow{\sigma_1} & Q +_P R \end{array}$$

where  $Q +_P R$  is the set-theoretic pushout, ordered by

$$\begin{aligned} \sigma_1(x) &\leq \sigma_1(y) && \text{iff } x \leq y \\ \sigma_1(x) &\leq \sigma_2(y) && \text{iff } \exists z . x = f(z) \wedge g(z) \leq y \\ \sigma_2(x) &\leq \sigma_2(y) && \text{iff } x \leq y \\ \sigma_2(x) &\leq \sigma_1(y) && \text{iff } \exists z . x = g(z) \wedge f(z) \leq y. \end{aligned}$$

In particular, the maps  $\sigma_1$  and  $\sigma_2$  are also sieve inclusions.

(ii) If  $U$  and  $V$  are sieves in a poset  $P$  then the square

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \downarrow & \lrcorner & \downarrow \\ U & \longrightarrow & U \cup V \end{array}$$

is a pushout in  $\mathbf{Cat}$ , where the sieves are equipped with the induced ordering.

**11.2. The fat small object argument.** Throughout this subsection let  $\mathcal{C}$  be a coclan.

We start by establishing some notation. Given a poset  $P$  and an element  $x \in P$ , we write  $P_{\leq x} = \{y \in P \mid y \leq x\}$  for the principal sieve generated by  $x$ , and  $P_{< x} = \{y \in P \mid y < x\}$  for its subset on elements that are strictly smaller than  $x$ . If  $x$  is a maximal element of  $P$ , we write  $P \setminus x$  for the sub-poset obtained by removing  $x$ . Given a diagram  $D : P \rightarrow \mathcal{C}$ , we write  $D_{\leq x}$ ,  $D_{< x}$ , and  $D \setminus x$  for the restrictions of  $D$  to  $P_{\leq x}$ ,  $P_{< x}$ , and  $P \setminus x$ , respectively. More generally we write  $D_U$  for the restriction of  $D$  to arbitrary sieves  $U \subseteq P$ .

Note that we have  $P_{\leq x} = P_{< x} \star 1$ , where  $\star$  is the *join* or *ordinal sum*, thus diagrams  $D : P_{\leq x} \rightarrow \mathcal{C}$  are in correspondence with cocones on  $D_{< x}$  with vertex  $D_x$ , and with arrows  $\text{colim}(D_{< x}) \rightarrow D_x$  whenever the colimit exists.

**DEFINITION 11.3.** A *finite  $\mathcal{C}$ -complex* is a pair  $(P, D)$  of a finite poset  $P$  and a diagram  $D : P \rightarrow \mathcal{C}$ , such that:

- (i)  $\text{colim}(D_{< x})$  exists for all  $x \in P$ , and the induced  $\alpha_x : \text{colim}(D_{< x}) \rightarrow D_x$  is codisplay.
- (ii) For  $x, y \in P$  we have  $x = y$  whenever  $P_{< x} = P_{< y}$ ,  $D_x = D_y$ , and  $\alpha_x = \alpha_y$ .

An *inclusion of finite  $\mathcal{C}$ -complexes*  $f : (P, D) \rightarrow (Q, E)$  is a sieve inclusion  $f : P \rightarrow Q$  such that  $D = E \circ f$ . We write  $\mathbf{FC}(\mathcal{C})$  for the category of finite  $\mathcal{C}$ -complexes and inclusions.

**REMARK 11.4.** We view a finite  $\mathcal{C}$ -complex as a construction of an object by a finite (though not necessarily linearly ordered) number of ‘cell attachments’, represented by the codisplay maps  $\alpha_x : \text{colim}(D_{< x}) \rightarrow D_x$ . Condition (ii) should be read as saying that ‘every cell can only be attached once at the same stage’. This is needed in Lemma 11.7 to show that  $\mathbf{FC}(\mathcal{C})$  is a preorder.

**LEMMA 11.5.** (i) The colimit  $\text{colim}(D)$  exists for every finite  $\mathcal{C}$ -complex  $(P, D)$ .  
 (ii) The induced functor

$$\text{Colim} : \mathbf{FC}(\mathcal{C}) \rightarrow \mathcal{C} \tag{11.1}$$

sends inclusions of finite  $\mathcal{C}$ -complexes to codisplay maps.

**PROOF.** The first claim is shown by induction on  $|P|$ . For empty  $P$  the statement is true since coclans have initial objects. For  $|P| = n + 1$  assume that  $x \in P$  is a maximal element. Then the square

$$\begin{array}{ccc} P_{< x} & \longrightarrow & P \setminus x \\ \downarrow & \lrcorner & \downarrow \\ P_{\leq x} & \longrightarrow & P \end{array}$$

is a pushout in  $\mathbf{Cat}$  by Lemma 11.2, which by the colimit decomposition formula 11.1 means that the pushout of the span

$$\begin{array}{ccc} \mathrm{colim}(D_{<x}) & \longrightarrow & \mathrm{colim}(D \setminus x) \\ \Downarrow & & \Downarrow \\ D_x & \dashrightarrow & \mathrm{colim}(D) \end{array} \quad (11.2)$$

—which exists since the left arrow is a codisplay map by 11.3(i)—is a colimit of  $D$  in  $\mathcal{C}$ .

For the second claim let  $f : (E, Q) \rightarrow (D, P)$  be an inclusion of finite  $\mathcal{C}$ -complexes. Since every inclusion of finite  $\mathcal{C}$ -complexes can be decomposed into ‘atomic’ inclusions with  $|P \setminus f[Q]| = 1$ , we may assume without loss of generality that  $Q = P \setminus x$  for some maximal  $x \in P$ . Then the image of  $f$  under  $\mathrm{Colim}$  is the right dashed arrow in (11.2), which is codisplay since codisplay maps are stable under pushout.  $\dashv$

REMARK 11.6. Lemma 11.5 implies that the assumption ‘ $\mathrm{colim}(D_{<x})$  exists’ in Definition 11.3(i) is redundant, since the colimits in question are colimits of finite subcomplexes.

LEMMA 11.7. *The category  $\mathrm{FC}(\mathcal{C})$  is an essentially small preorder with finite joins.*

PROOF.  $\mathrm{FC}(\mathcal{C})$  is essentially small as a collection of finite diagrams in a small category. To see that it is a preorder let  $f, g : (P, D) \rightarrow (Q, E)$  be inclusions of finite  $\mathcal{C}$ -complexes. We show that  $f(x) = g(x)$  by well-founded induction on  $x \in P$ . Let  $x \in P$  and assume that  $f(y) = g(y)$  for all  $y < x$ . Then since  $f$  and  $g$  are sieve inclusions we have  $Q_{<f(x)} = Q_{<g(x)}$  and since  $E f = D = E g$  we have the equalities

$$(E_y \rightarrow E_{f(x)})_{y < f(x)} = (D_y \rightarrow D_x)_{y < x} = (E_y \rightarrow E_{g(x)})_{y < g(x)}$$

of cocones, whence  $f(x) = g(x)$  by Definition 11.3(ii).

It remains to show that  $\mathrm{FC}(\mathcal{C})$  has finite suprema. The empty complex is clearly initial. We show that a supremum of  $(P, D)$  and  $(Q, E)$  exists by induction on  $|P|$ . The empty case is trivial, so assume that  $P$  is inhabited and let  $x$  be a maximal element. Let  $(R, F)$  be a supremum of  $(P \setminus x, D \setminus x)$  and  $(Q, E)$ , with inclusion maps  $f : (P \setminus x, D \setminus x) \rightarrow (R, F)$  and  $g : (Q, E) \rightarrow (R, F)$ . If there exists a  $y \in R$  such that  $R_{<y} = f[P_{<x}]$  and  $(D_z \rightarrow D_x)_{z < x} = (R_{f(z)} \rightarrow R_y)_{z < x}$  then ‘the cell-attachment corresponding to  $x$  is already contained in  $(R, F)$ ’, i.e.,  $f$  extends to an inclusion  $f' : (P, D) \rightarrow (R, F)$  of finite complexes with  $f'(x) = y$ , whence  $(R, F)$  is a supremum of  $(P, D)$  and  $(Q, E)$ .

If no such  $y$  exists then a supremum of  $(P, D)$  and  $(R, F)$  is given by  $(P +_{P \setminus x} R, [D, F])$ , as in the pushout diagram

$$\begin{array}{ccccc} P \setminus x & \xrightarrow{f} & R & & \\ \downarrow & & \downarrow & \searrow F & \\ P & \xrightarrow{\quad} & P +_{P \setminus x} R & \dashrightarrow [D, F] & \searrow \\ & & & & \searrow \\ & & & & D \xrightarrow{\quad} \mathcal{C} \end{array}$$

constructed as in Lemma 11.2.  $\dashv$

**THEOREM 11.8.** *The object  $C = \operatorname{colim}_{(P,D) \in \operatorname{FC}(\mathcal{C})} H(\operatorname{colim}(D))$  is a 0-extension in  $\mathcal{C}^{\operatorname{op}}\text{-Alg}$  and  $C \rightarrow 1$  is full.*

**PROOF.** To see that  $C \rightarrow 1$  is full, let  $e : I \twoheadrightarrow J$  be codisplay in  $\mathcal{C}$  and let  $f : H(I) \rightarrow C$ . Since  $\operatorname{FC}(\mathcal{C})$  is filtered and  $H(I)$  is compact,  $f$  factors through a colimit inclusion as

$$f = (H(I) \xrightarrow{H(g)} H(\operatorname{colim}(D)) \xrightarrow{\sigma_{(P,D)}} C)$$

for some finite complex  $(P, D)$ . We form the pushout

$$\begin{array}{ccc} I & \xrightarrow{g} & \operatorname{colim}(D) \\ e \downarrow & & \downarrow k \\ J & \longrightarrow & K \end{array}$$

and extend the finite complex  $(P, D)$  to  $(P \star 1, D \star k)$  where  $P \star 1$  is the join of  $P$  and  $1$ , and  $D \star k : P \star 1 \rightarrow \mathcal{C}$  is the diagram extending  $D$  with the cell-attachment  $k : \operatorname{colim}(D) \twoheadrightarrow K$ . Then  $K = \operatorname{colim}(D \star k)$  and  $k$  is the image of the inclusion  $(P, D) \hookrightarrow (P \star 1, D \star k)$  of finite complexes under the colimit functor (11.1), thus we obtain an extension of  $f$  along  $H(e)$  as in the following diagram.

$$\begin{array}{ccccc} & & f & & \\ & \nearrow & & \searrow & \\ H(I) & \xrightarrow{H(g)} & H(\operatorname{colim}(D)) & \xrightarrow{\sigma_{(P,D)}} & C \\ H(e) \downarrow & & \downarrow H(k) & & \uparrow \sigma_{(P \star 1, D \star k)} \\ H(J) & \longrightarrow & H(K) & & \end{array}$$

To see that  $C$  is a 0-extension, consider a full map  $f : Y \twoheadrightarrow X$  in  $\mathcal{C}^{\operatorname{op}}\text{-Alg}$  and an arrow  $h : C \rightarrow X$ . To show that  $h$  lifts along  $f$  we construct a lift of the cocone

$$\left( H(\operatorname{colim}(D)) \xrightarrow{\sigma_{(P,D)}} C \xrightarrow{h} X \right)_{(P,D) \in \operatorname{FC}(\mathcal{C})}$$

by induction over the preorder  $\operatorname{FC}(\mathcal{C})$  which is well-founded since every finite  $\mathcal{C}$ -complex has only finitely many subcomplexes. Given a finite complex  $(D, P)$  it is sufficient to exhibit a lift  $\kappa_{(P,D)} : H(\operatorname{colim}(D)) \rightarrow Y$  satisfying

$$f \circ \kappa_{(P,D)} = h \circ \sigma_{(P,D)} \quad \text{and} \quad (11.3)$$

$$\kappa_{(P,D)} \circ H(\operatorname{colim} j) = \kappa_{(Q,E)} \quad \text{for all subcomplexes } j : (Q, E) \rightarrow (P, D), \quad (11.4)$$

where we may assume that the  $\kappa_{(Q,E)}$  satisfy the analogous equations by induction hypothesis. We distinguish two cases:

1. If  $P$  has a greatest element  $x$  then we can take  $\kappa_{(P,D)}$  to be a lift in the square

$$\begin{array}{ccccc} H(\operatorname{colim}(D_{<x})) & \xrightarrow{\kappa_{(P_{<x}, D_{<x})}} & Y & & \\ \downarrow & \nearrow \kappa_{(P,D)} & \downarrow f & & \\ H(D_x) & \xrightarrow{\sigma_{(P,D)}} & C & \xrightarrow{h} & X \end{array}$$

whose left side is an extension by Lemma 11.5 and whose right side is full by assumption. Then (11.3) holds by construction, and (11.4) holds for all subcomplexes since it holds for the largest strict subcomplex  $(P_{<x}, D_{<x}) \rightarrow (P, D)$ .

2. If  $P$  doesn't have a greatest element we can write  $P = U \cup V$  as union of two strict sub-sieves, whence we have pushouts

$$\begin{array}{ccc} U \cap V \rightarrow V & & \text{colim}(D_{U \cap V}) \rightarrow \text{colim}(D_V) \\ \downarrow \quad \ulcorner \downarrow & \text{and} & \downarrow \quad \ulcorner \downarrow \\ U \longrightarrow P & & \text{colim}(D_U) \longrightarrow \text{colim}(D) \end{array}$$

by Lemma 11.2 and the CDF. This means that condition (11.4) forces us to define  $\kappa_{(P,D)}$  to be the unique arrow fitting into

$$\begin{array}{ccc} H(\text{colim}(D_{U \cap V})) & \xrightarrow{\phi_V^{U \cap V}} & H(\text{colim}(D_V)) \\ \phi_U^{U \cap V} \downarrow & \ulcorner \downarrow \phi_P^V & \searrow \kappa_{(V,D_V)} \\ H(\text{colim}(D_U)) & \xrightarrow{\phi_P^U} & H(\text{colim}(D)) \\ & \searrow \kappa_{(U,D_U)} & \dashrightarrow \kappa_{(P,D)} \searrow \\ & & Y \end{array}, \quad (11.5)$$

where for the remainder of the proof we write  $\phi_W^X : H(\text{colim}(D_X)) \rightarrow H(\text{colim}(D_W))$  for the canonical arrows induced by successive sieve inclusions  $X \subseteq W \subseteq P$ . Using the fact that the  $\phi_P^U$  and  $\phi_P^V$  are jointly epic it is easy to see that the  $\kappa_{(P,D)}$  defined in this way satisfies condition (11.3), and it remains to show that (11.4) is satisfied for arbitrary sieves  $W \subseteq P$ , i.e.,  $\kappa_{(P,D)} \circ \phi_P^W = \kappa_{(W,D_W)} : H(\text{colim}(D_W)) \rightarrow Y$ . Since

$$\begin{array}{ccc} H(\text{colim}(D_{U \cap V \cap W})) & \xrightarrow{\phi_{V \cap W}^{U \cap V \cap W}} & H(\text{colim}(D_{V \cap W})) \\ \phi_U^{U \cap V \cap W} \downarrow & \ulcorner \downarrow \phi_W^{V \cap W} & \\ H(\text{colim}(D_{U \cap W})) & \xrightarrow{\phi_W^{U \cap W}} & H(\text{colim}(D_W)) \end{array}$$

is a pushout it is enough to verify this equation after precomposing with  $\phi_W^{U \cap W}$  and  $\phi_W^{V \cap W}$ . We have

$$\begin{aligned} \kappa_{(P,D)} \circ \phi_P^W \circ \phi_W^{U \cap W} &= \kappa_{(P,D)} \circ \phi_P^U \circ \phi_U^{U \cap W} && \text{by functoriality} \\ &= \kappa_{(U,D_U)} \circ \phi_U^{U \cap W} && \text{by (11.5)} \\ &= \kappa_{(U \cap W, D_{U \cap W})} && \text{by (11.4)} \\ &= \kappa_{(W,D_W)} \circ \phi_W^{U \cap W} && \text{by (11.4)} \end{aligned}$$

and the case with  $\phi_W^{V \cap W}$  is analogous.  $\dashv$

**COROLLARY 11.9.** *For any clan  $\mathcal{T}$ , the 0-extensions in  $\mathcal{T}\text{-Mod}$  are flat.*

**PROOF.** Let  $E \in \mathcal{T}\text{-Mod}$  be a 0-extension. By applying Theorem 11.8 in  $\mathcal{T}\text{-Mod}/E$  (using Proposition 3.6), we obtain a full map  $f : F \twoheadrightarrow E$  where  $F$  is a 0-extension and  $f$  is a filtered colimit of arrows  $H(\Gamma) \rightarrow E$  in  $\mathcal{T}\text{-Mod}/A$ . Since  $\mathcal{T}\text{-Mod}/A \rightarrow \mathcal{T}\text{-Mod}$  creates colimits this means that  $F$  is a filtered colimit of representable models in



$\mathcal{T}\text{-Mod}$ , and therefore flat (Lemma 5.4). Since  $f$  is a full map into a 0-extension it has a section, thus  $E$  is a retract of  $F$  and therefore flat as well.  $\dashv$

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