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HARMONIC-MEASURE DISTRIBUTION FUNCTIONS AND RELATED FUNCTIONS FOR SIMPLY CONNECTED AND MULTIPLY CONNECTED TWO-DIMENSIONAL REGIONS

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The focus of this thesis is to compute explicitly two related functions $h, g : (0, \infty) \to [0, 1]$ associated with two-dimensional regions and to determine some of the properties of these functions. These functions are the *harmonic-measure distribution function*, or *h-function*, and its companion the *g-function*. They encode aspects of the behaviour of Brownian particles released from a specified basepoint within the region. In turn, this behaviour is influenced by the shape of the region's boundary and the location of the basepoint.

We compute the h-functions of numerous simply connected and multiply connected planar regions as well as g-functions of simply connected planar regions. We also compute the h-functions of several nonplanar regions lying on a cylindrical surface.

We make extensive use of complex analysis techniques, especially conformal mappings and the conformal invariance of harmonic measure, as well as a deep connection between Brownian motion, harmonic measure and the Dirichlet problem. For simply connected regions, the Riemann map plays a key role. For multiply connected regions, we bring to bear the Schottky–Klein prime function. We also cross-check the validity of many of the h-functions we find by rederiving them via numerical simulation of Brownian motion.

In Part I of this thesis, we determine the h-functions of several simply connected planar domains which have not been treated before and compute their asymptotics. Some of these new h-functions show behaviour that was not previously known to occur. We give examples of distinct regions whose h-functions coincide in part but not fully. Next, we identify several regions whose h-functions have a horizontal tangent

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at a given point within a neighbourhood where the h-function is otherwise strictly increasing. We also identify some regions for which not just the first derivatives but all the derivatives of the h-function vanish at a given point.

Part II is concerned with the h-functions of multiply connected planar regions. Until now, most of the existing theory of h-functions has been confined to simply connected domains. However, recently the h-function of the complement of finitely many collinear segments was computed, using the prime function. In this thesis, we extend this approach to treat bounded and unbounded regions whose boundaries consist of finitely many parallel but noncollinear line segments, or circles, or concentric circular slits (arcs), or polygons, with various locations of the basepoint. Two crucial ingredients of the approach are to compute various canonical conformal mappings and to construct certain harmonic functions, which are both expressed in terms of the prime function. Lastly, we present an explicit construction, via the prime function, of regions that generate a pre-designated step function as their h-function, thus providing a constructive solution to a longstanding open problem.

We validate some of our results by confirming that in each of several simple regions, we obtain the same h-function via a number of different conformal maps. Moreover, for all regions in Part II, we numerically approximate the h-function by a completely different method, namely simulating the movement of Brownian particles. In all cases, we obtain excellent agreement with our h-functions calculated via the prime function.

In Part III, we calculate the g-functions of several simply connected planar regions, and compute their asymptotics. We compare these g-functions with the h-functions of the same regions. We also show that the g-functions of two different regions can coincide in part but not fully, by exhibiting several pairs of regions with this behaviour.

Finally, in Part IV, we find the h-functions of four nonplanar regions, and calculate their asymptotics. The regions we consider here lie on the surface of a cylinder, with various geometries of the region's boundary.

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