DISCRETE LIAPUNOV FUNCTIONS WITH $\Delta^2 V > 0$

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Abstract

Consideration of functions whose second difference along the trajectories of a difference equation is positive gives a stability theorem for autonomous discrete-time systems. Such functions can be used to estimate domains of nonglobal stability.

1. Introduction

Recently Yorke [2] and Chow and Dunninger [1] have used conditions on the second derivative of Liapunov functions to establish stability theorems for autonomous differential equations. The main condition was that $\dot{V} > 0$, along trajectories, except at the origin. This note proves a discrete analogue of these results for systems

$$x_{k+1} = f(x_k), \quad k = 0, 1, 2, \ldots, \quad (1)$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ and $f(0) = 0$. The result is used to estimate local stability regions and a simple example of the technique is given. A feature is that no assumption on the definiteness of the Liapunov function is required.

2. Stability theorem

Let $p(k, x)$ denote the orbit of (1) for which $x_0 = x$. By $\Delta^2 V(x_k)$ is meant the second difference

$$V(x_{k+2}) - 2V(x_{k+1}) + V(x_k)$$

along the trajectories of $x_{k+1} = f(x_k)$. 

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THEOREM 1. Suppose that $V: \mathbb{R}^n \to \mathbb{R}$ is a continuous function and that $\Delta^2 V(x_k) > 0$ for $x_k \neq 0$. Then for any $x \in \mathbb{R}^n$:

either $p(k, x)$ is unbounded, or $p(k, x) \to 0$, as $k \to \infty$.

Likewise if

$\Delta^2 V(x_k) < 0$ for $x_k \neq 0$.

PROOF. Since $\Delta^2 V(x) > 0$, it follows that $V(p(k, x))$ is a monotone function of $k$, provided that $k$ is sufficiently large. To see this, observe that

$$V(x_{k+2}) - V(x_{k+1}) > V(x_{k+1}) - V(x_k).$$

If there exists an integer $K$ so that $V(x_{K+1}) - V(x_K) \geq 0$, then, $V(x_{K+1}) > V(x_K)$ for $k > K$. On the other hand, if there is no such integer $K$, $V(x_{k+1}) < V(x_k)$ for all $k \geq 0$.

Suppose $V(x_k)$ is nonincreasing as $k \to \infty$. The details for the nondecreasing case are very much the same. Define

$$L(x) = \{ u : p(k, x) \to u \text{ for some sequence } k \to \infty \}.$$ 

If $L(x)$ is empty, the Bolzano–Weierstrass theorem implies that $p(k, x)$ is unbounded as $k \to \infty$. If $L(x) \neq \emptyset$, for any $u \in L(x)$,

$$\lim_{k \to \infty} V(p(k, x)) = \lim_{t \to \infty} V(p(t, x)) = V(u).$$

since $V(p(k, x))$ is monotone in $k$ and $V$ continuous. It follows that $V(u)$ is a constant for all $u$ in $L(x)$. But

$$\lim_{t \to \infty} p(j+k_t, x) = \lim_{t \to \infty} p(j, p(k_t, x)) = p(j, u),$$

and so $p(j, u)$ is contained in $L(x)$ for all $j$. Hence $V(p(k, u)) = V(u)$. Since $\Delta^2 V(p(k, u)) > 0$, for $p(k, u) = 0$, then $L(x) = \{0\}$. The proof for $\Delta^2 V < 0$ is very similar.

As an example, consider the second-order difference equation:

$$x_{k+2} = x_k + f(x_{k+1}).$$

If

$$\Delta(x_k f(x_k)) > 0,$$

then either $x_k$ is unbounded as $k \to \infty$ or $x_k \to 0$ as $k \to \infty$.

To see this, put $y_k = x_{k+1}$, rewrite the system as

$$x_{k+1} = y_k, \quad y_{k+1} = x_k + f(y_k),$$

and define $V(x, y) = xy$. Then $\Delta^2 V(x_k, y_k) = \Delta(y_k f(y_k)) > 0$ and Theorem 1 applies. Observe that the Liapunov function $V$ is indefinite.

Clearly the theorem is valid for both forward and backward differences.
3. Nonglobal stability domains

Suppose $\Delta^2 V(x) > 0$ is known to hold only in some open region $H$ containing the origin. Then, in general, stability is not global but a finite stability domain exists. Set

$$\Delta_{\text{max}} = \max \{ \Delta V(x): x \in \text{bdy } H \}$$

and define

$$E_j = \{ x \in H: \Delta V(p(j, x)) > \Delta_{\text{max}} \}, \quad j = 0, 1, 2, \ldots.$$ 

**Theorem 2.** If the regions $E_j$ are bounded and nonempty, then they are domains of asymptotic stability for the system $x_{k+1} = f(x_k)$.

**Proof.** If $E_j$ is nonempty and $x \in E_j$, $p(j+k, x) \in H$ since $\Delta V(x)$ is nondecreasing along any trajectory of the system, in $E_j$. So

$$\Delta V(p(j+k+1, x)) - \Delta V(j+k, x)) = \Delta^2 V(p(j+k, x)) > 0,$$

and $\Delta V(p(j+k, x)) > \Delta V(p(j, x)) > \Delta_{\text{max}}$, $p(k, x) \in E_j$ and is thus bounded. By Theorem 1, $p(k, x) \to 0$ as $k \to \infty$.

**Notes**

1. A similar result holds for the regions

$$F_j = \{ x \in H: \Delta V(p(j, x)) < \Delta_{\text{min}} \},$$

$$\Delta_{\text{min}} = \min \{ \Delta V(x): x \in \text{bdy } H \},$$

provided $\Delta^2 V < 0$ in $H$ and $F_j$ is nonempty and bounded.

2. If $E_j$ is empty there may be no stability domain. Consider $x_{k+1} = y_k$, $y_{k+1} = x_k + f(y_k)$, where $f(y) = y + 1$ and $V(x, y) = xy$. Then

$$\Delta^2 V = (x+y+1)^2 + (x+y+1) - y(y+1)$$

$$= (x+y+1)^2 + x + 1 - y^2,$$

which is positive provided

$$y^2 > x + 1.$$ 

But the supremum of $\Delta V = y(y+1)$ on $y^2 = x + 1$ is $\Delta_{\text{max}} = \infty$, $E_j$ is empty for all $j$, and $p(j, x) \to \infty$, as stated as one alternative in Theorem 1.
4. Example

Consider the two-dimensional system

\[
\begin{align*}
x_{k+1} &= y_k, \\
y_{k+1} &= ax_k - y_k^2, \quad |a| < 3^{-1},
\end{align*}
\]

and the Liapunov function

\[V(x, y) = 2a^2 x^2/(1 + a^2) + y^2.\]

This system arises in a biological context. The equations are associated with the stability analysis of the population dynamics of a single species, two age-class model (see [3]).

We observe that

\[
\Delta^2 V(x, y) = y^2(1 - 3a^2 - 4ax - 2y^2)/(1 + a^2) + (ay + (ax - y^2)a)^2,
\]

and \(H = \{(x, y): 1 - 3a^2 - 4ax - 2y^2 > 0\}\) contains the origin. For definiteness set \(a = \frac{1}{4}\), and it is easy to show that \(\Delta_{\text{max}} = -0.0364\), whence

\[E_0 = \{(x, y): (y^2 - x/4)^2 - 2x^2/17 - 15y^2/17 > -0.0364\}.
\]

The regions \(H, E_0\) are shown in Fig. 1.

![Fig. 1. Local stability region for the example. The region \(H\) is a domain where \(\Delta^2 V > 0\) and \(E_0\) is a finite stability domain as in Theorem 2.](image-url)
References


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