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THETA BLOCK FOURIER EXPANSIONS, BORCHERDS PRODUCTS AND A SEQUENCE OF NEWMAN AND SHANKS

CRIS POOR, JERRY SHURMAN✉ and DAVID S. YUEN

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Abstract

The 'Borcherds products everywhere' construction [Gritsenko *et al.*, 'Borcherds products everywhere', *J. Number Theory* **148** (2015), 164–195] creates paramodular Borcherds products from certain theta blocks. We prove that the q-order of every such Borcherds product lies in a sequence $\{C_v\}$, depending only on the q-order v of the theta block. Similarly, the q-order of the leading Fourier–Jacobi coefficient of every such Borcherds product lies in a sequence $\{A_v\}$, and this is the sequence $\{a_n\}$ from work of Newman and Shanks in connection with a family of series for π. Our proofs use a combinatorial formula giving the Fourier expansion of any theta block in terms of its germ.

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1. Introduction

The papers [6, 8] by Newman and Shanks, with appended work by Zagier, feature a family of series for π that involve a sequence of positive integers a_n. For each positive integer N, and for a real algebraic number $U = U(N)$ determined by N,

$$\pi = \frac{1}{\sqrt{N}}\Big(-\log|U| - 24\sum_{n=1}^{\infty}(-1)^n\frac{a_n}{n}U^n\Big).$$

As noted in [8], the rapid convergence of some of these series is astonishing. The definition of a_n is rather complicated, but the sequence begins

$$\{a_n\}_{n=1}^{\infty} = \{1, 47, 2488, 138799, 7976456, 467232200, \ldots\}.$$

We show that this sequence of Newman and Shanks appears in the theory of paramodular Borcherds products.

Borcherds product theory can be used to construct meromorphic paramodular forms from weakly holomorphic Jacobi forms of weight zero that have integral Fourier coefficients. Such a Jacobi form $\psi \in \mathrm{J}^{\text{w.h.}}_{0,N}(\mathbb{Z})$, having Fourier expansion $\psi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n, r; \psi)q^n\zeta^r$ (see the next section for notation), determines an infinite product

$$q^A\zeta^B\xi^C \prod_{(m,n,r)\geq 0} (1 - q^n\zeta^r\xi^{Nm})^{c(nm,r;\psi)},$$

taken over $m, n, r \in \mathbb{Z}$ such that $m \geq 0$, and if $m = 0$ then $n \geq 0$ and if $m = n = 0$ then $r < 0$, and with the Weyl vector (A, B, C) given by

$$A = \frac{1}{24}\sum_{r\in\mathbb{Z}} c(0, r; \psi), \quad B = \frac{1}{2}\sum_{r\geq 1} rc(0, r; \psi), \quad C = \frac{1}{4}\sum_{r\in\mathbb{Z}} r^2c(0, r; \psi).$$

This product is normally convergent on some open set of values $Z = \left[\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right]$ in the Siegel space $\mathcal{H}_2$, with $q = \mathrm{e}(\tau)$, $\zeta = \mathrm{e}(z)$, $\xi = \mathrm{e}(\omega)$, and it defines $\mathrm{Borch}(\psi)$ by meromorphic continuation to all of $\mathcal{H}_2$ (see [1, Theorem 3.22]). In general $\mathrm{Borch}(\psi)$ transforms by a character of the paramodular group $\mathrm{K}(N)$, but restricting to integral A makes the character trivial and the weight $k' = \frac{1}{2}c(0, 0; \psi)$ integral, so that $\mathrm{Borch}(\psi)$ lies in $\mathcal{M}^{\text{mero}}_{k'}(\mathrm{K}(N))$, that is, it is a meromorphic paramodular form.

Even though the infinite product may converge normally only on a small open set, $\mathrm{Borch}(\psi)$ still has a Fourier–Jacobi expansion there,

$$\mathrm{Borch}(\psi)(Z) = \phi_1(\mathrm{Borch}(\psi))(\tau, z)\xi^C + \phi_2(\mathrm{Borch}(\psi))(\tau, z)\xi^{C+N} + \cdots.$$

The leading Fourier–Jacobi coefficient of $\mathrm{Borch}(\psi)$ is given by the infinite product

$$\phi_1(\mathrm{Borch}(\psi)) = q^A\zeta^B \prod_{(n,r)\geq 0} (1 - q^n\zeta^r)^{c(0,r;\psi)} = \eta(\tau)^{2k'} \prod_{r\geq 1}\left(\frac{\vartheta(\tau, rz)}{\eta(\tau)}\right)^{c(0,r;\psi)},$$

where $(n, r) \geq 0$ means that $n, r \in \mathbb{Z}$ and $n \geq 0$ and if $n = 0$ then $r < 0$; these meromorphic Jacobi forms are examples of *theta blocks*. Thus theta blocks arise naturally as the leading Fourier–Jacobi coefficients of Borcherds products. See Gritsenko *et al.* [5] for the general theory of theta blocks. When the powers of $\vartheta(\tau, rz)/\eta(\tau)$ are nonnegative for all $r \geq 1$, the theta block is *without denominator*.

It is natural to ask for weakly holomorphic Jacobi forms that produce *holomorphic* paramodular forms. The *Borcherds products everywhere* construction from [4] ensures holomorphy. Recall the index-raising Hecke operators $V_\ell : \mathrm{J}^{\text{w.h.}}_{k,N} \longrightarrow \mathrm{J}^{\text{w.h.}}_{k,\ell N}$ from [2]. Take a theta block without denominator $\phi \in \mathrm{J}_{k,N}$, and set $\psi = (-1)^\nu \phi|V_2/\phi \in \mathrm{J}^{\text{w.h.}}_{0,N}$, where $\nu = \mathrm{ord}_q\,\phi$ is the leading power of q in the Fourier expansion of ϕ. The resulting Borcherds product is holomorphic, that is, $\mathrm{Borch}(\psi) \in \mathcal{M}_{k'}(\mathrm{K}(N))$. In particular, the character of $\mathrm{Borch}(\psi)$ is trivial, and k', A and C/N are integral. In this way we associate a holomorphic Borcherds product with every theta block without denominator that is actually a Jacobi form. It is helpful to keep in mind that there are two theta blocks in play when we use the Borcherds products everywhere construction. The initial theta block ϕ has weight k, index N and q-order ν. We set $\psi = (-1)^\nu \phi|V_2/\phi$ and construct $\mathrm{Borch}(\psi)$. The leading Fourier–Jacobi coefficient $\phi_1(\mathrm{Borch}(\psi))$ of this Borcherds product is a secondary theta block, of weight k', index C and q-order A.

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We will prove that only in the case $\nu = 1$ do we have $\phi_1(\mathrm{Borch}(\psi)) = \phi$, because in the Borcherds products everywhere construction the q-order A grows quickly with ν. Our main result, Theorem 5.2, is that, unexpectedly, C/N and A depend only on ν, with no reference to any other particulars of the suitable theta block ϕ. Thus we may define two sequences of positive integers,

$$A_\nu = \mathrm{ord}_q\, \phi_1(\mathrm{Borch}(\psi)), \quad C_\nu = \mathrm{ord}_q\, \mathrm{Borch}(\psi),$$

by choosing any theta block ϕ without denominator of q-order ν, and by setting $\psi = (-1)^\nu\, \phi|V_2\, /\phi$. As intimated, we prove that $\{A_\nu\}$ is the sequence of Newman and Shanks. We do not know whether our sequence $\{C_\nu\}$ arises in some other context.

Our proofs rely on a grand theta block Fourier expansion formula, expressing the Fourier expansion in terms of double partitions, which we hope will be useful beyond this one application. The grand theta block formula is given in Proposition 3.1.

2. Notation

We assume that the reader has some acquaintance with the notation in [4], and with Jacobi forms, whose theory can be found in [2] and [3]. The space of Jacobi forms of weight $k \in \frac{1}{2}\mathbb{Z}$ and nonnegative integral index N is denoted $\mathrm{J}_{k,N}$. The Fourier expansion of such a Jacobi form, $\phi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n, r; \phi)q^n\zeta^r$, is supported only on (n, r) with $4nN - r^2 \geq 0$. For $\mathrm{J}^{\mathrm{cusp}}_{k,N}$ the support condition is $4nN - r^2 > 0$, for $\mathrm{J}^{\mathrm{weak}}_{k,N}$ it is $n \geq 0$ and for $\mathrm{J}^{\mathrm{w.h.}}_{k,N}$ it is that n be bounded from below. The space $\mathrm{J}^{\mathrm{mero}}_{k,N}$ of meromorphic Jacobi forms consists of the quotients of holomorphic Jacobi forms, the weight and index being the differences of those in the numerator and denominator. When ϕ has integral Fourier coefficients we write $\phi \in \mathrm{J}_{k,N}(\mathbb{Z})$, and similarly for other subrings of $\mathbb{C}$. When ϕ transforms by a multiplier we indicate so in the notation; for example, the odd Jacobi theta function ϑ lies in $\mathrm{J}^{\mathrm{cusp}}_{1/2,1/2}(\epsilon^3 v_H)$, where ϵ is the multiplier of the Dedekind eta function and v_H is a certain character of order two [3]. Elliptic modular forms are considered to be Jacobi forms of index zero, so that, for example, the Dedekind eta function lies in $\mathrm{J}^{\mathrm{cusp}}_{1/2,0}(\epsilon)$.

For paramodular forms we refer to [4]. The Siegel upper half space of degree two is denoted $\mathcal{H}_2$. For $N \in \mathbb{Z}_{\geq 1}$ the paramodular group $\mathrm{K}(N)$ is the stabiliser of $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$ (as a column vector) in $\mathrm{Sp}_2(\mathbb{Q})$. The $\mathbb{C}$-vector space of weight k paramodular forms for $\mathrm{K}(N)$ is denoted $\mathcal{M}_k(\mathrm{K}(N))$, and $\mathcal{M}^{\mathrm{mero}}_k(\mathrm{K}(N))$ is given by quotients of holomorphic paramodular forms for $\mathrm{K}(N)$ whose respective difference of weights is k.

3. Theta block Fourier expansion

We quickly review some basic terminology of theta blocks. Throughout, τ is a variable from the complex upper half plane and z is a complex variable, and $q = \mathrm{e}(\tau) = e^{2\pi i\tau}$ and $\zeta = e(z) = e^{2\pi iz}$. The Dedekind eta function and the odd Jacobi theta function are

$$\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n) \in \mathrm{J}^{\mathrm{cusp}}_{1/2,0}(\epsilon),$$

$$\begin{aligned}\vartheta(\tau,z) &= \sum_{n\in\mathbb{Z}}(-1)^n q^{(n+1/2)^2/2}\zeta^{n+1/2} \\ &= q^{1/8}(\zeta^{1/2}-\zeta^{-1/2})\prod_{n\geq 1}(1-q^n\zeta)(1-q^n\zeta^{-1})(1-q^n) \in \mathrm{J}^{\mathrm{cusp}}_{1/2,1/2}(\epsilon^3 v_H).\end{aligned}$$

For any $r \in \mathbb{Z}_{\geq 1}$, define $\vartheta_r(\tau,z) = \vartheta(\tau, rz) \in \mathrm{J}^{\mathrm{cusp}}_{1/2,r^2/2}(\epsilon^3 v_H^r)$, so that

$$\vartheta_r(\tau,z)/\eta(\tau) = q^{1/12}(\zeta^{r/2}-\zeta^{-r/2})\prod_{n\geq 1}(1-q^n\zeta^r)(1-q^n\zeta^{-r}) \in \mathrm{J}^{\mathrm{cusp}}_{0,r^2/2}(\epsilon^2 v_H^r).$$

A *theta block* is a meromorphic function of the form

$$\mathrm{TB}(\tau,z) = \mathrm{TB}(\varphi)(\tau,z) = \eta(\tau)^{\varphi(0)}\prod_{r\geq 1}(\vartheta_r(\tau,z)/\eta(\tau))^{\varphi(r)},$$

where $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$ is even and finitely supported. A theta block such that $\varphi(r) \geq 0$ for each $r \in \mathbb{Z}_{\geq 1}$ is a theta block *without denominator*. The theta block of φ has the product form

$$\mathrm{TB}(\tau,z) = q^{\nu} b(\zeta)\prod_{i\geq 1, r\in\mathbb{Z}}(1-q^i\zeta^r)^{\varphi(r)}, \tag{3.1}$$

where the leading exponent of q is

$$\nu = \frac{1}{24}\sum_{r\in\mathbb{Z}}\varphi(r),$$

and the *baby theta block* is defined by $b(\zeta) = \prod_{r\geq 1}(\zeta^{r/2}-\zeta^{-r/2})^{\varphi(r)}$, or by

$$b(\zeta) = \zeta^{-B}\prod_{r\geq 1}(\zeta^r - 1)^{\varphi(r)}, \quad B = \frac{1}{2}\sum_{r\geq 1} r\varphi(r).$$

Introduce also the weight $k = \frac{1}{2}\varphi(0)$ and the index $N = \frac{1}{2}\sum_{r\geq 1} r^2\varphi(r)$ of TB so that $\mathrm{TB}(\varphi) \in \mathrm{J}^{\mathrm{mero}}_{k,N}(\epsilon^{24\nu} v_H^{2B})$. The multiplicity function φ determines a *germ*

$$G(\zeta) = G(\varphi)(\zeta) = \sum_{r\in\mathbb{Z}}\varphi(r)\zeta^r,$$

which itself determines the theta block $\mathrm{TB}(\varphi)$. Wanting to think of the germ as a function either of the multiplicity function or of the theta block, we freely write it both ways, $G(\zeta) = G(\varphi)(\zeta) = G(\mathrm{TB}(\varphi))(\zeta)$. The germ has the properties $G(1) = 24\nu$, $G'(1) = 0$ and $G''(1) = 4N$, recovering the q-order and index of the theta block. We know that this germ determines the coefficients $B_n(\varphi)$ in the q-expansion of the double product in (3.1),

$$\prod_{i\geq 1, r\in\mathbb{Z}}(1-q^i\zeta^r)^{\varphi(r)} = \sum_{n\geq 0} B_n(\varphi)(\zeta) q^n.$$

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The following proposition is important because it calculates the q-expansion of an arbitrary theta block by giving the coefficients $B_n(\varphi)$ in terms of universal polynomials b_n in the functions $G(\zeta), G(\zeta^2), \ldots, G(\zeta^n)$. Our calculations use the *partitions* and the *double partitions* of nonnegative integers,

$$\mathrm{P}(n) = \Big\{d : \mathbb{Z}_{\geq 1} \longrightarrow \mathbb{Z}_{\geq 0} : n = \sum_i d(i)i\Big\},$$

$$\mathrm{DP}(n) = \Big\{u : \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \longrightarrow \mathbb{Z}_{\geq 0} : n = \sum_{i,j} u(i,j)ij\Big\}.$$

The *sign* of a partition d is $\mathrm{sgn}(d) = (-1)^{\sum_i d(i)}$, and similarly the *sign* of a double partition u is $\mathrm{sgn}(u) = (-1)^{\sum_{i,j} u(i,j)}$. Addition gives $\mathrm{P}(\ell) \times \mathrm{P}(m) \longrightarrow \mathrm{P}(\ell + m)$, with $\mathrm{sgn}(d_1 + d_2) = \mathrm{sgn}(d_1)\,\mathrm{sgn}(d_2)$ for partitions d_1, d_2, and similarly for double partitions, with $\mathrm{sgn}(u_1 + u_2) = \mathrm{sgn}(u_1)\,\mathrm{sgn}(u_2)$.

Proposition 3.1 (Grand theta block formula). *For any even, finitely supported function $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}$, define a corresponding product*

$$P(\varphi) = P(\varphi)(\tau, z) = \prod_{i\geq 1,\, r\in\mathbb{Z}} (1 - q^i\zeta^r)^{\varphi(r)}.$$

This product has a formal q-expansion

$$P(\varphi)(\tau, z) = \sum_{n\in\mathbb{Z}_{\geq 0}} B_n(\varphi)(\zeta)q^n,$$

given as follows. For each $n \in \mathbb{Z}_{\geq 0}$, the q^n-coefficient of $P(\varphi)$ is

$$B_n(\varphi)(\zeta) = \sum_{u\in\mathrm{DP}(n)} \mathrm{sgn}(u) \prod_{i,j\geq 1} \frac{1}{u(i,j)!}\Big(\frac{G(\varphi)(\zeta^j)}{j}\Big)^{u(i,j)}.$$

Thus $B_n(\varphi)(\zeta) = b_n(G(\varphi)(\zeta), G(\varphi)(\zeta^2), \ldots, G(\varphi)(\zeta^n))$, where the polynomial b_n lies in $(1/n!)\mathbb{Z}[x_1, x_2, \ldots, x_n]$ and is independent of φ.

Before proving the proposition, we give the first few B_n. Omitting φ from the notation for brevity, $B_0(\zeta) = 1$ and then

$$\begin{aligned}
B_1(\zeta) &= -G(\zeta),\\
B_2(\zeta) &= \tfrac{1}{2}(-G(\zeta^2) + G(\zeta)^2 - 2G(\zeta)),\\
B_3(\zeta) &= \tfrac{1}{6}(-2G(\zeta^3) + 3G(\zeta^2)G(\zeta) - G(\zeta)^3 + 6G(\zeta)^2 - 6G(\zeta)),\\
B_4(\zeta) &= \tfrac{1}{24}\left(\begin{aligned} &-6G(\zeta^4) + 8G(\zeta^3)G(\zeta) - 6G(\zeta^2)G(\zeta)^2 + 12G(\zeta^2)G(\zeta)\\ &+ 3G(\zeta^2)^2 - 12G(\zeta^2) + G(\zeta)^4 - 12G(\zeta)^3 + 36G(\zeta)^2 - 24G(\zeta)\end{aligned}\right).
\end{aligned}$$

The corresponding polynomials b_n are easily read off from these.

PROOF. First consider the case where φ is nonnegative, that is, $\varphi(r) \geq 0$ for all $r \in \mathbb{Z}$. For each $i \in \mathbb{Z}_{\geq 0}$, view the finite product $\prod_{r\in\mathbb{Z}}(1 - q^i\zeta^r)^{\varphi(r)}$ as a polynomial in q^i whose reciprocal roots are the multiset that contains $\varphi(r)$ copies of ζ^r for each $r \in \mathbb{Z}$. This multiset is independent of i, and its power-sum functions are $s_j = \sum_{r\in\mathbb{Z}} \varphi(r)\zeta^{jr} = G(\varphi)(\zeta^j)$ for $j \in \mathbb{Z}_{\geq 0}$. Now, letting $\deg(\varphi) = \sum_{r\in\mathbb{Z}} \varphi(r)$ and letting σ_d denote the dth elementary symmetric function over the multiset defined above, the finite product is $\prod_{r\in\mathbb{Z}}(1 - q^i\zeta^r)^{\varphi(r)} = \sum_{d=0}^{\deg(\varphi)}(-1)^d\sigma_d q^{di}$. Thus the given double product $\prod_{i,r}(1 - q^i\zeta^r)^{\varphi(r)}$ in the proposition is

$$P(\varphi)(\tau, z) = \prod_{i\geq 1} \sum_{d=0}^{\deg(\varphi)} (-1)^d\sigma_d q^{di},$$

and so its q^n-coefficients are

$$B_n(\varphi)(\zeta) = \sum_{d\in \mathrm{P}(n)} \mathrm{sgn}(d) \prod_{i\geq 1} \sigma_{d(i)}, \quad n \in \mathbb{Z}_{\geq 0}.$$

Newton's identities $\sum_{\ell=0}^{m-1}(-1)^\ell\sigma_\ell s_{m-\ell} + (-1)^m m\sigma_m = 0$ give the elementary symmetric functions in terms of the power-sum functions,

$$\sigma_d = (-1)^d \sum_{u\in \mathrm{P}(d)} \prod_{j\geq 1} \frac{1}{u(j)!}\left(\frac{-s_j}{j}\right)^{u(j)}.$$

Since the power-sum functions are $s_j = G(\varphi)(\zeta^j)$ in our setting, the values of $B_n(\varphi)$ for any nonnegative φ as n varies through $\mathbb{Z}_{\geq 0}$ are given by

$$B_n(\varphi)(\zeta) = \sum_{d\in \mathrm{P}(n)} \prod_{i\geq 1} \sum_{u(i,\cdot)\in \mathrm{DP}(d(i))} \prod_{j\geq 1} \frac{1}{u(i, j)!}\left(\frac{-G(\varphi)(\zeta^j)}{j}\right)^{u(i,j)},$$

and these are just the values given in the proposition,

$$B_n(\varphi)(\zeta) = \sum_{u\in \mathrm{DP}(n)} \mathrm{sgn}(u) \prod_{i,j\geq 1} \frac{1}{u(i, j)!}\left(\frac{G(\varphi)(\zeta^j)}{j}\right)^{u(i,j)}. \tag{3.2}$$

The corresponding polynomial $b_n(x_1, \ldots, x_n)$ with x_j in place of $G(\varphi)(\zeta^j)$ lies in $(1/n!)\mathbb{Z}[x_1, \ldots, x_n]$, because $u(i, j)!j^{u(i,j)}$ divides $(u(i, j)ij)!$ and the multinomial coefficient $(n/\{u(i, j)ij\})$ is integral.

We now drop the restriction that φ is nonnegative. The product-formation map $\varphi \mapsto P(\varphi)$ takes sums to products, and the relation $P(\varphi + \tilde{\varphi}) = P(\varphi)P(\tilde{\varphi})$ shows that $P(\varphi)$ is determined by $P(\varphi + \tilde{\varphi})$ and $P(\tilde{\varphi})$. This says that the sequence $\{B_n(\varphi)\}$ is determined by the sequences $\{B_n(\varphi + \tilde{\varphi})\}$ and $\{B_n(\tilde{\varphi})\}$ and by the convolution relations

$$B_n(\varphi + \tilde{\varphi}) = \sum_{\ell+m=n} B_\ell(\varphi)B_m(\tilde{\varphi}), \quad n \in \mathbb{Z}_{\geq 0}. \tag{3.3}$$

In particular, we decompose any φ as $\varphi = \varphi_+ - \varphi_-$, with $\varphi_+(r) = \max\{\varphi(r), 0\}$ and $\varphi_-(r) = -\min\{\varphi(r), 0\}$, and we take $\varphi + \tilde{\varphi} = \varphi_+$ and $\tilde{\varphi} = \varphi_-$ in (3.3). Because the $\varphi_\pm$

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are nonnegative, the coefficient sequences $\{B_n(\varphi_\pm)\}$ are given by (3.2), and what needs to be shown is that the coefficient formula (3.2), which makes sense for general φ, also satisfies the convolution relation (3.3). Because Laurent polynomial formation $\varphi \mapsto G(\varphi)$ takes sums to sums, that is, $G(\varphi+\tilde\varphi) = G(\varphi) + G(\tilde\varphi)$, the binomial theorem gives

$$\begin{aligned}
&\frac{1}{u(i,j)!}\Big(\frac{G(\varphi+\tilde\varphi)(\zeta^j)}{j}\Big)^{u(i,j)}\\
&\quad = \sum_{v(i,j)+w(i,j)=u(i,j)} \frac{1}{v(i,j)!}\Big(\frac{G(\varphi)(\zeta^j)}{j}\Big)^{v(i,j)} \frac{1}{w(i,j)!}\Big(\frac{G(\tilde\varphi)(\zeta^j)}{j}\Big)^{w(i,j)},
\end{aligned}$$

and the desired result follows,

$$\begin{aligned}
B_n(\varphi+\tilde\varphi) &= \sum_{u\in \mathrm{DP}(n)} \mathrm{sgn}(u) \prod_{i,j\geq 1} \frac{1}{u(i,j)!}\Big(\frac{G(\varphi+\tilde\varphi)(\zeta^j)}{j}\Big)^{u(i,j)}\\
&= \sum_{\ell+m=n} \Big(\sum_{v\in \mathrm{DP}(\ell)} \mathrm{sgn}(v) \prod_{i,j\geq 1} \frac{1}{v(i,j)!}\Big(\frac{G(\varphi)(\zeta^j)}{j}\Big)^{v(i,j)}\Big)\\
&\qquad \cdot \Big(\sum_{w\in \mathrm{DP}(m)} \mathrm{sgn}(w) \prod_{i,j\geq 1} \frac{1}{w(i,j)!}\Big(\frac{G(\tilde\varphi)(\zeta^j)}{j}\Big)^{w(i,j)}\Big)\\
&= \sum_{\ell+m=n} B_\ell(\varphi)(\zeta) B_m(\tilde\varphi).
\end{aligned}$$

□

4. The polynomials f_n

We define the sequences $\{A_\nu\}$ and $\{C_\nu\}$ in this section and explain them conceptually in the next.

Given any sequence of polynomials b_n with $b_0 = 1$, $b_n \in (1/n!)\mathbb{Z}[x_1, x_2, \ldots, x_n]$, the conditions $f_0 = 1$ and $b_n(x) = \sum_{2\ell+m=n} b_\ell(x) f_m(x)$ recursively define another such sequence f_n. These conditions are equivalent to an equality of formal series,

$$\frac{\sum_{n\geq 0} b_n(x_1, x_2, \ldots, x_n) q^{n/2}}{\sum_{n\geq 0} b_n(x_1, x_2, \ldots, x_n) q^n} = \sum_{n\geq 0} f_n(x_1, x_2, \ldots, x_n) q^{n/2}.$$

Definition 4.1. Let $b_n \in (1/n!)\mathbb{Z}[x_1, x_2, \ldots, x_n]$ be defined by

$$b_n(x_1, x_2, \ldots, x_n) = \sum_{u\in \mathrm{DP}(n)} \mathrm{sgn}(u) \prod_{i,j\geq 1} \frac{1}{u(i,j)!}\Big(\frac{x_j}{j}\Big)^{u(i,j)}.$$

Define $f_n \in (1/n!)\mathbb{Z}[x_1, x_2, \ldots, x_n]$ recursively by the conditions $f_0 = 1$ and $b_n(x) = \sum_{2\ell+m=n} b_\ell(x) f_m(x)$. Define two sequences,

$$\begin{aligned}
A_\nu &= \tfrac{1}{24}(-1)^\nu f_\nu(24\nu, 24\nu, \ldots, 24\nu),\\
C_\nu &= (-1)^\nu \sum_{i=1}^{\nu} i^2 \partial_i f_\nu(24\nu, \ldots, 24\nu).
\end{aligned}$$

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The first few polynomials f_n after $f_0 = 1$ are

$$\begin{aligned} f_1(x_1) &= -x_1,\\ f_2(x_1,x_2) &= \tfrac{1}{2}(x_1^2 - x_2),\\ f_3(x_1,x_2,x_3) &= \tfrac{1}{6}(-2x_3 + 3x_1x_2 - 6x_1 - x_1^3),\\ f_4(x_1,x_2,x_3,x_4) &= \tfrac{1}{24}(-6x_4 + 8x_1x_3 - 6x_1^2x_2 + 24x_1^2 + 3x_2^2 + x_1^4). \end{aligned}$$

These give the first few values A_ν and C_ν,

$$\begin{aligned} \{A_\nu\} &= \{1, 47, 2488, 138799, \ldots\},\\ \{C_\nu\} &= \{1, 46, 2416, 134236, \ldots\}. \end{aligned}$$

From the definition only the rationality of the sequences is clear for now, but the terms will be proven to be positive integers in the next section.

5. The main theorem

Although the introduction to this article emphasised holomorphic Borcherds products produced by the Borcherds products everywhere construction, the construction can be applied more generally if $\psi = (-1)^\nu \phi|V_2/\phi$ is weakly holomorphic. As shown in [7], weak holomorphy holds precisely when the baby theta block of ϕ satisfies the condition $b(\zeta) \mid b(\zeta^2)$ in $\mathbb{Z}[\zeta, \zeta^{-1}]$. This divisibility condition holds automatically for a theta block ϕ without denominator. This section considers a meromorphic Borcherds product $\mathrm{Borch}(\psi) \in \mathcal{M}_{k'}^{\mathrm{mero}}(\mathrm{K}(N))$ made by the Borcherds product everywhere construction from a theta block $\phi \in \mathrm{J}_{k,N}^{\mathrm{w.h.}}$ satisfying $b(\zeta) \mid b(\zeta^2)$ and having $\mathrm{ord}_q\, \phi = \nu$. The idea is to construct the germ of the theta block $\phi_1(\mathrm{Borch}(\psi))$ from the germ of the theta block ϕ. From this it will follow that A_ν is the q-order of $\phi_1(\mathrm{Borch}(\psi))$ and that C_ν is the q-order of $\mathrm{Borch}(\psi)$.

Lemma 5.1. *Let $k, \nu \in \mathbb{Z}_{\geq 1}$ and $N \in \mathbb{Z}_{\geq 0}$. Let $\phi \in \mathrm{J}_{k,N}^{\mathrm{w.h.}}$ be a theta block with $b(\zeta) \mid b(\zeta^2)$ in $\mathbb{C}[\zeta, \zeta^{-1}]$ and with $\mathrm{ord}_q\, \phi = \nu$. Let G be the germ of ϕ. Then $\psi = (-1)^\nu \phi|V_2/\phi \in \mathrm{J}_{0,N}^{\mathrm{w.h.}}(\mathbb{Z})$. Define $\mathcal{G} : \mathbb{C}^* \to \mathbb{C}$ by $\mathcal{G}(\zeta) = \mathrm{Coeff}(\psi, q^0)$. Then*

$$\mathcal{G}(\zeta) = (-1)^\nu\, \mathrm{Coeff}\Big(\frac{\phi(\tau/2, z)}{\phi(\tau, z)}, q^0\Big) = (-1)^\nu f_\nu(G(\zeta), G(\zeta^2), \ldots, G(\zeta^\nu)),$$

and $\mathcal{G}(1) = 24A_\nu$, $\mathcal{G}'(1) = 0$ and $\mathcal{G}''(1) = 4NC_\nu$.

Proof. Theorem 4.2 in [7] shows that $b(\zeta) \mid b(\zeta^2)$ implies $\psi \in \mathrm{J}_{0,N}^{\mathrm{w.h.}}(\mathbb{Z})$, including the case $N = 0$. Write the theta block ϕ as a double product,

$$\phi(\tau, z) = q^\nu b(\zeta) \prod_{i \geq 1, r \in \mathbb{Z}} (1 - q^i \zeta^r)^{\varphi(r)},$$

and recall that the operator V_2 acts on $\phi \in \mathrm{J}_{k,N}^{\mathrm{w.h.}}$ as

$$(\phi|V_2)(\tau, z) = 2^{k-1}\phi(2\tau, 2z) + \tfrac{1}{2}(\phi(\tau/2, z) + \phi(\tau/2 + 1/2, z)).$$

We see that $\phi(2\tau, 2z)/\phi(\tau, z) = q^\nu b(\zeta^2)/b(\zeta) \prod_{j \geq 1, r \in \mathbb{Z}} (1 + q^j \zeta^r)^{\varphi(r)}$ contains only positive powers of q because $\nu \geq 1$. Further, because $\phi(\tau/2, z)$ contains the same terms

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with integral powers of q as $(1/2)(\phi(\tau/2,z)+\phi(\tau/2+1/2,z))$, we deduce

$$\mathcal{G}=\mathrm{Coeff}(\psi,q^0)=(-1)^v\,\mathrm{Coeff}\Big(\frac{\phi(\tau/2,z)}{\phi(\tau,z)},q^0\Big).$$

Furthermore, using the Fourier expansion of ϕ from Proposition 3.1 in terms of $B_n(\zeta)=b_n(G(\zeta),G(\zeta^2),\dots,G(\zeta^n))$, and using the functions f_n introduced in the previous section,

$$\begin{aligned}\frac{\phi(\tau/2,\zeta)}{\phi(\tau,\zeta)}&=\frac{q^{v/2}b(\zeta)\sum_{n\in\mathbb{Z}_{\geq 0}}B_n(\zeta)q^{n/2}}{q^v b(\zeta)\sum_{n\in\mathbb{Z}_{\geq 0}}B_n(\zeta)q^n}\\&=q^{-v/2}\sum_{n\in\mathbb{Z}_{\geq 0}}f_n(G(\zeta),G(\zeta^2),\dots,G(\zeta^n))q^{n/2}.\end{aligned}$$

Therefore $\mathcal{G}(\zeta)=(-1)^v f_v(G(\zeta),G(\zeta^2),\dots,G(\zeta^v))$ as asserted. As a consequence of the relations $G(1)=24v$, $G'(1)=0$, $G''(1)=4N$, and basic differentiation,

$$\begin{aligned}\mathcal{G}(1)&=(-1)^v f_v(24v,\dots,24v)=24A_v,\\ \mathcal{G}'(1)&=0,\\ \mathcal{G}''(1)&=4N\sum_{i=1}^{v}i^2\partial_i f_v(24v,\dots,24v)=4NC_v,\end{aligned}$$

and these are the last three statements of the lemma. □

THEOREM 5.2 (Main theorem). *Let $k,v,N\in\mathbb{Z}_{\geq 1}$. Let $\phi\in \mathrm{J}^{\text{w.h.}}_{k,N}$ be a theta block with $b(\zeta)\mid b(\zeta^2)$ in $\mathbb{C}[\zeta,\zeta^{-1}]$ and with $\mathrm{ord}_q\,\phi=v$. Let $G(\phi)$ be the germ of ϕ. Then $\psi=(-1)^v\phi|V_2/\phi\in \mathrm{J}^{\text{w.h.}}_{0,N}(\mathbb{Z})$ and $\mathrm{Borch}(\psi)\in\mathcal{M}^{\text{mero}}_{k'}(\mathrm{K}(N))$ for some $k'\in\mathbb{Z}$. The leading theta block $\phi_1(\mathrm{Borch}(\psi))\in \mathrm{J}^{\text{mero}}_{k',C}$ has its germ given by*

$$G(\phi_1(\mathrm{Borch}(\psi)))(\zeta)=(-1)^v f_v(G(\phi)(\zeta),G(\phi)(\zeta^2),\dots,G(\phi)(\zeta^v)).$$

Furthermore,

- *the q-order of $\mathrm{Borch}(\psi)$ depends only on v and is C_v;*
- *the q-order of $\phi_1(\mathrm{Borch}(\psi))$ depends only on v and is A_v;*
- *the sequences $\{A_v\}$ and $\{C_v\}$ consist of positive integers and $A_v\geq C_v$.*

REMARK 5.3. The q-order and the ξ^N-order of $\mathrm{Borch}(\psi)$ are both C/N. This follows from the fact that $\mathrm{Borch}(\psi)$ is an eigenform under an involution that sends $(q,\zeta,\xi^N)\mapsto(\xi^N,\zeta^{-1},q)$ (see [3, Theorem 2.1]).

PROOF. We have $\psi\in \mathrm{J}^{\text{w.h.}}_{0,N}(\mathbb{Z})$ by [7]. Then $\mathrm{Borch}(\psi)$ is a meromorphic paramodular form for some character of $\mathrm{K}(N)$ and some weight $k'\in\frac{1}{2}\mathbb{Z}$ by the general theory (see [3, Theorem 2.1]). Eventually we will show that the character is trivial and the weight is integral. First we show that the germ of $\phi_1(\mathrm{Borch}(\psi))$ is $\mathrm{Coeff}(\psi,q^0)$.

There exists some $\lambda>0$ such that the infinite product for $\mathrm{Borch}(\psi)$ is normally convergent, as an infinite product, on the set $U=\{Z\in\mathcal{H}_2:\mathrm{Im}(Z)>\lambda I_2\}$. The expansion of this product into a Fourier–Jacobi series is normally convergent, as a series, on the complement of the zeros and poles of $\mathrm{Borch}(\psi)$ in U. Thus the infinite

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product for $\mathrm{Borch}(\psi)$ gives the first Fourier–Jacobi coefficient $\phi_1(\mathrm{Borch}(\psi))(\tau, z)$ as

$$q^A \zeta^B \prod_{(n,r)\geq 0} (1 - q^n \zeta^r)^{c(0,r;\psi)} = \eta(\tau)^{c(0,0;\psi)} \prod_{r\geq 1} (\vartheta_r(\tau, z)/\eta(\tau))^{c(0,r;\psi)},$$

so that

$$G(\phi_1(\mathrm{Borch}(\psi))) = \sum_{r\in\mathbb{Z}} c(0, r; \psi)\zeta^r = \mathrm{Coeff}(\psi, q^0).$$

This shows that the germ of $\phi_1(\mathrm{Borch}(\psi))$ is $\mathcal{G}$ in the notation of Lemma 5.1. The q-order of $\phi_1(\mathrm{Borch}(\psi))$ is A because the factors $(1 - q^n\zeta^r)^{c(0,r;\psi)}$ in the infinite product have only nonnegative powers of q. Thus, recovering the q-order from the germ $\mathcal{G}$ and using Lemma 5.1 gives $24A = \mathcal{G}(1) = 24A_\nu$, or $\mathrm{ord}_q\, \phi_1(\mathrm{Borch}(\psi)) = A = A_\nu$. The ξ-order of $\mathrm{Borch}(\psi)$ is C because the factors $(1 - q^n\zeta^r\xi^{Nm})^{c(nm,r;\psi)}$ in the infinite product have only nonnegative powers of ξ. The value of C is also the index of $\phi_1(\mathrm{Borch}(\psi))$. Thus, recovering the index from the germ $\mathcal{G}$ and using Lemma 5.1 gives $4C = \mathcal{G}''(1) = 4NC_\nu$, or $\mathrm{ord}_{\xi^N} \mathrm{Borch}(\psi) = C/N = C_\nu$. This implies that the q-order of $\mathrm{Borch}(\psi)$ is also C_ν. Because the factors $(1 - q^n\zeta^r\xi^{Nm})^{c(nm,r;\psi)}$ may contain negative powers of q, we have $\mathrm{ord}_q\, \mathrm{Borch}(\psi) \leq A$, or $C_\nu \leq A_\nu$.

We now use the identification of A_ν and C_ν as q-orders to prove that they are integral and nonnegative. We exhibit, for each $\nu \in \mathbb{Z}_{\geq 1}$, one Jacobi form that is a theta block without denominator and has q-order ν. For example, we may take $\phi_o = \vartheta^{8\nu} \in \mathrm{J}_{4\nu,4\nu}$. By the Borcherds products everywhere theorem [4], we know that $\mathrm{Borch}(\psi_o) \in \mathcal{M}_{4\nu}(\mathrm{K}(4\nu))$ is holomorphic with trivial character, and that $\phi_1(\mathrm{Borch}(\psi_o))$ is holomorphic with trivial character and integral index. Since $\mathrm{Borch}(\psi)$ has trivial character, $A_\nu = A$ is integral; since $\phi_1(\mathrm{Borch}(\psi_o))$ is holomorphic, its q-order A_ν is nonnegative. The index C of $\phi_1(\mathrm{Borch}(\psi_o))$ is necessarily divisible by $N = 4\nu$, as is the index of every Fourier–Jacobi coefficient of a paramodular form in $\mathcal{M}_{4\nu}(\mathrm{K}(N))$, and so $C_\nu = C/N$ is integral; since $\mathrm{Borch}(\psi)$ is holomorphic, its q-order C_ν is nonnegative. Having established the integrality of $A = A_\nu$ by a specific example, we return to the general case and conclude that $\mathrm{Borch}(\psi)$ has trivial character and integral weight k', that is, $\mathrm{Borch}(\psi) \in \mathcal{M}^{\mathrm{mero}}_{k'}(\mathrm{K}(N))$ with $k' \in \mathbb{Z}$.

Finally, we show that $C_\nu > 0$ by showing that $\phi_1(\mathrm{Borch}(\psi_o))$ is a theta block without denominator and with nonconstant germ $\mathcal{G}_o(\zeta)$; a theta block without denominator has index zero if and only if its germ is a constant function. By Lemma 5.1, $\mathcal{G}_o$ is the q^0-coefficient of $(-1)^\nu\phi_o(\tau/2, z)/\phi_o(\tau, z)$, which here is given by

$$(-1)^\nu q^{-\nu/2} \prod_{\text{odd } n\geq 1} (1 - q^{n/2}\zeta)^{8\nu}(1 - q^{n/2})^{8\nu}(1 - q^{n/2}\zeta^{-1})^{8\nu}.$$

The support of $\mathcal{G}_o$ arises from partitions $\nu = n_1 + \cdots + n_\ell$ of ν into positive odd numbers, and each such partition gives, in general, many terms. Some terms are easy to compute; for example, ζ^ν arises in a unique way from the partition of ν into all ones, and so $\mathcal{G}_o$ is not constant. In general, each partition contributes terms that are $(-1)^\nu$ multiplied by all possible products of ℓ choices from $\{-\zeta, -1, -\zeta^{-1}\}$. Since ℓ has the same parity as ν, all the coefficients of $\mathcal{G}_o$ are nonnegative and hence $\phi_1(\mathrm{Borch}(\psi_o))$ is a theta block without denominator. Hence $C_\nu > 0$ because $\mathcal{G}_o$ is nonconstant. □

6. Equality of the sequences $\{A_\nu\}$ and $\{a_n\}$

Now we show that the sequence $\{A_\nu\}$ is the sequence $\{a_n\}$ of Newman and Shanks. Since each A_ν is a positive integer, this gives another proof that the a_n of Newman and Shanks are positive integers, as shown in [6, Theorem 2].

Proposition 6.1. *$A_\nu = a_\nu$ for all $\nu \in \mathbb{Z}_{\geq 1}$.*

Proof. Fix $\nu \geq 1$. By Lemma 5.1, A_ν can be computed from any theta block ϕ that has q-order ν, and we simply take $\phi(\tau, z) = \eta(\tau)^{24\nu} = q^\nu \prod_{i\geq 1}(1-q^i)^{24\nu}$, of index zero. Thus $A_\nu = \mathcal{G}(1)/24$, where $\mathcal{G}(\zeta)$ is the q^0-coefficient of

$$\begin{aligned}(-1)^\nu \frac{\phi(\tau/2, z)}{\phi(\tau, z)} &= (-1)^\nu q^{-\nu/2} \frac{\prod_{i\geq 1}(1-q^{i/2})^{24\nu}}{\prod_{i\geq 1}(1-q^i)^{24\nu}} \\ &= (-1)^\nu q^{-\nu/2} \prod_{i\geq 1}(1-q^{(2i-1)/2})^{24\nu}.\end{aligned}$$

The q^0-coefficient is unchanged if q is replaced by q^2 and then q is replaced by $-q$. These substitutions give

$$A_\nu = \frac{1}{24}\,\mathrm{Coeff}\Big(\prod_{i\geq 1}(1+q^{2i-1})^{24\nu}, q^\nu\Big),$$

and this is the formula for a_ν in [6, Theorem 1]. □

In particular, using only the $i = 1$ term of the product gives the weak bound $A_\nu \geq \binom{24\nu}{\nu}/24$, which is [6, Equation (55)], and which we use below.

We do not know whether the sequence $\{C_\nu\}$ corresponds to any previously studied sequence. Here are the first 14 C_ν values:

$$\begin{aligned}&1, 46, 2416, 134236, 7695136, 450001696, 26681441536, 1598114568376,\\ &96466710289216, 5858827139417536, 357603570891951616,\\ &21916784219466266176, 1347879537846576487936, 83138677749569762960896.\end{aligned}$$

7. An application

We conclude with an application to Gritsenko lifts. For $\phi \in \mathrm{J}_{k,N}$, in the particular case $\nu = \mathrm{ord}_q\,\phi \geq 1$, the Gritsenko lift $\mathrm{Grit}(\phi) \in \mathcal{M}_k(\mathrm{K}(N))$ is given by the convergent series

$$\mathrm{Grit}(\phi)(Z) = \sum_{\ell=1}^{\infty}(\phi|V_\ell)(\tau, z)\xi^{\ell N} = \phi(\tau, z)\xi^N + (\phi|V_2)(\tau, z)\xi^{2N} + \cdots.$$

In [4], the authors investigated when a Gritsenko lift of a theta block ϕ having q-order ν is also the Borcherds lift of $\psi = (-1)^\nu \phi|V_2/\phi$, and the conjecture was that for a theta block without denominator, the condition $\nu = 1$ is sufficient.

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Conjecture 7.1. Let $\phi \in J_{k,N}$ be a theta block without denominator, having positive index N and q-order 1. Then $\mathrm{Grit}(\phi) = \mathrm{Borch}(\psi)$ for $\psi = -\phi|V_2/\phi$.

The hypotheses imply $k < 12$. This conjecture was proven in [4] for weights 4 through 11, leaving weights 2 and 3 open. With the results of this article, we can prove a partial converse result that $\nu > 1$ is impossible, subject to even weaker hypotheses on the theta block.

Proposition 7.2. *Let $\phi \in \mathrm{J}_{k,N}$ be a theta block such that $b(\zeta)|b(\zeta^2)$ in $\mathbb{C}[\zeta, \zeta^{-1}]$, and let ν denote the q-order of ϕ. Assume that the theta block has weight $k \geq 1$ and that $\nu \geq 1$. If the Gritsenko lift* $\mathrm{Grit}(\phi)$ *is a Borcherds lift as well, then $\nu = 1$.*

Proof. Suppose that $\mathrm{Grit}(\phi)$ is a Borcherds lift, forcing it to be $\mathrm{Borch}(-\phi|V_2/\phi)$. If ν is even then $\mathrm{Grit}(\phi) = \mathrm{Borch}(-(-1)^\nu\phi|V_2/\phi) = 1/\mathrm{Borch}((-1)^\nu\phi|V_2/\phi)$ has negative q-order $-A_\nu$, which is impossible, so ν is odd and $-\phi|V_2/\phi = (-1)^\nu\phi|V_2/\phi$. Now the leading theta block of $\mathrm{Borch}(-\phi|V_2/\phi)$ has q-order A_ν, but because $\mathrm{Borch}(-\phi|V_2/\phi) = \mathrm{Grit}(\phi)$, this leading theta block is ϕ itself, which has q-order ν. Thus $A_\nu = \nu$, and this holds only for $\nu = 1$ in consequence of the weak bound $A_\nu \geq \binom{24\nu}{\nu}/24$ noted above. □

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CRIS POOR, Department of Mathematics,
Fordham University, Bronx, NY 10458, USA
e-mail: poor@fordham.edu

JERRY SHURMAN, Department of Mathematics,
Reed College, Portland, OR 97202, USA
e-mail: jerry@reed.edu

DAVID S. YUEN, Department of Mathematics,
University of Hawaii, Honolulu, HI 96822, USA
e-mail: yuen@math.hawaii.edu

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