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CURRENT-MODIFIED EVOLUTION EQUATION FOR A BROADER BANDWIDTH CAPILLARY–GRAVITY WAVE PACKET

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Abstract

We derive a higher order nonlinear evolution equation for a broader bandwidth three-dimensional capillary–gravity wave packet, in the presence of a surface current produced by an internal wave. Instead of a set of coupled equations, a single nonlinear evolution equation is obtained by eliminating the velocity potential for the wave-induced slow motion. Finally, the equation is expressed in an integro-differential equation form, similar to Zakharov's integral equation. Using the evolution equation derived here, we show that the two sidebands of a surface capillary–gravity wave get excited as a result of resonance with an internal wave, all propagating in the same direction. It is also shown that surface waves can grow exponentially with time at the expense of the energy of the internal wave.

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1. Introduction

The effect of surface current on propagation of wave packets on the surface of water is of considerable interest. A nonuniform current produced by long internal waves interacts with short surface waves and causes patterns on the surface. This topic was considered by Peregrine [14], Jonsson [12], Stocker and Peregrine [17, 18], Ruban [15], Toffoli et al. [19] and others. In order to study the effect of surface current due to an internal wave on a surface-wave pattern, Stocker and Peregrine [17] derived an $O(\epsilon^4)$ nonlinear evolution equation for a three-dimensional surface gravity wave packet. In their paper and also in the present paper, ϵ is the slowness parameter which measures the smallness of nonlinearity and the weakness of dispersion. The equation derived by Stocker and Peregrine [17] is the same as that derived by Dysthe [7] with

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extra terms added to it due to surface current. Here the surface current is assumed to be of $O(\epsilon^2)$. The previous work in this approach is by Bakhanov et al. [1], who derived an $O(\epsilon^3)$ current-modified nonlinear Schrödinger equation in one horizontal direction.

Since the current-modified nonlinear evolution equation derived by Stocker and Peregrine [17] is valid for narrow-bandwidth waves, they pointed out the importance of an equation valid for a broader bandwidth wave. Such an equation in the absence of surface current was obtained by Trulsen and Dysthe [20]. It is known that the narrow-bandwidth constraint limits the applicability of the cubic nonlinear Schrödinger equation and its fourth-order modification [7] while describing the evolution of real ocean wave fields. This is because the continental shelf wave packets are not necessarily of narrow bandwidth. Moreover, Stokes-wave solutions of these equations have regions of instability extending outside the narrow-bandwidth constraint. Zakharov's [23] integral equation does not require any assumption on bandwidth, but this equation is not very suitable for numerical solution. Keeping the above points in view, Trulsen and Dysthe [20] assumed a bandwidth $|\Delta k|/|k|$ to be of $O(\epsilon^{1/2})$, and derived an evolution equation correct up to $O(\epsilon^{7/2})$ which includes higher order linear dispersive terms. Using this evolution equation, Trulsen and Dysthe [20] carried out a linear stability analysis of the uniform wave solution, and showed that the region of instability for perturbations of a uniform Stokes wave was in better agreement with the exact results of McLean et al. [13]. Later on, Debsarma and Das [4] derived an evolution equation with further relaxation in bandwidth. In that paper, the authors assumed a bandwidth of $O(\epsilon^{1/3})$ and derived an evolution equation correct up to $O(\epsilon^{11/3})$. They found that the region of instability fits very nicely with the corresponding figures of McLean et al. [13]. To describe the evolution of weakly nonlinear waves, Trulsen et al. [21] suggested a new model that incorporates any power of ϵ by considering exact linear dispersive behaviour in the evolution equation.

Ruban [15] derived a modified nonlinear Schrödinger equation for surface gravity waves in the presence of steady nonuniform current without the assumption of relative smallness of the current velocity. His method of derivation is based on a Hamiltonian formulation [23]. He also reported that the evolution equation obtained by Stocker and Peregrine [17] does not satisfy the principle of conservation of the wave action. He argued that the reason for this inconsistency is due to measuring the smallness of the surface-wave amplitude and the variation of the current velocity by the same slowness parameter.

In the present investigation, we have considered a modification of the evolution equation obtained by Trulsen and Dysthe [20] to include the effect of surface current produced by an internal wave and the effect of capillarity. So, the equation derived here is valid for broader bandwidth waves. The equation for the evolution of the amplitude of the wave derived by Stocker and Peregrine [17] remains coupled with the velocity potential for the wave-induced slow motion. This velocity potential is given by an additional set of equations. Eliminating this potential, a single nonlinear evolution equation is obtained in the present paper and this equation is also expressed in an integro-differential equation form. The corresponding equations for the particular case of a surface gravity wave packet are obtained in the absence of capillarity.

Trulsen and Mei [22] studied a problem of wave propagation on an opposing current of increasing strength. They showed that a wave in the presence of current can suffer a reflection of its energy. They also showed that the effect of capillarity may result in repeated reflection. In the present paper, we assume the presence of surface current described by the velocity potential, ϕ_c. While deriving the current-modified evolution equation, we do not take into account the possibility of wave reflection. So, the evolution equation derived here remains valid so long as there is no reflection of energy in the medium.

Hasselmann [9] studied the problem of resonant interaction between one finite internal gravity wave mode $(\mathbf{k_0}, \omega_0)$ and two infinitesimal internal wave modes, $(\mathbf{k_1}, \omega_1)$ and $(\mathbf{k_2}, \omega_2)$. He found that waves became unstable as a result of sum interaction: $\mathbf{k_1} + \mathbf{k_2} = \mathbf{k_0}$, $\omega_1 + \omega_2 = \omega_0$, and waves remained neutrally stable for the case of difference interaction: $\mathbf{k_1} - \mathbf{k_2} = \mathbf{k_0}$, $\omega_1 - \omega_2 = \omega_0$.

In resonant interaction of three waves, the amplitude of each varies slowly with time. If one of the three resonantly interacting waves has an amplitude larger than the amplitudes of the other two and its amplitude does not change appreciably with time, then such a wave in the process of three-wave interaction is called a *pump wave* and the other two waves are called *sidebands*. It is known that when one of the three resonantly interacting waves is a pump wave, the two sidebands grow exponentially with time. For example, in case of stimulated Brillouin scattering two of the three interacting waves are electromagnetic waves, and the third one is an ion-acoustic wave. One of the two electromagnetic waves is a pump wave. As a result of resonant interaction, the pump wave decays into an ion-acoustic wave together with another electromagnetic wave propagating in the opposite direction (see Chen [2]).

It is of importance to study – if three-wave-resonant interaction can take place in an ocean medium – when a capillary–gravity wave packet propagates at the free water surface and a surface current is produced due to the presence of an internal wave. Whether or not three-wave-resonant interactions will take place depends on the dispersion relations of the participating waves. At the end of this paper, we have considered resonant interaction between two surface capillary–gravity waves and surface current produced by an internal wave in a two-layer model of stratified ocean. Hogan [10] derived an evolution equation for a surface capillary–gravity wave packet and showed that sideband instability takes place as the wave packet propagates as a result of weakly nonlinear interaction. In this paper, we find that sideband instability can occur as a result of three-wave-resonant interaction in which an internal wave acts as the pump wave and sideband instability occurs in the surface capillary–gravity wave packet. Some feasible wave numbers undergoing the resonance condition are given in Tables 1 and 2. Stable–unstable regions are plotted in the perturbed wave-number plane. We also observe that for a given resonant-interaction sideband, instability is possible if the surface-current amplitude exceeds a certain critical value which is proportional to the square of the carrier-wave amplitude of the surface capillary–gravity wave packet.

2. Basic equations

Assuming that the waves and currents are irrotational (so that we can work with velocity potentials), the governing equations for the flow are the following:

$$\nabla^2\phi = 0, \quad -\infty < z \le \zeta, \tag{2.1}$$

$$\frac{\partial\phi}{\partial z} - \frac{\partial\zeta}{\partial t} = \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y} \quad \text{at } z = \zeta, \tag{2.2}$$

$$\frac{\partial\phi}{\partial t} + g\zeta = -\frac{1}{2}(\nabla\phi)^2 + \frac{\nu\{\zeta_{xx}(1+\zeta_y^2) + \zeta_{yy}(1+\zeta_x^2) - 2\zeta_{xy}\zeta_x\zeta_y\}}{(1+\zeta_x^2+\zeta_y^2)^{3/2}} \quad \text{at } z = \zeta, \tag{2.3}$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{as } z \to -\infty. \tag{2.4}$$

Here g is the gravitational acceleration; ϕ is the velocity potential for the flow; ζ is the elevation of the free surface from its unperturbed state, which is taken as the $z = 0$ plane with z axis pointing in the vertically upward direction; $\nu = T/\rho$ with T and ρ being the surface tension and density of water, respectively.

Expanding the terms in equations (2.2) and (2.3) using Taylor's formula about $z = 0$,

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=0} - \frac{\partial\zeta}{\partial t} = a, \tag{2.5}$$

$$\left(\frac{\partial\phi}{\partial t}\right)_{z=0} + g\zeta - \nu(\zeta_{xx} + \zeta_{yy}) = b, \tag{2.6}$$

where a and b are nonlinear terms; their expressions up to cubic nonlinear terms are given in Appendix A.

Let the velocity potential and surface elevation for surface current be ϕ_c and ζ_c, respectively, given by the following (Stocker and Peregrine [17]):

$$\nabla^2\phi_c = 0 \quad \text{for } z \le 0, \tag{2.7}$$

$$\left(\frac{\partial\phi_c}{\partial z}\right)_{z=0} - \frac{\partial\zeta_c}{\partial t} = 0, \tag{2.8}$$

$$\left(\frac{\partial\phi_c}{\partial t}\right)_{z=0} + g\zeta_c = 0. \tag{2.9}$$

We look for solutions for ϕ and ζ in the form (Stocker and Peregrine [17], Davey and Stewartson [3])

$$\phi = \phi_c + \phi_0 + \sum_{n=1}^{\infty}[\phi_n \exp(in\psi) + \phi_n^* \exp(-in\psi)],$$

$$\zeta = \zeta_c + \zeta_0 + \sum_{n=1}^{\infty}[\zeta_n \exp(in\psi) + \zeta_n^* \exp(-in\psi)], \quad i = \sqrt{-1}, \tag{2.10}$$

where $\psi = kx - \omega t$ and ω, k satisfy the linear dispersion relation

$$D(\omega, k) \equiv \omega^2 - gk - \nu k^3 = 0$$

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for a capillary–gravity wave; $\phi_0, \phi_c, \phi_n, \phi_n^*$ are functions of $x_1 = \epsilon^{1/2}x, y_1 = \epsilon^{1/2}y, t_1 = \epsilon^{1/2}t, z$; and $\zeta_0, \zeta_c, \zeta_n, \zeta_n^*$ are functions of x_1, y_1, t_1. In equation (2.10), the summation is taken over all positive integer values of n. Here we are considering a surface capillary–gravity wave packet to have a bandwidth of $O(\epsilon^{1/2})$. Following Trulsen and Dysthe [20], we call this a *broader bandwidth capillary–gravity wave packet*.

3. Derivation of evolution equation

For derivation of the evolution equation, we shall follow the paper by Dhar and Das [5]. Substituting the expansion for ϕ given by (2.10) in equation (2.1) and then equating coefficients of $\exp(in\psi)$ for $n = 0, 1, 2, \ldots$, we get the following equations for $\phi_n, n = 0, 1, 2, \ldots$:

$$\frac{d^2\phi_n}{dz^2} - \Delta_n^2\phi_n = 0, \quad n = 0, 1, 2, \tag{3.1}$$

where the operator

$$\Delta_n = \left[\left(nk - i\epsilon^{1/2}\frac{\partial}{\partial x_1}\right)^2 - \epsilon\frac{\partial^2}{\partial y_1^2}\right]^{1/2}.$$

The solutions of the three equations in (3.1) satisfying the boundary condition (2.4) are

$$\phi_n = e^{z\Delta_n}A_n \quad \text{for } n = 1, 2, \tag{3.2}$$

$$\overline{\phi}_0 = e^{\epsilon^{1/2}\overline{k}z}\overline{A}_0 \quad \text{for } n = 0. \tag{3.3}$$

In the above equations, A_1, A_2 are functions of x_1, y_1, t_1; the operator

$$e^{z\Delta_n} = 1 + z\Delta_n + \frac{1}{2!}z^2\Delta_n^2 + \cdots;$$

and $\overline{\phi}_0$ is the Fourier transform of ϕ_0 defined by

$$\overline{\phi}_0 = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \phi_0(x_1, y_1, z, t_1)e^{i(k_x x_1 + k_y y_1)}\,dx_1\,dy_1,$$

where $\overline{k}^2 = k_x^2 + k_y^2$. For the sake of convenience, we have taken the Fourier transform of the equations corresponding to $n = 0$.

Since ζ_c is a function of x_1, y_1, t_1 and ϕ_c is a function of x_1, y_1, z, t_1, from equations (2.8) and (2.9),

$$\zeta_c = -\frac{\epsilon^{3/2}}{g}\left(\frac{\partial\phi_c}{\partial t_1}\right)_{z=0}, \tag{3.4}$$

$$\left(\frac{\partial\phi_c}{\partial z}\right)_{z=0} = -\frac{\epsilon^2}{g}\left(\frac{\partial^2\phi_c}{\partial t_1^2}\right)_{z=0}, \tag{3.5}$$

where we assume that $\phi_c = O(\epsilon)$ (Stocker and Peregrine [17]). Substituting the expansions (2.10) in equations (2.5) and (2.6), and then equating coefficients of $\exp(in\psi)$ on both sides, we obtain the following three sets of equations in which we substitute the solutions (3.2) and (3.3) for $\phi_1, \phi_2, \overline{\phi}_0$.

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(i) For $n = 1$,

$$\Delta_1 A_1 + i\Big(\omega + i\epsilon^{1/2}\frac{\partial}{\partial t_1}\Big)\zeta_1 = a_1, \tag{3.6}$$

$$-i\Big(\omega + i\epsilon^{1/2}\frac{\partial}{\partial t_1}\Big)A_1 + g\zeta_1 + \nu\Delta_1^2 A_1 = b_1. \tag{3.7}$$

(ii) For $n = 2$,

$$\Delta_2 A_2 + i\Big(2\omega + i\epsilon^{1/2}\frac{\partial}{\partial t_1}\Big)\zeta_2 = a_2, \tag{3.8}$$

$$-i\Big(2\omega + i\epsilon^{1/2}\frac{\partial}{\partial t_1}\Big)A_2 + g\zeta_2 + \nu\Delta_2^2 A_2 = b_2. \tag{3.9}$$

(iii) For $n = 0$,

$$\Big(\frac{\partial\overline{\phi}_0}{\partial z}\Big)_{z=0} - \epsilon^{1/2}\frac{\partial\overline{\zeta}_0}{\partial t_1} = \overline{a}_0, \tag{3.10}$$

$$\epsilon^{1/2}\Big(\frac{\partial\overline{\phi}_0}{\partial t_1}\Big)_{z=0} + g\overline{\zeta}_0 + \epsilon\nu\overline{k}^{-2}\overline{\zeta}_0 = \overline{b}_0. \tag{3.11}$$

Here $a_n, b_n, n = 0, 1, 2$, are contributions from nonlinear terms.

Eliminating A_1 between (3.6) and (3.7), we get the following equation for ζ_1:

$$\Big[\Big(\omega + i\epsilon^{1/2}\frac{\partial}{\partial t_1}\Big)^2 - g\Delta_1 - \nu\Delta_1^3\Big]\zeta_1 = -\Big(\omega + i\epsilon^{1/2}\frac{\partial}{\partial t_1}\Big)a_1 - \Delta_1 b_1. \tag{3.12}$$

This equation will produce the desired nonlinear evolution equation when terms up to $O(\epsilon^{7/2})$ are retained, assuming that $\zeta_1 = O(\epsilon)$ and $\phi_c = O(\epsilon)$.

In order to calculate a_1 and b_1 up to $O(\epsilon^{7/2})$ terms, we require solutions for A_1, A_2, $\zeta_2, \zeta_0, (\phi_0)_{z=0}, (\partial\phi_0/\partial z)_{z=0}$ up to $O(\epsilon^{3/2}), O(\epsilon^{5/2}), O(\epsilon^{5/2}), O(\epsilon^{5/2}), O(\epsilon^2), O(\epsilon^{5/2})$ terms, respectively.

The solution for A_1 up to $O(\epsilon^{3/2})$ terms can be obtained from (3.6) by neglecting nonlinear terms; the solution thus obtained is

$$A_1 = -\frac{i\epsilon\omega}{k}\zeta_1 + \epsilon^{3/2}\Big(\frac{\omega}{k^2}\frac{\partial\zeta_1}{\partial x_1} + \frac{1}{k}\frac{\partial\zeta_1}{\partial t_1}\Big). \tag{3.13}$$

The equations (3.8) and (3.9) produce the following solutions for A_2 and ζ_2, both correct up to $O(\epsilon^{5/2})$ terms, where we use solution (3.13) for A_1.

$$A_2 = i\epsilon^2\Big(\omega - \frac{2\omega^3}{f}\Big)\zeta_1^2 - \epsilon^{5/2}\Big\{1 + 4\omega^3\Big(\frac{g + 12\nu k^2}{f^2}\Big)\zeta_1\frac{\partial\zeta_1}{\partial x_1} + 4\omega^2\Big(\frac{gk + 10\nu k^3}{f^2}\Big)\zeta_1\frac{\partial\zeta_1}{\partial t_1}\Big\}, \tag{3.14}$$

$$\zeta_2 = 2\epsilon^2\Big(\frac{k\omega^2}{f}\Big)\zeta_1^2 - 8i\epsilon^{5/2}\Big\{\omega^2\Big(\frac{\omega^2 + 4\nu k^3}{f^2}\Big)\zeta_1\frac{\partial\zeta_1}{\partial x_1} + k^2\omega\Big(\frac{g + 4\nu k^2}{f^2}\Big)\zeta_1\frac{\partial\zeta_1}{\partial t_1}\Big\}, \tag{3.15}$$

where

$$f = D(2\omega, 2k).$$

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From equations (3.10) and (3.11), we get the following solutions for ζ_0, $(\phi_0)_{z=0}$ and $(\partial\phi_0/\partial z)_{z=0}$, which are correct up to $O(\epsilon^{5/2})$, $O(\epsilon^2)$ and $O(\epsilon^{5/2})$ terms, respectively.

$$\zeta_0 = 2\epsilon^{5/2}\frac{\omega c_g}{g}H\Big\{\frac{\partial}{\partial x_1}(\zeta_1\zeta_1^*)\Big\}, \tag{3.16}$$

$$(\phi_0)_{z=0} = 2\epsilon^2\omega H(\zeta_1\zeta_1^*), \tag{3.17}$$

$$\Big(\frac{\partial\phi_0}{\partial z}\Big)_{z=0} = 2\epsilon^{5/2}\omega\frac{\partial}{\partial x_1}(\zeta_1\zeta_1^*), \tag{3.18}$$

where H is the Hilbert transform operator given by

$$H\{h(x_1,y_1)\} = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{(x_1'-x_1)}{[(x_1'-x_1)^2+(y_1'-y_1)^2]^{3/2}}h(x_1',y_1')\,dx_1'\,dy_1'.$$

Since the lowest-order term on the right-hand side of (3.12) is of $O(\epsilon^{5/2})$, this equation, when written correct up to $O(\epsilon^2)$ terms, becomes

$$2i\epsilon^{3/2}\omega\Big(\frac{\partial\zeta_1}{\partial t_1}+c_g\frac{\partial\zeta_1}{\partial x_1}\Big)+\epsilon^2\Big(-\frac{\partial^2\zeta_1}{\partial t_1^2}+3\nu k\frac{\partial^2\zeta_1}{\partial x_1^2}+\frac{\omega c_g}{k}\frac{\partial^2\zeta_1}{\partial y_1^2}\Big)=0.$$

Solving this equation in a perturbative way for $\partial\zeta_1/\partial t_1$,

$$\epsilon^{3/2}\frac{\partial\zeta_1}{\partial t_1} = -\epsilon^{3/2}c_g\frac{\partial\zeta_1}{\partial x_1}+i\epsilon^2\Big\{\Big(\frac{3\nu k-c_g^2}{2\omega}\Big)\frac{\partial^2\zeta_1}{\partial x_1^2}+\frac{c_g}{2k}\frac{\partial^2\zeta_1}{\partial y_1^2}\Big\}, \tag{3.19}$$

which is correct up to $O(\epsilon^2)$ terms, and expresses the time derivative of ζ_1 in terms of its space derivatives.

Using the solutions for A_1, A_2, ζ_2, $\zeta_0, (\phi_0)_{z=0}, (\partial\phi_0/\partial z)_{z=0}, \zeta_c, (\partial\phi_c/\partial z)_{z=0}$ given by (3.13)–(3.15), (3.16)–(3.18) and (3.4)–(3.5), respectively, the expression on the right-hand side of (3.12) is obtained correct up to $O(\epsilon^{7/2})$ terms. On the left-hand side, keeping terms also up to $O(\epsilon^{7/2})$, we get the following nonlinear evolution equation correct up to $O(\epsilon^{7/2})$ terms for a wider bandwidth capillary–gravity wave packet modified by the surface current due to an internal wave:

$$\begin{aligned}
&2i\epsilon^{3/2}\omega\Big(\frac{\partial\zeta_1}{\partial t_1}+c_g\frac{\partial\zeta_1}{\partial x_1}\Big)+\epsilon^2\Big(3\nu k\frac{\partial^2\zeta_1}{\partial x_1^2}+\frac{\omega c_g}{k}\frac{\partial^2\zeta_1}{\partial y_1^2}-\frac{\partial^2\zeta_1}{\partial t_1^2}\Big)\\
&\quad+i\epsilon^{5/2}\Big(-\nu\frac{\partial^3\zeta_1}{\partial x_1^3}+\frac{g-3\nu k^2}{2k^2}\frac{\partial^3\zeta_1}{\partial x_1\partial y_1^2}\Big)+\epsilon^3\Big(-\frac{g}{2k^3}\frac{\partial^4\zeta_1}{\partial x_1^2\partial y_1^2}+\frac{g-3\nu k^2}{8k^3}\frac{\partial^4\zeta_1}{\partial y_1^4}\Big)\\
&\quad+i\epsilon^{7/2}\Big(\frac{3g-3\nu k^2}{8k^4}\frac{\partial^5\zeta_1}{\partial x_1\partial y_1^4}-\frac{g}{2k^4}\frac{\partial^5\zeta_1}{\partial x_1^3\partial y_1^2}\Big)\\
&=\epsilon^3d_1\zeta_1^2\zeta_1^*+i\epsilon^{7/2}d_2\zeta_1\zeta_1^*\frac{\partial\zeta_1}{\partial x_1}+i\epsilon^{7/2}d_3\zeta_1^2\frac{\partial\zeta_1^*}{\partial x_1}+4\epsilon^{7/2}k\omega^2\zeta_1H\Big\{\frac{\partial}{\partial x_1}(\zeta_1\zeta_1^*)\Big\}+E,
\end{aligned} \tag{3.20}$$

where E is an expression involving surface-current terms. Its expression together with the expressions for the coefficients d_1, d_2, d_3 are given in Appendix B. In the nonlinear

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expression of equation (3.20), the time derivative of ζ_1 has been replaced by its space derivatives by using equation (3.19). Solving equation (3.20) in a perturbative way for $\partial\zeta_1/\partial t_1$, we get the following expression for $\partial\zeta_1/\partial t_1$, correct up to $O(\epsilon^3)$ terms:

$$\begin{aligned}
\epsilon^{3/2}\frac{\partial\zeta_1}{\partial t_1} &= -\epsilon^{3/2}c_g\frac{\partial\zeta_1}{\partial x_1} + i\epsilon^2\Big(\frac{3\nu k - c_g^2}{2\omega}\frac{\partial^2\zeta_1}{\partial x_1^2} + \frac{c_g}{2k}\frac{\partial^2\zeta_1}{\partial y_1^2}\Big)\\
&\quad + \epsilon^{5/2}\Big\{\frac{1}{2\omega^2}(\nu\omega + c_g^3 - 3\nu k c_g)\frac{\partial^3\zeta_1}{\partial x_1^3} - \frac{1}{4k^2\omega}(g - 3\nu k^2 + 2kc_g^2)\frac{\partial^3\zeta_1}{\partial x_1\partial y_1^2}\Big\}\\
&\quad + i\epsilon^3\Big\{\frac{1}{8\omega^3}(4\nu\omega c_g + 9\nu^2k^2 - 18\nu k c_g^2 + 5C_g^4)\frac{\partial^4\zeta_1}{\partial x_1^4}\\
&\quad - \frac{1}{4k^3\omega^2}(g\omega + gkc_g - 6\nu k^3 c_g + 3k^2c_g^3)\frac{\partial^4\zeta_1}{\partial x_1^2\partial y_1^2}\\
&\quad + \frac{1}{16k^3\omega}(g - 3\nu k^2 + 2kc_g^2)\frac{\partial^4\zeta_1}{\partial y_1^4}\Big\} - i\epsilon^{5/2}k\zeta_1\Big(\frac{\partial\phi_c}{\partial x_1}\Big)_{z=0}\\
&\quad - \epsilon^3\Big\{\frac{\partial\zeta_1}{\partial x_1}\Big(\frac{\partial\phi_c}{\partial x_1}\Big)_{z=0} + \frac{\partial\zeta_1}{\partial y_1}\Big(\frac{\partial\phi_c}{\partial y_1}\Big)_{z=0}\Big\}\\
&\quad + \epsilon^3\Big(\frac{kc_g}{2\omega} - 1\Big)\zeta_1\Big(\frac{\partial^2\phi_c}{\partial x_1^2}\Big)_{z=0} - \frac{1}{2}\epsilon^3\zeta_1\Big(\frac{\partial^2\phi_c}{\partial y_1^2}\Big)_{z=0} - i\epsilon^3\Big(\frac{d_1}{2\omega}\Big)\zeta_1^2\zeta_1^*. \qquad (3.21)
\end{aligned}$$

Using equation (3.21), the term $-\epsilon^2(\partial^2\zeta_1/\partial t_1^2)$ on the left-hand side of (3.20) can be replaced by an expression where there exists no time derivative of ζ_1. With this expression for $-\epsilon^2(\partial^2\zeta_1/\partial t_1^2)$, the nonlinear evolution equation assumes the following dimensionless form by introducing the dimensionless variables: $\xi = kx_1, \eta = ky_1, \tau = \omega t_1, \zeta = k\zeta_1$ and $\phi_c' = (k^2/\omega)\phi_c$. Finally, dropping primes on ϕ_c' yields

$$\begin{aligned}
&i\Big(\frac{\partial\zeta}{\partial\tau} + \beta_0\frac{\partial\zeta}{\partial\xi}\Big) + \epsilon^{1/2}\Big(\beta_1\frac{\partial^2\zeta}{\partial\xi^2} + \beta_2\frac{\partial^2\zeta}{\partial\eta^2}\Big) + i\epsilon\Big(\beta_3\frac{\partial^3\zeta}{\partial\xi^3} + \beta_4\frac{\partial^3\zeta}{\partial\xi\partial\eta^2}\Big)\\
&\quad + \epsilon^{3/2}\Big(\beta_5\frac{\partial^4\zeta}{\partial\xi^4} + \beta_6\frac{\partial^4\zeta}{\partial\xi^2\partial\eta^2} + \beta_7\frac{\partial^4\zeta}{\partial\eta^4}\Big) + i\epsilon^2\Big(\beta_8\frac{\partial^5\zeta}{\partial\xi^5} + \beta_9\frac{\partial^5\zeta}{\partial\xi^3\partial\eta^2} + \beta_{10}\frac{\partial^5\zeta}{\partial\xi\partial\eta^4}\Big)\\
&= \epsilon^{3/2}\lambda_1\zeta^2\zeta^* + i\epsilon^2\Big(\lambda_2\zeta\zeta^*\frac{\partial\zeta}{\partial\xi} + \lambda_3\zeta^2\frac{\partial\zeta^*}{\partial\xi}\Big) + 2\epsilon^2\zeta H\frac{\partial}{\partial\xi}(\zeta\zeta^*) + \Big[\epsilon\zeta\Big(\frac{\partial\phi_c}{\partial\xi}\Big)_{z=0}\\
&\quad - i\epsilon^{3/2}\Big\{\frac{\partial\zeta}{\partial\xi}\Big(\frac{\partial\phi_c}{\partial\xi}\Big)_{z=0} + \frac{\partial\zeta}{\partial\eta}\Big(\frac{\partial\phi_c}{\partial\eta}\Big)_{z=0} - \mu_1\zeta\Big(\frac{\partial^2\phi_c}{\partial\xi^2}\Big)_{z=0} - \mu_2\zeta\Big(\frac{\partial^2\phi_c}{\partial\eta^2}\Big)_{z=0}\Big\}\\
&\quad + \epsilon^2\Big\{\mu_3\zeta\Big(\frac{\partial^3\phi_c}{\partial\xi^3}\Big)_{z=0} + \mu_4\zeta\Big(\frac{\partial^3\phi_c}{\partial\xi\partial\eta^2}\Big)_{z=0} + \mu_5\zeta\Big(\frac{\partial^3\phi_c}{\partial\tau\partial\xi^2}\Big)_{z=0} + \mu_6\zeta\Big(\frac{\partial^3\phi_c}{\partial\tau\partial\eta^2}\Big)_{z=0}\\
&\quad + \mu_7\zeta\Big(\frac{\partial^3\phi_c}{\partial\tau^3}\Big)_{z=0} + \mu_8\frac{\partial\zeta}{\partial\xi}\Big(\frac{\partial^2\phi_c}{\partial\xi^2}\Big)_{z=0} + \mu_9\frac{\partial\zeta}{\partial\eta}\Big(\frac{\partial^2\phi_c}{\partial\xi\partial\eta}\Big)_{z=0}\Big\}\Big]. \qquad (3.22)
\end{aligned}$$

The coefficients β_k, λ_k and μ_k appearing in the above equation are given in Appendix C. The equation (3.22) is the desired current-modified nonlinear evolution

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equation for a broader bandwidth capillary–gravity wave packet correct up to $O(\epsilon^{7/2})$ terms. So, in this equation ζ represents the wave envelope correct up to this order of accuracy. Equation (3.22) gives the modified version of equation (24) of Stocker and Peregrine [17] in order to include the capillarity effect for the case of a broader bandwidth wave packet in infinite-depth water. In the absence of surface currents and dispersive terms involving fourth- and fifth-order derivatives, the resulting equation becomes the same as the equation (2.20) of Hogan [10]. The terms within the third bracket on the right-hand side of equation (3.22) are due to interaction of surface waves with surface current. The velocity potential ϕ_c for surface current, appearing in equation (3.22), is governed by equations (2.7)–(2.9). Thus, for numerical simulation of equation (3.22) one must use equations (2.7)–(2.9) along with equation (3.22). The evolution equation (3.22) can also be written in an integro-differential equation form, following Janssen [11].

In the absence of capillarity, that is, with $m = \nu k^2/g = 0$, the evolution equation (3.22) becomes the following, which is, therefore, the current-modified nonlinear evolution equation for a broader bandwidth surface gravity wave packet:

$$\begin{aligned}
&i\Big(\frac{\partial\zeta}{\partial\tau} + \frac{1}{2}\frac{\partial\zeta}{\partial\xi}\Big) + \epsilon^{1/2}\Big(-\frac{1}{8}\frac{\partial^2\zeta}{\partial\xi^2} + \frac{1}{4}\frac{\partial^2\zeta}{\partial\eta^2}\Big) + i\epsilon\Big(-\frac{1}{16}\frac{\partial^3\zeta}{\partial\xi^3} + \frac{3}{8}\frac{\partial^3\zeta}{\partial\xi\partial\eta^2}\Big)\\
&\quad + \epsilon^{3/2}\Big(\frac{5}{128}\frac{\partial^4\zeta}{\partial\xi^4} - \frac{15}{32}\frac{\partial^4\zeta}{\partial\xi^2\partial\eta^2} + \frac{3}{32}\frac{\partial^4\zeta}{\partial\eta^4}\Big)\\
&\quad + i\epsilon^2\Big(\frac{7}{256}\frac{\partial^5\zeta}{\partial\xi^5} - \frac{35}{64}\frac{\partial^5\zeta}{\partial\xi^3\partial\eta^2} + \frac{21}{64}\frac{\partial^5\zeta}{\partial\xi\partial\eta^4}\Big)\\
&= 2\epsilon^{3/2}\zeta^2\zeta^* + i\epsilon^2\Big(-6\zeta\zeta^*\frac{\partial\zeta}{\partial\xi} - \zeta^2\frac{\partial\zeta^*}{\partial\xi}\Big) + 2\epsilon^2\zeta H\frac{\partial}{\partial\xi}(\zeta\zeta^*) + \epsilon\zeta\Big(\frac{\partial\phi_c}{\partial\xi}\Big)_{z=0}\\
&\quad - i\epsilon^{3/2}\Big\{\frac{\partial\zeta}{\partial\xi}\Big(\frac{\partial\phi_c}{\partial\xi}\Big)_{z=0} + \frac{\partial\zeta}{\partial\eta}\Big(\frac{\partial\phi_c}{\partial\eta}\Big)_{z=0} + \frac{3}{4}\zeta\Big(\frac{\partial^2\phi_c}{\partial\xi^2}\Big)_{z=0} + \frac{1}{2}\zeta\Big(\frac{\partial^2\phi_c}{\partial\eta^2}\Big)_{z=0}\Big\}\\
&\quad + \epsilon^2\Big\{\frac{1}{8}\zeta\Big(\frac{\partial^3\phi_c}{\partial\xi^3}\Big)_{z=0} + \frac{1}{8}\zeta\Big(\frac{\partial^3\phi_c}{\partial\tau\partial\xi^2}\Big)_{z=0} - \frac{1}{2}\zeta\Big(\frac{\partial^3\phi_c}{\partial\tau^3}\Big)_{z=0} - \frac{1}{2}\frac{\partial\zeta}{\partial\eta}\Big(\frac{\partial^2\phi_c}{\partial\xi\partial\eta}\Big)_{z=0}\Big\}.
\end{aligned} \tag{3.23}$$

In the absence of surface currents and in the absence of dispersive terms involving fourth- and fifth-order derivatives, the equation (3.23) becomes the same as equation (14) of Stiassnie [16]. In addition, if the terms due to surface current are neglected then equation (3.23) matches the evolution equation (21) of Trulsen and Dysthe [20] for a broad-band surface gravity wave packet in the limiting case of infinite-depth water.

The two evolution equations (3.22) and (3.23) remain valid as long as reflection of energy does not take place. Trulsen and Mei [22] considered propagation of a wave packet, taking into account the reflection of waves. They showed that when a train of gravity waves encounters an opposing current, the wavelength is shortened and the waves may be reflected. For the case of capillary–gravity waves, the shortened waves may again be reflected and, as a result, may undergo further shortening.

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4. Sideband instability

In this section, we show that the resonance condition can be satisfied by two sidebands of the surface capillary–gravity waves and an internal wave, all propagating in the same direction.

When surface waves and internal waves both propagate along the x direction, the evolution equation (3.22) reduces to the following equation:

$$i\frac{\partial\zeta}{\partial\tau}+i\beta_0\frac{\partial\zeta}{\partial\xi}+\beta_1\frac{\partial^2\zeta}{\partial\xi^2}+i\beta_3\frac{\partial^3\zeta}{\partial\xi^3}+\beta_5\frac{\partial^4\zeta}{\partial\xi^4}+i\beta_8\frac{\partial^5\zeta}{\partial\xi^5}$$
$$=\lambda_1\zeta^2\zeta^*+i\lambda_2\zeta\zeta^*\frac{\partial\zeta}{\partial\xi}+i\lambda_3\zeta^2\frac{\partial\zeta^*}{\partial\xi}+2\zeta H\frac{\partial}{\partial\xi}(\zeta\zeta^*)+\zeta\Big(\frac{\partial\phi_{\rm c}}{\partial\xi}\Big)_{z=0}$$
$$-i\frac{\partial\zeta}{\partial\xi}\Big(\frac{\partial\phi_{\rm c}}{\partial\xi}\Big)_{z=0}+i\mu_1\zeta\Big(\frac{\partial^2\phi_{\rm c}}{\partial\xi^2}\Big)_{z=0}+\mu_3\zeta\Big(\frac{\partial^3\phi_{\rm c}}{\partial\xi^3}\Big)_{z=0}+\mu_5\zeta\Big(\frac{\partial^3\phi_{\rm c}}{\partial\tau\partial\xi^2}\Big)_{z=0}$$
$$+\mu_7\zeta\Big(\frac{\partial^3\phi_{\rm c}}{\partial\tau^3}\Big)_{z=0}+\mu_8\frac{\partial\zeta}{\partial\xi}\Big(\frac{\partial^2\phi_{\rm c}}{\partial\xi^2}\Big)_{z=0},\tag{4.1}$$

in which we have set $\epsilon=1$.

The Stokes uniform wave solution [7, 20] of equation (4.1) in the absence of surface current is given by

$$\zeta=\zeta^{(\rm s)}=\alpha_{\rm s}\exp(-i\Omega_{\rm s}\tau),\tag{4.2}$$

where $\alpha_{\rm s}$ is a constant and $\Omega_{\rm s}=\lambda_1|\alpha_{\rm s}|^2$. We now seek a nonuniform solution of equation (4.1) in the following form:

$$\zeta=\zeta^{(\rm n)}=\alpha\exp(-i\Omega_{\rm s}\tau),\tag{4.3}$$

where α is a function of ξ and τ, satisfying the equation

$$i\frac{\partial\alpha}{\partial\tau}+\Omega_0\alpha+i\beta_0\frac{\partial\alpha}{\partial\xi}+\beta_1\frac{\partial^2\alpha}{\partial\xi^2}+i\beta_3\frac{\partial^3\alpha}{\partial\xi^3}+\beta_5\frac{\partial^4\alpha}{\partial\xi^4}+i\beta_8\frac{\partial^5\alpha}{\partial\xi^5}$$
$$=\lambda_1\alpha^2\alpha^*+i\lambda_2\alpha\alpha^*\frac{\partial\alpha}{\partial\xi}+i\lambda_3\alpha^2\frac{\partial\alpha^*}{\partial\xi}+2\alpha H\frac{\partial}{\partial\xi}(\alpha\alpha^*)+\alpha\Big(\frac{\partial\phi_{\rm c}}{\partial\xi}\Big)_{z=0}$$
$$-i\frac{\partial\alpha}{\partial\xi}\Big(\frac{\partial\phi_{\rm c}}{\partial\xi}\Big)_{z=0}+i\mu_1\alpha\Big(\frac{\partial^2\phi_{\rm c}}{\partial\xi^2}\Big)_{z=0}+\mu_3\alpha\Big(\frac{\partial^3\phi_{\rm c}}{\partial\xi^3}\Big)_{z=0}$$
$$+\mu_5\alpha\Big(\frac{\partial^3\phi_{\rm c}}{\partial\tau\partial\xi^2}\Big)_{z=0}+\mu_7\alpha\Big(\frac{\partial^3\phi_{\rm c}}{\partial\tau^3}\Big)_{z=0}+\mu_8\frac{\partial\alpha}{\partial\xi}\Big(\frac{\partial^2\phi_{\rm c}}{\partial\xi^2}\Big)_{z=0}.\tag{4.4}$$

We now assume the space–time dependence of α to be of the form $\exp[i(\widetilde{\omega}\tau-k\xi)]$. Substituting (4.3) in equation (4.4) and then linearizing, we get the following linear dispersion relation for a sideband of a Stokes wave train given by equation (4.2):

$$\widetilde{\omega}(k)=\Omega_{\rm s}+\beta_0k-\beta_1k^2-\beta_3k^3+\beta_5k^4+\beta_8k^5.\tag{4.5}$$

Assuming the presence of two sidebands (ω_1,k_1) and (ω_2,k_2) of the Stokes wave train given by equation (4.2), α can be expressed as

$$\alpha=\alpha_{\rm s}+\alpha_1\exp(i\psi_1)+\alpha_2\exp(i\psi_2),\tag{4.6}$$

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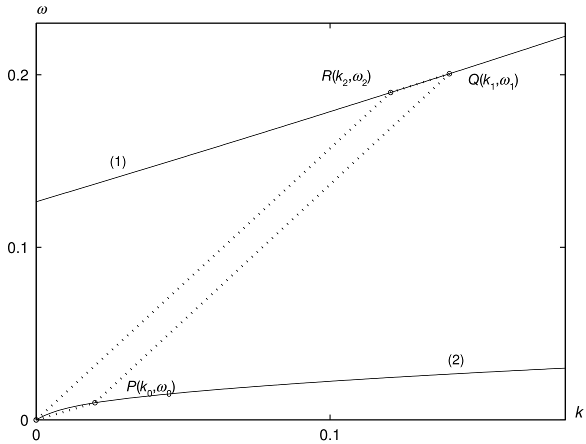

FIGURE 1. Dispersion curves: (1) for surface capillary–gravity wave, (2) for internal waves; $\Gamma = 0.01$, $m = 0.01, d = 100, \alpha_s = 0.25$; $OPQR$ is a parallelogram establishing equation (4.7).

where α_1, α_2 are functions of τ only and

$$\psi_1 = \omega_1\tau - k_1\xi, \quad \psi_2 = \omega_2\tau - k_2\xi.$$

Here (ω_1, k_1) and (ω_2, k_2) satisfy the following dispersion relations:

$$\omega_1 = \widetilde{\omega}(k_1), \quad \omega_2 = \widetilde{\omega}(k_2).$$

We further assume that the surface current is generated by an underlying internal wave [6, 17]. For the presence of an internal wave, we assume a simple model – the two-layer model for a stratified ocean. In the presence of a thermocline at depth d, the dispersion relation for an internal wave in dimensionless form is given by

$$\omega_0^2 = [\omega_0(k_0)]^2 = \frac{\Gamma k_0 \tanh(k_0 d)}{1 + \tanh(k_0 d)}.$$

Here $(\omega_0,\ k_0)$ are, respectively, the frequency and wavenumber of the internal wave and $\Gamma = \delta\rho/\rho$, $\delta\rho$ being the increment in density across the thermocline. For the resonance of two sidebands of surface capillary–gravity waves and an internal wave, the following condition is to be satisfied:

$$k_0 + k_2 = k_1, \quad \omega_0 + \omega_2 = \omega_1. \tag{4.7}$$

Here the suffix “0” corresponds to the internal wave, and the suffixes “1” and “2” correspond to the two sidebands of the surface capillary–gravity waves. The resonance condition given in (4.7) can be simplified as follows:

$$R(k_0, k_2) \equiv \omega_0(k_0) + \widetilde{\omega}(k_2) - \widetilde{\omega}(k_0 + k_2) = 0.$$

Figure 1 shows the triad resonance among the wave number and frequencies (k_0, ω_0), (k_1, ω_1), (k_2, ω_2) satisfying condition (4.7). A plot of dispersion curves for internal

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TABLE 1. Some wave numbers undergoing resonance condition (4.7), $\Gamma = 0.01, m = 0.01, d = 100, \alpha_s = 0.25$.

(k_0, ω_0)	(k_2, ω_2)	(k_1, ω_1)
(0.001, 0.000952)	(0.7996, 0.665226)	(0.8006, 0.666179)
(0.002, 0.001816)	(0.7574, 0.626025)	(0.7594, 0.627840)
(0.003, 0.002602)	(0.7157, 0.589083)	(0.7187, 0.591684)
(0.004, 0.003319)	(0.6741, 0.553867)	(0.6781, 0.557186)
(0.005, 0.003975)	(0.6321, 0.519836)	(0.6371, 0.523811)
(0.006, 0.004579)	(0.5901, 0.487211)	(0.5961, 0.491709)
(0.007, 0.005135)	(0.5476, 0.455512)	(0.5546, 0.460647)
(0.008, 0.005650)	(0.5048, 0.424811)	(0.5128, 0.430461)
(0.009, 0.006129)	(0.4615, 0.394895)	(0.4705, 0.401023)
(0.010, 0.006575)	(0.4176, 0.365636)	(0.4276, 0.372211)
(0.011, 0.006993)	(0.3730, 0.336920)	(0.3840, 0.343913)
(0.012, 0.007386)	(0.3276, 0.308642)	(0.3396, 0.316028)
(0.013, 0.007757)	(0.2814, 0.280765)	(0.2944, 0.288522)
(0.014, 0.008108)	(0.2340, 0.253023)	(0.2480, 0.261131)
(0.015, 0.008442)	(0.1855, 0.225457)	(0.2005, 0.233898)

TABLE 2. Some wave numbers undergoing resonance condition (4.7), $\Gamma = 0.01, m = 0.8, d = 100, \alpha_s = 0.25$.

(k_0, ω_0)	(k_2, ω_2)	(k_1, ω_1)
(0.001, 0.000952)	(0.162, 0.001463)	(0.163, 0.002338)
(0.002, 0.001816)	(0.162, 0.001463)	(0.164, 0.003212)
(0.003, 0.002602)	(0.180, 0.017143)	(0.183, 0.019743)
(0.004, 0.003319)	(0.275, 0.097752)	(0.279, 0.101070)
(0.005, 0.003975)	(0.896, 0.575880)	(0.901, 0.579855)
(0.006, 0.004579)	(0.807, 0.506770)	(0.813, 0.511348)
(0.007, 0.005135)	(0.663, 0.398709)	(0.670, 0.403923)

wave and surface capillary–gravity waves is shown in Figure 1 for $\alpha_s = 0.25$, $m = 0.01$, $\Gamma = 0.01$ and $d = 100$. If P, Q, R are the three points representing (k_0, ω_0), (k_1, ω_1) and (k_2, ω_2), respectively, then it follows from (4.7) that $\overrightarrow{PR} + \overrightarrow{RQ} = \overrightarrow{PQ}$. Also, the three points P, Q, R along with the origin O form a parallelogram $OPQR$.

In Tables 1 and 2, some of the feasible values of (k_0, ω_0), (k_1, ω_1) and (k_2, ω_2) are given when the resonance condition (4.7) is satisfied for $\Gamma = 0.01$, $d = 100$ and $\alpha_s = 0.25$. In Table 1, $m = 0.01$, while in Table 2, $m = 0.8$. The value $m = 0.01$ corresponds to surface capillary–gravity wave number $k = 0.37\ \mathrm{cm}^{-1}$, wavelength $\lambda = 17$ cm and frequency $\omega = 19\ \mathrm{rad\ s}^{-1}$. Since all the variables are made dimensionless using k and

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ω, the dimensional values of the entries of Table 1 can be found by multiplying those wavenumbers and frequencies, respectively, by the characteristic values $k = 0.37\ \mathrm{cm}^{-1}$ and $\omega = 19\ \mathrm{rad\ s}^{-1}$. In this table, k_0 actually represents the ratio of the wave number of the internal wave to the wave number of the surface capillary–gravity wave. For Table 2, the wavelength of the surface capillary–gravity wave is $\lambda = 1.92$ cm and the frequency is $\omega = 76\ \mathrm{rad\ s}^{-1}$.

For a physically relevant situation, one should consider up to several hundred wavelengths of the short waves on one wavelength of an internal wave. Stocker and Peregrine [17] reported that in an estuarine channel, experimental data are available, for which the wavelength λ of a surface wave is 40 cm and the wavelength of a surface current is $\Lambda = 12\,000$ cm. Thus, $\Lambda/\lambda = 300$. A pycnocline may be formed at a depth of 600 cm with $\Gamma = 0.0025$ and the surface current may have a maximum value of $4\ \mathrm{cm\ s}^{-1}$. For numerical computations, Stocker and Peregrine [17] chose $\Lambda/\lambda = 20$, which models the physical situation in which 20 wavelengths of a short wave arise on an 800 cm internal wave with a maximum current of $4.7\ \mathrm{cm\ s}^{-1}$. Such a physical situation may be observed when a pycnocline is formed at a depth of 77 cm with freshwater having low salinity of 10 parts per thousand overlying sea water with salinity content of 35 parts per thousand.

For the internal wave, we take ϕ_c in the following form:

$$\phi_c = \hat{\phi}_c \exp(i\psi_0) + \hat{\phi}_c^{\,*} \exp(-i\psi_0), \tag{4.8}$$

in which $\hat{\phi}_c$ is a constant and $\psi_0 = \omega_0\tau - k_0\xi$. Substituting (4.6) and (4.8) in equation (4.4) and assuming that α_1, α_2 are infinitesimal in magnitude having time dependence of the form $\exp(-i\Omega\tau)$, we get the following equation for finding Ω:

$$[\Omega - (2\lambda_1 + \lambda_2 k_1 - 2|k_1|)|\alpha_s|^2][\Omega - (2\lambda_1 + \lambda_2 k_2 - 2|k_2|)|\alpha_s|^2] = BC|\hat{\phi}_c|^2, \tag{4.9}$$

where

$$B = k_0 - \mu_1 k_0^2 - \mu_3 k_0^3 + \mu_5 k_0^2\omega_0 + \mu_7\omega_0^3 - k_0 k_1 + \mu_8 k_0^2 k_1,$$
$$C = k_0 + \mu_1 k_0^2 - \mu_3 k_0^3 + \mu_5 k_0^2\omega_0 + \mu_7\omega_0^3 - k_0 k_2 - \mu_8 k_0^2 k_2.$$

Equation (4.9) determines Ω as

$$[\Omega - 2\lambda_1|\alpha_s|^2 - \{\lambda_2(k_1 + k_2) - 2(|k_1| + |k_2|)\}|\alpha_s|^2]^2 = A^2|\alpha_s|^4 + 4BC|\hat{\phi}_c|^2, \tag{4.10}$$

where

$$A = \lambda_2(k_1 - k_2) - 2(|k_1| - |k_2|).$$

The frequency Ω given by (4.10) will have an imaginary part if the conditions

$$BC < 0 \tag{4.11}$$

and

$$|\hat{\phi}_c| > \frac{|A|}{2\sqrt{|BC|}}|\alpha_s|^2 \tag{4.12}$$

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are satisfied. The conditions (4.11) and (4.12) are then the conditions for sideband instability, in which case the two sidebands of capillary–gravity waves start growing exponentially with time. The condition (4.11) is equivalent to $BC = [Gk_0 + O(\epsilon^2)] < 0$, where

$$G \equiv k_0 - k_0(k_1 + k_2) + k_0k_1k_2 - (\mu_1 - \mu_8)k_0^2(k_1 - k_2) - (\mu_1^2 + 2\mu_3)k_0^3 + 2\mu_5k_0^2\omega_0 + 2\mu_7\omega_0^3. \tag{4.13}$$

Retaining terms correct up to $O(\epsilon^{7/2})$, we get $Gk_0 < 0$ for instability, which gives $G < 0$, since $k_0 > 0$.

The lowest order at which the condition (4.13) is satisfied is at $O(\epsilon^3)$. Thus, the minimal model that produces instability for a perturbation of the uniform wave solution can be obtained from equation (3.22) by dropping terms multiplied by ϵ^2.

Using equation (3.4), we can rewrite the condition (4.12) as

$$|\zeta_c| > \frac{|A|}{2\sqrt{|BC|}}\omega_0|\alpha_s|^2. \tag{4.14}$$

The condition in (4.14) shows that for a given set of wave numbers of internal wave and surface capillary–gravity waves and for a given amplitude of the carrier wave of a surface capillary–gravity wave packet, sideband instability is possible when the surface-current amplitude exceeds a certain critical value $\zeta_c^{(c)}$ given by

$$\zeta_c^{(c)} = \frac{|A|}{2\sqrt{|BC|}}\omega_0|\alpha_s|^2.$$

In Figures 2 and 3, the region of stability ($G > 0$) and the region of instability ($G < 0$) have been shown in the (k_0, k_2) plane. In these two figures, resonance curves satisfying condition (4.7) are also displayed. Thus, in Figures 2 and 3, we find wave numbers k_0 of the internal wave and wave numbers k_2 of the surface capillary–gravity wave undergoing the resonance condition with wave numbers k_1 of the surface capillary–gravity wave and, as a result, the Stokes wave being stable or unstable according as $G > 0$ or $G < 0$.

In Figures 4 and 5, $\zeta_c^{(c)}$ has been plotted against α_s for different values of thermocline depth. These two figures reveal the dependence of $\zeta_c^{(c)}$ on thermocline depth d.

5. Conclusion

The main aim of this paper is to derive the equations (3.22) and (3.23), which are the broader bandwidth nonlinear evolution equations, respectively, for capillary–gravity and surface gravity wave packets modified by surface currents due to an internal wave. The importance of such equations was pointed out by Stocker and Peregrine [17]. The derived evolution equation has also been expressed in integro-differential equation form, suitable for deriving a wave kinetic equation. Using the

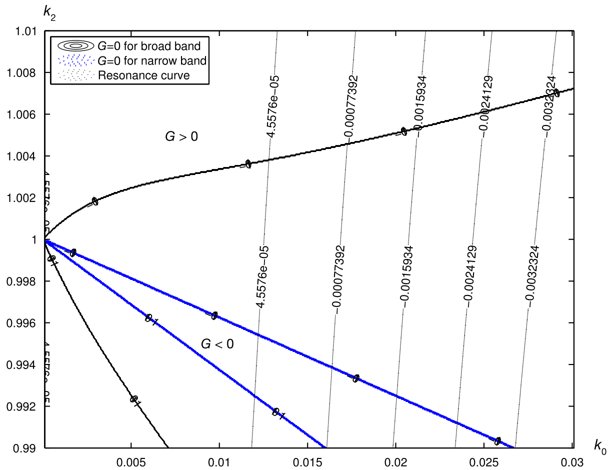

FIGURE 2. Regions of stability and instability; $G = 0$ for broad-band wave packet, $G = 0$ for narrow-band wave packet, contour plot of resonance curves satisfying condition (4.7); $\Gamma = 0.004, m = 3, d = 150, \alpha_s = 0.2$.

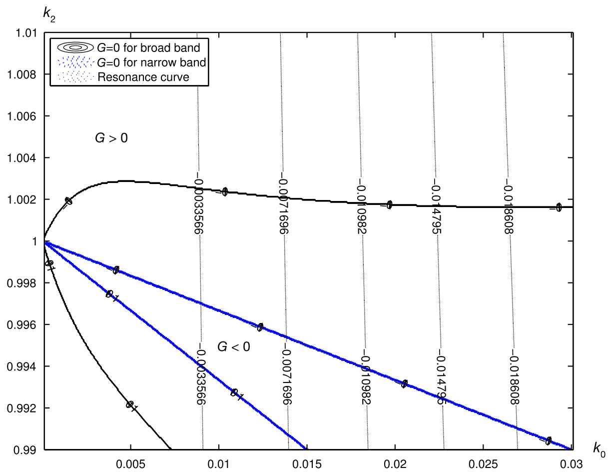

FIGURE 3. Regions of stability and instability; $G = 0$ for broad-band wave packet, $G = 0$ for narrow-band wave packet, contour plot of resonance curves satisfying condition (4.7); $\Gamma = 0.01, m = 0.2, d = 300, \alpha_s = 0.2$.

evolution equation derived here, we have made an investigation of sideband instability of a surface capillary–gravity wave packet when it undergoes resonant interaction with a surface current due to an internal wave.

Dysthe and Das [8] studied a theoretical model to present the coupling of an internal wave with a surface-wave spectrum. They considered a simple three-layer model of the ocean with a (shallow) seasonal thermocline region separating homogeneous

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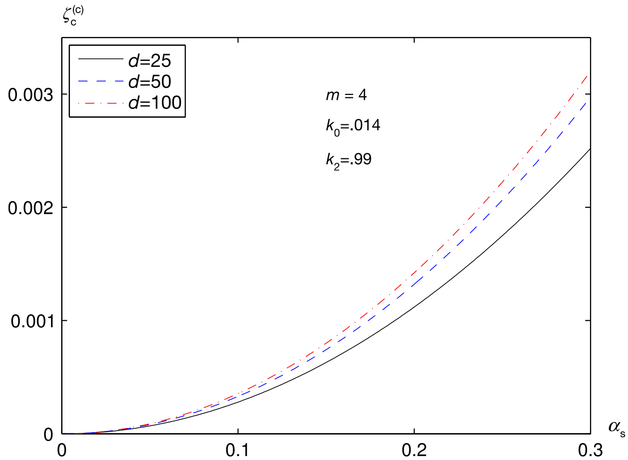

FIGURE 4. Plot of $\zeta_c^{(c)}$ against α_s for different thermocline depths d; $\Gamma = 0.004$, $d = 25$, $d = 50$, $d = 100$.

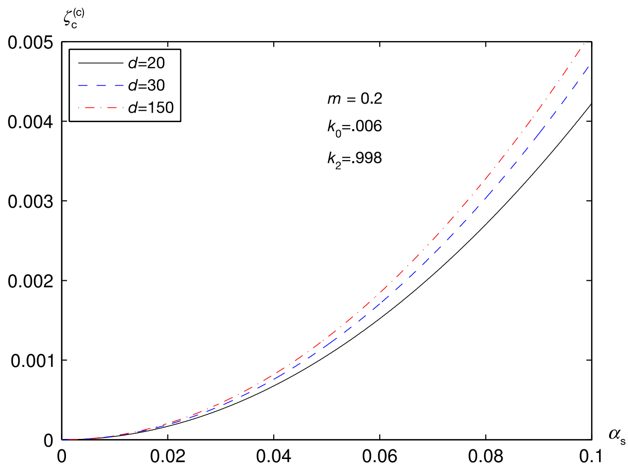

FIGURE 5. Plot of $\zeta_c^{(c)}$ against α_s for different thermocline depths d; $\Gamma = 0.01$, $d = 20$, $d = 30$, $d = 150$.

waters above and below. With this model, they showed that the nonuniform surface current induced by an internal wave causes refraction and thereby causes modulation of the surface-wave spectrum. This modulation introduces nonuniform radiation stress on the internal wave. They derived a transport equation for the spectral density of a surface-wave spectrum and showed that modulational instability may persist because of a coupling between a modulational mode of the surface-wave spectrum and an internal wave. In the present paper, we have followed a deterministic approach and proved that sideband instability of surface capillary–gravity waves occurs as a result of resonant interaction with an internal wave, all propagating in the same direction.

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Appendix A. Nonlinear terms

$$a = \left[-\zeta\left(\frac{\partial^2\phi}{\partial z^2}\right)_{z=0} + \frac{\partial\zeta}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)_{z=0} + \frac{\partial\zeta}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)_{z=0}\right]$$

$$+ \left[-\frac{1}{2}\zeta^2\left(\frac{\partial^3\phi}{\partial z^3}\right)_{z=0} + \zeta\frac{\partial\zeta}{\partial x}\left(\frac{\partial^2\phi}{\partial x\partial z}\right)_{z=0} + \zeta\frac{\partial\zeta}{\partial y}\left(\frac{\partial^2\phi}{\partial y\partial z}\right)_{z=0}\right],$$

$$b = \left[-\zeta\left(\frac{\partial^2\phi}{\partial t\partial z}\right)_{z=0} - \frac{1}{2}\left(\nabla\phi\right)^2_{z=0}\right] + \left[-\frac{1}{2}\zeta^2\left(\frac{\partial^3\phi}{\partial t\partial z^2}\right)_{z=0} - \zeta\left(\frac{\partial\phi}{\partial x}\right)_{z=0}\left(\frac{\partial^2\phi}{\partial x\partial z}\right)_{z=0}\right.$$

$$- \zeta\left(\frac{\partial\phi}{\partial y}\right)_{z=0}\left(\frac{\partial^2\phi}{\partial y\partial z}\right)_{z=0} - \zeta\left(\frac{\partial\phi}{\partial z}\right)_{z=0}\left(\frac{\partial^2\phi}{\partial z^2}\right)_{z=0}$$

$$\left. + \nu\left\{\zeta_{xx}\zeta_y^2 + \zeta_{yy}\zeta_x^2 - 2\zeta_{xy}\zeta_x\zeta_y - \frac{3}{2}(\zeta_x^2 + \zeta_y^2)(\zeta_{xx} + \zeta_{yy})\right\}\right].$$

Appendix B. Some coefficients and terms in (3.20)

$$d_1 = 2k^2\omega^2 - \frac{3}{2}\nu k^5 + 4k^2\omega^4 f^{-1},$$

$$d_2 = -6k\omega^2 + 9\nu k^4 - 12k\omega^4 f^{-1} - 4k\omega^4 f^{-2}(2gk + 24\nu k^3)$$
$$- c_g\{4k^2\omega - 4k^2\omega^3 f^{-2}(2gk + 20\nu k^3) + 4k^2\omega^3 f^{-1}\},$$

$$d_3 = -(2k\omega^2 - \tfrac{3}{2}\nu k^4 + 4k\omega^4 f^{-1}),$$

$$E = 2\epsilon^{5/2}k\omega\zeta_1\left(\frac{\partial\phi_c}{\partial x_1}\right)_{z=0} + \epsilon^3\left\{ik\zeta_1\left(\frac{\partial^2\phi_c}{\partial t_1\partial x_1}\right)_{z=0} - 2i\omega\zeta_1\left(\frac{\partial^2\phi_c}{\partial x_1^2}\right)_{z=0} - i\omega\zeta_1\left(\frac{\partial^2\phi_c}{\partial y_1^2}\right)_{z=0}\right.$$

$$- 2i(\omega + kc_g)\frac{\partial\zeta_1}{\partial x_1}\left(\frac{\partial\phi_c}{\partial x_1}\right)_{z=0} - 2i\omega\frac{\partial\zeta_1}{\partial y_1}\left(\frac{\partial\phi_c}{\partial y_1}\right)_{z=0} + \frac{\partial\zeta_1}{\partial y_1}\left(\frac{\partial^2\phi_c}{\partial t_1\partial y_1}\right)_{z=0}$$

$$\left. - c_g\frac{\partial^2\zeta_1}{\partial x_1\partial y_1}\left(\frac{\partial\phi_c}{\partial y_1}\right)_{z=0}\right\} + \epsilon^{7/2}\left\{\zeta_1\left(\frac{\partial^3\phi_c}{\partial t_1\partial x_1^2}\right)_{z=0} + \left(1 - \frac{\omega^2}{2kg}\right)\zeta_1\left(\frac{\partial^3\phi_c}{\partial t_1\partial y_1^2}\right)_{z=0}\right.$$

$$- \frac{\omega}{2k}\zeta_1\left(\frac{\partial^3\phi_c}{\partial x_1\partial y_1^2}\right)_{z=0} + \frac{\partial\zeta_1}{\partial x_1}\left(\frac{\partial^2\phi_c}{\partial t_1\partial x_1}\right)_{z=0} + \frac{\partial\zeta_1}{\partial y_1}\left(\frac{\partial^2\phi_c}{\partial t_1\partial y_1}\right)_{z=0} - 2c_g\frac{\partial\zeta_1}{\partial x_1}\left(\frac{\partial^2\phi_c}{\partial x_1^2}\right)_{z=0}$$

$$- c_g\frac{\partial\zeta_1}{\partial x_1}\left(\frac{\partial^2\phi_c}{\partial y_1^2}\right)_{z=0} - \frac{2\omega}{k}\frac{\partial\zeta_1}{\partial y_1}\left(\frac{\partial^2\phi_c}{\partial x_1\partial y_1}\right)_{z=0} - \left(\frac{3\nu k^2 - kc_g^2 + 2\omega c_g}{\omega}\right)\frac{\partial^2\zeta_1}{\partial x_1^2}\left(\frac{\partial\phi_c}{\partial x_1}\right)_{z=0}$$

$$- \left(\frac{g + 3\nu k^2}{2\omega}\right)\frac{\partial^2\zeta_1}{\partial y_1^2}\left(\frac{\partial\phi_c}{\partial x_1}\right)_{z=0} - 2c_g\frac{\partial^2\zeta_1}{\partial x_1\partial y_1}\left(\frac{\partial\phi_c}{\partial y_1}\right)_{z=0}$$

$$\left. + i\left(\frac{3\nu k - c_g^2}{2\omega}\right)\frac{\partial^3\zeta_1}{\partial x_1^2\partial y_1}\left(\frac{\partial\phi_c}{\partial y_1}\right)_{z=0} + i\left(\frac{g + 3\nu k^2}{4k\omega}\right)\frac{\partial^3\zeta_1}{\partial y_1^3}\left(\frac{\partial\phi_c}{\partial y_1}\right)_{z=0}\right\}.$$

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Appendix C. Coefficients of evolution equation (3.22)

$$m = \frac{\nu k^2}{g}, \quad \beta_0 = \frac{1+3m}{2(1+m)}, \quad \beta_1 = -\frac{1-6m-3m^2}{8(1+m)^2}, \quad \beta_2 = \frac{1+3m}{4(1+m)},$$

$$\beta_3 = -\frac{(1-m)(1+6m+m^2)}{16(1+m)^3}, \quad \beta_4 = \frac{3+2m+3m^2}{8(1+m)^2},$$

$$\beta_5 = \frac{5+20m+70m^2-28m^3-3m^4}{128(1+m)^4}, \quad \beta_6 = -\frac{15+35m+5m^2+9m^3}{32(1+m)^3},$$

$$\beta_7 = \frac{3+2m+3m^2}{32(1+m)^2}, \quad \beta_8 = \frac{7+33m+54m^2+210m^3-45m^4-3m^5}{256(1+m)^5},$$

$$\beta_9 = -\frac{35+120m+150m^2+15m^4}{64(1+m)^4}, \quad \beta_{10} = \frac{3(7+15m+5m^2+5m^3)}{64(1+m)^3},$$

$$\lambda_1 = \frac{8+m+2m^2}{4(1+m)(1-2m)}, \quad \lambda_2 = -\frac{3(8-m+9m^2-4m^3-4m^4)}{4(1+m)^2(1-2m)^2},$$

$$\lambda_3 = -\frac{(1-m)(8+m+2m^2)}{8(1+m)^2(1-2m)},$$

$$\mu_1 = -\frac{3+m}{4(1+m)}, \quad \mu_2 = -\frac{1}{2},$$

$$\mu_3 = \frac{1+8m+3m^2}{8(1+m)^2}, \quad \mu_4 = \frac{m}{2(1+m)}, \quad \mu_5 = \frac{1+3m}{8(1+m)},$$

$$\mu_6 = -\frac{m}{4}, \quad \mu_7 = -\frac{1+m}{2}, \quad \mu_8 = \frac{m}{(1+m)^2}, \quad \mu_9 = -\frac{1-m}{2(1+m)}.$$

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