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EXISTENCE OF DIRICHLET INFINITE HARMONIC MEASURES ON THE UNIT DISC

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The primary purpose of this paper is to give an affirmative answer to a problem posed by Ohtsuka [13] whether there exists a p-harmonic measure on the unit disc in the 2-dimensional Euclidean space $\mathbf{R}^2$ with an infinite p-Dirichlet integral for the exponent $1 < p < 2$.

To clarify the meaning of the problem we start by explaining the background of the problem. We say that $\mathscr{A}$ is a strictly monotone elliptic operator on the Euclidean space $\mathbf{R}^d$ of dimension $d \geqq 2$ with exponent $p \in (1, d]$ if $\mathscr{A}$ is a mapping of $\mathbf{R}^d \times \mathbf{R}^d$ to $\mathbf{R}^d$ satisfying the following assumption for some constants $0 < \alpha \leqq \beta < \infty$:

(1) the function $h \mapsto \mathscr{A}(x, h)$ is continuous for almost every fixed $x \in \mathbf{R}^d$, and the function $x \mapsto \mathscr{A}(x, h)$ is measurable for all fixed $h \in \mathbf{R}^d$;

for almost every $x \in \mathbf{R}^d$ and for all $h \in \mathbf{R}^d$

$$\mathscr{A}(x, h) \cdot h \geqq \alpha |h|^p, \tag{2}$$

$$|\mathscr{A}(x, h)| \leqq \beta |h|^{p-1}, \tag{3}$$

$$(\mathscr{A}(x, h_1) - \mathscr{A}(x, h_2)) \cdot (h_1 - h_2) > 0 \tag{4}$$

whenever $h_1 \neq h_2$, and

$$\mathscr{A}(x, \lambda h) = |\lambda|^{p-2} \lambda \mathscr{A}(x, h) \tag{5}$$

for all $\lambda \in \mathbf{R} \setminus \{0\}$. Here $|x|$ indicates the length of a vector $x = (x^1, \ldots, x^d)$ in $\mathbf{R}^d$. The class of all operators $\mathscr{A}$ on $\mathbf{R}^d$ satisfying (1)-(5) with exponent $p \in (1, d]$ will be denoted by $\mathscr{A}_p(\mathbf{R}^d)$.

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Using an $\mathscr{A} \in \mathscr{A}_p(\mathbf{R}^d)$ we consider a quasilinear elliptic equation

$$-\nabla \cdot \mathscr{A}(x, \nabla u(x)) = 0 \tag{6}$$

on $\mathbf{R}^d$. A function u on an open subset U of $\mathbf{R}^d$ is a weak solution of (6) if $u \in$ loc $W_p^1(U)$ and

$$\int_U \mathscr{A}(x, \nabla u(x)) \cdot \nabla \varphi(x)\, dx = 0$$

for every $\varphi \in C_0^\infty(U)$ where $W_p^1(U)$ is the Sobolev space on U consisting of functions $f \in L_p(U) = L_p(U; \mathbf{R})$ with distributional gradients $\nabla f \in L_p(U) = L_p(U; \mathbf{R}^d)$ and $dx = dx^1 \cdots dx^d$. A weak solution u of (6) (possibly modified on a set of zero measure dx) is actually continuous. We say that a function u is *$\mathscr{A}$-harmonic* on U if u is a continuous weak solution of (6) on U. We denote by $H_{\mathscr{A}}(U)$ the class of all $\mathscr{A}$-harmonic functions on U. The simplest and the most typical operotor $\mathscr{A}$ in $\mathscr{A}_p(\mathbf{R}^d)$ is the p-Laplacian $\mathscr{A}(x, h) = |h|^{p-2}h$ so that the corresponding elliptic equation is the p-Laplace equation

$$-\nabla \cdot (|\nabla u(x)|^{p-2} \nabla u(x)) = 0. \tag{7}$$

In this case we use the term *p-harmonic* instead of $\mathscr{A}$-harmonic and the notation $H_p(U)$ in place of $H_{\mathscr{A}}(U)$.

The greatest $\mathscr{A}$-harmonic minorant $u \wedge v$ on U, if it exists, of two $\mathscr{A}$-harmonic functions u and v on U is the $\mathscr{A}$-harmonic function $u \wedge v$ on U characterized by the following two conditions: (i) $u \wedge v \leq u$ and $u \wedge v \leq v$ on U; (ii) if there is an $\mathscr{A}$-harmonic function h on U such that $h \leq u$ and $h \leqq v$ on U, then $h \leqq u \wedge v$ on U. A function w is said to be an *$\mathscr{A}$-harmonic measure* on U in the sense of Heins [3] if w is $\mathscr{A}$-harmonic on U and satisfies

$$w \wedge (1 - w) = 0 \tag{8}$$

on U. An $\mathscr{A}$-harmonic measure always satisfies $0 \leqq w \leqq 1$ on U; $w \equiv 0$ or $w \equiv 1$ are $\mathscr{A}$-harmonic measures on U; when U is a region, an $\mathscr{A}$-harmonic measure w on U is nonconstant if and only if $0 < w < 1$ on U.

Our main concern in this paper is the *p-Dirichlet integral*

$$D_p(w) = D_p(w; B^d) = \int_{B^d} |\nabla w(x)|^p dx \leqq \infty$$

of each $\mathscr{A}$-harmonic measure w on the unit ball $B^d = \{x \in \mathbf{R}^d; |x| < 1\}$ with $\mathscr{A} \in \mathscr{A}_p(\mathbf{R}^d)$. We say that w is *p-Dirichlet finite* (*infinite*, resp.) if $D_p(w) < \infty$ ($D_p(w) = \infty$, resp.). We have the following result:

9. THEOREM. *If $2 \leqq p \leqq d$, then every noncontsant $\mathscr{A}$-harmonic measure on the unit ball B^d is p-Dirichlet infinite for every $\mathscr{A}$ in $\mathscr{A}_p(\mathbf{R}^d)$.*

We say that a subdivision $\delta_0 \cup \delta_1$ of ∂B^d gives rise to an electric condenser $(B^d; \delta_0, \delta_1)$ surrounded by two electrodes δ_0 and δ_1 if the unit potential difference can be produced between δ_0 and δ_1 by putting a charge of finite energy on δ_1 when δ_0 is grounded. The intuitive meaning of the above result is that B^d cannot be made to an electric condenser no matter how we decompose the boundary ∂B^d of B^d into two parts. The above result in its present final form was obtained and proved in [11]. The result in the special case of $p = 2$ and the classical Laplace operator $\mathscr{A}(x, h) = h$ was proved in [9] based on a different view point. If $p = d = 2$ and $\mathscr{A}(x, h) = h$, then the above result has been known in the frame of the theory of functions and its proof is found in various sources (cf. e.g. [8], [13], etc.). If $p = 2$ and $\mathscr{A}(x, h) = h$, then the above result is the one in the linear potential theory. From this view point we remark that (6) can be nonlinear for $p = 2$ and even for the borderline conformal case $p = d = 2$ (see Appendix at the end of this paper).

In contrast with the case $2 \leqq p \leqq d$, we have proved the following result in the same paper [11] cited above:

10. THEOREM. *If $1 < p < 2$, then there exist nonconstant p-Dirichlet finite $\mathscr{A}$-harmonic measures on the unit ball B^d for every $\mathscr{A}$ in $\mathscr{A}_p(\mathbf{R}^d)$.*

We turn to the final question in the case $1 < p < 2$ whether there are p-Dirichlet infinite $\mathscr{A}$-harmonic measures on the unit ball B^d for every $\mathscr{A}$ in $\mathscr{A}_p(\mathbf{R}^d)$, which is the main theme of this paper. For a technical reason we restrict ourselves to the case of the dimension $d = 2$ in the remainder of this paper. We view $\mathbf{R}^2$ also as the complex plane by identifying the point (x^1, x^2) in $\mathbf{R}^2$ with the complex number $x = x^1 + ix^2$ $(i = \sqrt{-1})$. For simplicity we denote by Δ the unit disc in $\mathbf{R}^2 : \Delta = B^2 = \{x \in \mathbf{R}^2 : |x| < 1\}$.

Take two sequences $(a_n) = (a_n : 1 \leqq n < N + 1)$ and $(b_n) = (b_n : 1 \leqq n < N + 1)$ of real numbers a_n and b_n such that

$$0 < a_n < b_n < a_{n+1} < b_{n+1} < \pi \quad (1 \leqq n < N) \tag{11}$$

so that (a_n) and (b_n) are finite sequences of N terms if $1 \leqq N < \infty$ and infinite sequences if $N = \infty$. With these two sequences (a_n) and (b_n) we associate the sequence $(A_n) = (A_n : 1 \leqq n < N + 1)$ of *main arcs* A_n in $\partial\Delta = \{x \in \mathbf{R}^2 : |x| = 1\}$ given by

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$$A_n = \{e^{i\theta} : a_n < \theta < b_n\} \quad (1 \leqq n < N+1)$$

and the sequence $(B_n) = (B_n : 1 \leqq n < N+1)$ of *subsidiary arcs* B_n in $\partial\Delta$ given by

$$B_n = \{e^{i\theta} : b_n < \theta < a_{n+1}\} \quad (1 \leqq n < N).$$

Finally we consider the open subset A in $\partial\Delta$ associated with sequences (a_n) and (b_n) given by

$$A = A((a_n), (b_n)) = \bigcup_{n=1}^{N} A_n.$$

The function $\omega(A, \Delta; \mathscr{A})$ on Δ given by

(12) $$\omega(A, \Delta; \mathscr{A})(x) = \sup\{h(x) : h \in C(\bar{\Delta}) \cap H_{\mathscr{A}}(\Delta),\ h|\bar{\Delta} \leqq 1,\ h|(\partial\Delta \setminus A) \leqq 0\}$$

for $x \in \Delta$ is referred to as the *$\mathscr{A}$-harmonic measure of A with respect to Δ* for $\mathscr{A} \in \mathscr{A}_p(\mathbf{R}^2)$ with $1 < p \leqq 2$. In this case of an open set A in $\partial\Delta$ the definition of $\omega(A, \Delta; \mathscr{A})$ in (12) coincides with the one given by Martio ([4], [2, Chap. 11]). We will see later in **44** that $\omega(A, \Delta; \mathscr{A})$ is actually an $\mathscr{A}$-harmonic measure on Δ in the sense of Heins characterized by (8).

If $1 < p < 2$, $\mathscr{A}(x, h) = |h|^{p-2}h$, and $N < \infty$, i.e. A is the union of a finite number of mutually disjoint open arcs in $\partial\Delta$, then we know that the p-harmonic measure $\omega(A, \Delta; \mathscr{A})$ of A with respect to Δ is p-Dirichlet finite (Ohtsuka [13], [10]; also see Theorem 14 below). In view of this fact one might feel that every p-harmonic measure on Δ is p-Dirichlet finite for every $1 < p < 2$. Thus we are naturally led to ask the following question originally raised by Ohtsuka [13, Chap. VIII] in terms of extremal distances in an equivalent to but superfacially different from our present setting:

13. OHTSUKA'S PROBLEM. *Does there exist a p-Dirichlet infinite p-harmonic measure on Δ for each $1 < p < 2$? Or more generally, does there exist a p-Dirichlet infinite $\mathscr{A}$-harmonic measure on Δ for every $\mathscr{A} \in \mathscr{A}_p(\mathbf{R}^2)$ with each $1 < p < 2$?*

The purpose of this paper is to give an affirmative answer to the above problem of Ohtsuka by proving the following result.

14. MAIN THEOREM. *If $N < \infty$ or if $N = \infty$ and either the sequence $(|A_n| : 1 \leqq n < \infty)$ or $(|B_n| : 1 \leqq n < \infty)$ converges to zero so rapidly as to satisfy the condition*

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(15) $$\min\left(\sum_{n=1}^{\infty} |A_n|^{2-p},\ \sum_{n=1}^{\infty} |B_n|^{2-p}\right) < \infty,$$

where $|A_n|$ *denotes the length of* A_n, *then the* $\mathscr{A}$*-harmonic measure* $\omega(A, \Delta; \mathscr{A})$ *is* p*-Dirichlet finite for every* $\mathscr{A}$ *in* $\mathscr{A}_p(\mathbf{R}^2)$ *with* $1 < p < 2$. *If the sequences* $(|A_n| : 1 \leqq n < \infty)$ *and* $(|B_n| : 1 \leqq n < \infty)$ *converge to zero so slowly as to satisfy the condition*

(16) $$\sum_{n=1}^{\infty} \min(|A_n|^{2-p}, |B_n|^{2-p}) = \infty,$$

then the $\mathscr{A}$*-harmonic measure* $\omega(A, \Delta; \mathscr{A})$ *is* p*-Dirichlet infinite for every* $\mathscr{A}$ *in* $\mathscr{A}_p(\mathbf{R}^2)$ *with each* $1 < p < 2$.

The proof of this theorem will be given later in **51** after a series of preparations starting from **22**. The latter half of the above result takes the following more applicable form.

17. COROLLARY. *If the sequences* $(|A_n| : 1 \leqq n < \infty)$ *and* $(|B_n| : 1 \leqq n < \infty)$ *satisfy the condition*

(18) $$\liminf_{n\to\infty} |B_n|/|A_n| > 0 \quad (\liminf_{n\to\infty} |A_n|/|B_n| > 0,\ \textit{resp.})$$

and also the condition

(19) $$\sum_{n=1}^{\infty} |A_n|^{2-p} = \infty \quad \left(\sum_{n=1}^{\infty} |B_n|^{2-p} = \infty,\ \textit{resp.}\right),$$

then the $\mathscr{A}$*-harmonic measure* $\omega(A, \Delta; \mathscr{A})$ *is* p*-Dirichlet infinite for every* $\mathscr{A}$ *in* $\mathscr{A}_p(\mathbf{R}^2)$ *with each* $1 < p < 2$.

Proof. Condition (18) assures the existence of a constant $C > 0$ such that

$$|B_n| \geqq C|A_n| \quad (|A_n| \geqq C|B_n|,\ \text{resp.})\ (n = 1,2,\cdots).$$

Then we see that

$$\min(|A_n|^{2-p}, |B_n|^{2-p}) \geqq \min(|A_n|^{2-p}, C^{2-p}|A_n|^{2-p})$$
$$= \min(1, C^{2-p})|A_n|^{2-p}$$
$$(\min(|A_n|^{2-p}, |B_n|^{2-p}) \geqq \min(C^{2-p}|B_n|^{2-p}, |B_n|^{2-p})$$
$$= \min(1, C^{2-p})|B_n|^{2-p},\ \text{resp.}).$$

Hence (19) implies (16) and thus Theorem 14 yields the above conclusion. □

We are now able to give an affirmative answer to Problem 13 as an applicaton of Corollary 17 by giving the following example.

20. EXAMPLE. *Choose sequences* $(a_n : 1 \leqq n < \infty)$ *and* $(b_n : 1 \leqq n < \infty)$ *so as to satisfy the condition*

(21) $$a_{n+1} - b_n = b_n - a_n = n^{-1/(2-p)}$$

for sufficiently large n. *Then the* $\mathscr{A}$*-harmonic measure* $\omega(A, \Delta; \mathscr{A})$ *is* p*-Dirichlet infinite for every* $\mathscr{A}$ *in* $\mathscr{A}_p(\mathbf{R}^2)$ *with each* $1 < p < 2$.

Proof. Since $0 < 2 - p < 1$, the series $\sum_{n\geqq 1} n^{-1/(2-p)} < \infty$ and therefore we can choose sequences (a_n) and (b_n) satisfying conditions (11) and (21). Then $|A_n| = |B_n| = n^{-1/(2-p)}$ for sufficiently large n and hence (18) and (19) are trivially satisfied. Thus Corollary 17 assures that the corresponding $\mathscr{A}$-harmonic measure $\omega(A, \Delta; \mathscr{A})$ is p-Dirichlet infinite for every $\mathscr{A}$ in $\mathscr{A}_p(\mathbf{R}^2)$ with each $1 < p < 2$. □

22. Trace

For simplicity we denote by $\Gamma = \partial\Delta$ the unit circle $\{x \in \mathbf{R}^2 : |x| = 1\}$. The Sobolev space $W_p^1(G)$ $(1 < p \leqq 2)$ is a Banach space equipped with the norm

$$\| f ; W_p^1(G) \| = \| f ; L_p(G) \| + \| \nabla f ; L_p(G) \|,$$

where G is an open set in $\mathbf{R}^2$. The Sobolev null space $W_{p,0}^1(G)$ is the closure of $C_0^\infty(G)$ in $W_p^1(G)$ with respect to the above norm.

There exists a unique continuous linear operator γ of $W_p^1(\Delta)$ into $L_p(\Gamma)$ such that $\gamma f = f | \Gamma$ for every f in $C(\bar{\Delta}) \cap W_p^1(\Delta)$. The function γf defined a.e. on Γ and belonging to $L_p(\Gamma)$ is referred to as the *trace* on Γ of f in $W_p^1(G)$. It is seen that the expression

(23) $$(\gamma f)(\zeta) = \lim_{r \uparrow 1} f(r\zeta)$$

holds for a.e. ζ in Γ (cf. e.g. [6, p.47]).

Concerning the kernel $\operatorname{Ker}\gamma = \gamma^{-1}(0)$ and the image $\operatorname{Im}\gamma = \gamma(W_p^1(\Delta))$ of γ we have the following fundamental results. First, $\operatorname{Ker}\gamma$ characterizes the Sobolev null space (cf. e.g. [7, p.187]):

(24) $$W_{p,0}^1(\Delta) = \operatorname{Ker}\gamma = \{f \in W_p^1(\Delta) : \gamma f = 0\}.$$

Second, we denote $\operatorname{Im}\gamma = \gamma(W_p^1(\Delta))$ by $\Lambda_p(\Gamma)$. It is seen that the space $\Lambda_p(\Gamma)$ forms a Banach space under the norm

$$(25)\qquad \| \varphi ; \Lambda_p(\Gamma) \| = \| \varphi ; L_p(\Gamma) \| + \left(\int\int_{\Gamma\times\Gamma} \frac{|\varphi(\zeta) - \varphi(\eta)|^p}{|\zeta - \eta|^p} ds_\zeta ds_\eta \right)^{1/p}$$

where ds is the line element on Γ. The theorem of Gagliardo [1] assures the existence of a constant $C \geqq 1$ such that

$$(26)\qquad C^{-1} \| \varphi ; \Lambda_p(\Gamma) \| \leqq \inf_{\gamma f = \varphi} \| f ; W_p^1(\Delta) \| \leqq C \| \varphi ; \Lambda_p(\Gamma) \|$$

for every φ in $\Lambda_p(\Gamma)$. The quantity $\| \varphi ; \Lambda_p(\Gamma) \|$ will be referred to as the *Gagliardo norm* of φ in this paper.

Hereafter we sometimes use the same letter C to denote positive constants which may differ from each other from line to line and even in the same line.

27. Dirichlet problem

Let G be a bounded region in $\mathbf{R}^2$. We will mainly consider the case $G = \Delta$ but G is supposed to be a general bounded region for a while. For any f in $W_p^1(G)$ there exists a *unique* u in the space $H_{\mathscr{A}}(G) \cap W_p^1(G)$ such that $u - f$ belongs to $W_{p,0}^1(G)$ (cf. Maz'ya [5]). This fact can be reformulated as the Maz'ya decomposition of $W_p^1(G)$:

$$(28)\qquad W_p^1(G) = (H_{\mathscr{A}}(G) \cap W_p^1(G)) \oplus W_{p,0}^1(G),$$

i.e. any f in $W_p^1(G)$ can be expressed as the sum of the $\mathscr{A}$-harmonic part u in $H_{\mathscr{A}}(G) \cap W_p^1(G)$ and the "potential part" g in $W_{p,0}^1(G) : f = u + g$. We denote by $\pi_{\mathscr{A}}^G$ the projection operator of $W_p^1(G)$ to $H_{\mathscr{A}}(G) \cap W_p^1(G)$ determined by $\pi_{\mathscr{A}}^G f = u$. We say that G is *$\mathscr{A}$-regular* if

$$(29)\qquad \lim_{x\in G, x\to y} (\pi_{\mathscr{A}}^G f)(x) = f(y)$$

for any f in $C(\bar{G}) \cap W_p^1(G)$ and for every y in ∂G. If G is bounded by a finite number of mutually disjoint smooth Jordan curves, then G is $\mathscr{A}$-regular (cf. [5]). The disc Δ is the most typical example of $\mathscr{A}$-regular regions.

We also use the following extremal property of $\pi_{\mathscr{A}}^G$: the quasi Dirichlet principle is valid in the sense that $\pi_{\mathscr{A}}^G f$ quasiminimizes the p-Dirichlet integral:

$$(30)\qquad \int_G |\nabla (\pi_{\mathscr{A}}^G f)(x)|^p dx \leqq (\beta/\alpha)^p \int_G |\nabla f(x)|^p dx.$$

In fact, since $u=\pi^G_{\mathscr{A}}f$ is a weak solution of (6) and $u-f$ belongs to $W^1_{p,0}(G)$ in which $C^\infty_0(G)$ is $\|\cdot;W^1_p(G)\|$-dense, we have

$$\int_G \mathscr{A}(x,\nabla u(x))\cdot\nabla(u-f)(x)\,dx=0.$$

By (2), (3) and the Hölder inequality we have

$$\begin{aligned}\alpha\int_G|\nabla u(x)|^p dx &\leqq \int_G\mathscr{A}(x,\nabla u(x))\cdot\nabla u(x)\,dx=\int_G\mathscr{A}(x,\nabla u(x))\cdot\nabla f(x)\,dx\\ &\leqq\Big(\int_G|\mathscr{A}(x,\nabla u(x))|^{p/(p-1)}dx\Big)^{(p-1)/p}\cdot\Big(\int_G|\nabla f(x)|^p dx\Big)^{1/p}\\ &\leqq\beta\Big(\int_G|\nabla u(x))|^p dx\Big)^{(p-1)/p}\cdot\Big(\int_G|\nabla f(x)|^p dx\Big)^{1/p},\end{aligned}$$

by which we can conclude the inequality (30).

We now restrict ourselves to the case $G=\Delta$. We use the abbreviation $\pi=\pi_{\mathscr{A}}=\pi^\Delta_{\mathscr{A}}$. We say that an f in $W^1_p(G)$ has an essential limit α at ξ in $\Gamma=\partial\Delta$,

$$\alpha=\operatorname*{ess\,lim}_{x\in\Delta,x\to\xi}f(x)$$

in notation, if

$$\lim_{\varepsilon\downarrow 0}\|f-\alpha;L_\infty(\Delta(\xi,\varepsilon)\cap\Delta)\|=0$$

where $\Delta(\xi,\varepsilon)$ is the disc of radius $\varepsilon>0$ centered at ξ. As a localized version of (29) we have

$$\lim_{x\in\Delta,x\to\xi}(\pi f)(x)=\operatorname*{ess\,lim}_{x\in\Delta,x\to\xi}f(x)$$

at a point ξ in Γ for every f in $L_\infty(\Delta)\cap W^1_p(\Delta)$ for which the right hand side of the above exists at a ξ in Γ (cf. [12]). Although the operator $\pi=\pi_{\mathscr{A}}=\pi^\Delta_{\mathscr{A}}$ is homogeneous but not linear, we see that π is *monotone* (cf. [11]), i.e. $f_1\geqq f_2$ a.e. on Δ for any f_1 and f_2 in $W^1_p(\Delta)$, then $\pi f_1\geqq\pi f_2$ on Δ.

In view of the relation (24) and the uniqueness of the Maz'ya decomposition (28) we can define the operator

$$\tau=\pi\circ\gamma^{-1}:\Lambda_p(\Gamma)\to H_{\mathscr{A}}(\Delta)\cap W^1_p(\Delta).$$

Clearly the operator $\tau=\tau_{\mathscr{A}}=\tau^\Delta_{\mathscr{A}}$ is *bijective*. Moreover we have the following result.

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31. PROPOSITION. *The operator τ is monotone, i.e. if $\varphi_1 \geqq \varphi_2$ a.e. on Γ for any φ_1 and φ_2 in $\Lambda_p(\Gamma)$, then $\tau\varphi_1 \geqq \tau\varphi_2$ everywhere on Δ.*

Proof. Choose an arbitrary g_i in $W_p^1(\Delta)$ with $\gamma g_i = \varphi_i$ $(i = 1,2)$. We denote by $F \cup G$ the function given by $(F \cup G)(x) = \max(F(x), G(x))$ for any two functions F and G. Then $(g_1 - g_2) \cup 0$ belongs to $W_p^1(\Delta)$ by the lattice property of $W_p^1(\Delta)$. By (23) we see that

$$\gamma((g_1 - g_2) \cup 0) = (\gamma(g_1 - g_2)) \cup 0 = (\varphi_1 - g_2) \cup 0 = \varphi_1 - \varphi_2.$$

If we set $f_2 = g_2$ and $f_1 = g_2 + (g_1 - g_2) \cup 0$, then $\gamma f_2 = \gamma g_2 = \varphi_2$ and

$$\gamma f_1 = \gamma g_2 + \gamma((g_1 - g_2) \cup 0) = \varphi_2 + (\varphi_1 - g_2) = \varphi_1.$$

Then $\tau\varphi_1 = \pi f_1$, $\tau\varphi_2 = \pi f_2$ and $f_1 \geqq f_2$ on Δ imply that $\tau\varphi_1 \geqq \tau\varphi_2$ on Δ by the monotoneity of π. □

Beside the defining boundary behavior $\gamma(\tau\varphi) = \varphi$ of $\tau\varphi$, we have the following more precise boundary behavior of $\tau\varphi$ if an additional condition is imposed upon φ:

32. LEMMA. *If $\varphi \in L_\infty(\Gamma) \cap \Lambda_p(\Gamma)$ is continuous at a point $\xi \in \Gamma$ in the sense that $\operatorname{ess\,lim}_{\eta\in\Gamma,\eta\to\xi}\varphi(\eta) = \varphi(\xi)$, then $\tau\varphi$ has a boundary value $\varphi(\xi)$ at ξ.*

Proof. We only have to show that $\lim_{x\in\Delta,x\to\xi}(\tau\varphi)(x) = \varphi(\xi)$. Since $\tau(\varphi - \varphi(\xi)) = \tau\varphi - \varphi(\xi)$, we may suppose $\varphi(\xi) = \operatorname{ess\,lim}_{\eta\in\Gamma,\eta\to\xi}\varphi(\eta) = 0$ to show the above identity. Let $|\varphi| \leqq K$ a.e. on Γ for a positive constant K and $\rho(x) = |x - \xi|$ on $\mathbf{R}^2$. Clearly ρ belongs to the class $C(\bar{\Delta}) \cap W_p^1(G)$ and $\tau(\rho\,|\,\Gamma) = \pi\rho$, or roughly $\tau\rho = \pi\rho$. Hence by (29) we have

$$\lim_{x\in\Delta,x\to\xi}(\tau\rho)(x) = 0.$$

For any $\varepsilon > 0$ there is a $\delta > 0$ such that $|\varphi(\eta)| < \varepsilon$ for a.e. η in $\Delta(\xi, \delta) \cap \Gamma$. Since $(K/\delta)\rho \geqq K$ for every η in $\Gamma \setminus \Delta(\xi, \delta)$, we see that

$$-\frac{K}{\delta}\rho(\eta) - \varepsilon \leqq \varphi(\eta) \leqq \frac{K}{\delta}\rho(\eta) + \varepsilon$$

a.e. on Γ. By Proposition 31, we have

$$-\frac{K}{\delta}(\tau\rho)(x) - \varepsilon \leqq (\tau\varphi)(x) \leqq \frac{K}{\delta}(\tau\rho)(x) + \varepsilon \quad (x \in \Delta).$$

On letting x in Δ tend to ξ, we see by $(\tau\rho)(x)\to 0$ that

$$-\varepsilon \leqq \liminf_{x\in\Delta, x\to\xi} (\tau\varphi)(x) \leqq \limsup_{x\in\Delta, x\to\xi} (\tau\varphi)(x) \leqq \varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we finally conclude the required identity $\lim_{x\in\Delta, x\to\xi}(\tau\varphi)(x)=0$. □

33. Estimate of Gagliardo norms

For two measurable subsets X and Y in Γ and mostly for open or closed subarcs X and Y in Γ we consider the set function

$$S(X, Y)=\int\int_{X\times Y}|\xi-\eta|^{-p}ds_\xi ds_\eta \tag{34}$$

where ds is the arc element on Γ. The following elementary properties of S are easily checked and will be used without making any further mention of them: S is symmetric, i.e. $S(X, Y)=S(Y, X)$; S is rotationally invariant, i.e. $S(e^{i\theta}X, e^{i\theta}Y)=S(X, Y)$ where $e^{i\theta}X=\{e^{i\theta}\xi:\xi\in X\}$; S is additive, i.e. $X=\cup_{j=1}^n X_j$ is a finite disjoint union, then

$$S(\bigcup_{j=1}^n X_j, Y)=\sum_{j=1}^n S(X_j, Y);$$

S is increasing, i.e. if $X\subset X'$ and $Y\subset Y'$, then $S(X, Y)\leqq S(X', Y')$.

We denote by Γ^+ the upper half circle $\{e^{i\theta}:0\leqq\theta\leqq\pi\}$. For a measurable subset X and mostly for open or closed subarc X in Γ we set

$$X^\wedge=\{x\in[0, 2\pi):e^{ix}\in X\}$$

which is a measurable subset of the real line and actually the interval $[0,2\pi)$. We consider the auxiliary set function

$$T(X, Y)=\int\int_{X^\wedge\times Y^\wedge}|x-y|^{-p}\,dxdy \tag{35}$$

which is comparable to (34) for X and Y in Γ^+:

$$T(X, Y)\leqq S(X, Y)\leqq(\pi/2)^p T(X, Y)\quad(X, Y\subset\Gamma^+). \tag{36}$$

To see this relation we observe that

$$S(X, Y)=\int\int_{X\times Y}|\xi-\eta|^{-p}\,ds_\xi ds_\eta=\int\int_{X^\wedge\times Y^\wedge}|e^{ix}-e^{iy}|^{-p}\,dxdy.$$

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Replacing $|e^{ix}-e^{iy}|$ in the above by $|x-y|$ or $(2/\pi)|x-y|$ based on the following inequalities

$$(2/\pi)|x-y| \leqq |e^{ix}-e^{iy}| \leqq |x-y| \quad (x, y \in [0, \pi]),$$

we deduce the required inequalities (36).

We choose arbitrary open or closed arcs I and J in Γ^+ such that $(\mathrm{int}\, I) \cap (\mathrm{int}\, J) = \emptyset$ where $\mathrm{int}\, I$ is the interior of I considered in Γ. We denote by $|I|$ the length of the arc I. Let $\rho = \rho(I, J)$ be the distance between I and J considered in the Riemannian metric in Γ. We then deduce the following fundamental relation:

37. IDENTITY. *The auxiliary set function $T(I, J)$ is given by*

$$T(I, J) = C_p\{(|I|+\rho)^{2-p} + (|J|+\rho)^{2-p} - (|I|+|J|+\rho)^{2-p} - \rho^{2-p}\}$$

$(1 < p < 2)$ *where* $C_p = 1/(p-1)(2-p)$.

Proof. Let the closures of intervals $I^\wedge$ and $J^\wedge$ be $[a, b]$ and $[c, d]$, respectively. Since $T(I, J) = T(J, I)$, we may assume that $0 \leqq a < b \leqq c < d \leqq \pi$. Then

$$T(I, J) = \int\int_{I^\wedge \times J^\wedge} |x-y|^{-p}\, dxdy = \int_a^b \Big(\int_c^d (y-x)^{-p}\, dy\Big) dx$$

$$= (p-1)^{-1} \int_a^b \{(c-x)^{1-p} - (d-x)^{1-p}\} dx$$

$$= \{(p-1)(2-p)\}^{-1} \cdot \{(c-a)^{2-p} - (c-b)^{2-p} + (d-b)^{2-p} - (d-a)^{2-p}\}.$$

Since $c-a = |I| + \rho$, $d-b = |J| + \rho$, $d-a = |I| + |J| + \rho$ and $c-b = \rho$, we deduce the identity 37. □

For an arbitrary open or closed arc I in Γ we denote by I^c the complement of I with respect to Γ so that $I^c = \Gamma \setminus I$. Then we have the following relation:

38. ESTIMATE. $S(I, I^C) \leqq (2^{p-1} + 3^{p-1} C_p)\pi^p |I|^{2-p} \quad (1 < p < 2)$.

Proof. Let $I = \cup_{j=1}^6 I_j$ be the decomposition of I into 6 arcs I_j such that $(\mathrm{int}\, I_j) \cap (\mathrm{int}\, I_k) = \emptyset$ and $|I_j| = |I_k|$ for $j, k = 1, 2, \cdots, 6$ with $j \neq k$. Take the arc J in Γ^+ such that the midpoint of J is $i = (0, 1)$ and $|J| = |I_j| = |I|/6$ for $j = 1, 2, \cdots, 6$. We denote by J_1 and J_2 the two arcs which are components of $\Gamma^+ \setminus J$ and set $J_3 = \Gamma^- = \{e^{i\theta} : \pi \leqq \theta \leqq 2\pi\}$. We estimate $S(I, I^c)$ as follows:

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$$S(I, I^c) = S(\bigcup_{j=1}^{6} I_j, I^c) = \sum_{j=1}^{6} S(I_j, I^c) \leqq \sum_{j=1}^{6} S(I_j, I_j^c)$$
$$= 6S(J, J^c) = 6S(J, \bigcup_{j=1}^{3} J_j) = 6 \sum_{j=1}^{3} S(J, J_j).$$

By (36) and (37) we see that

$$S(J, J_j) \leqq (\pi/2)^p T(J, J_j)$$
$$= (\pi/2)^p C_p \{|J|^{2-p} + |J_j|^{2-p} - (|J| + |J_j|)^{2-p}\}$$
$$\leqq (\pi/2)^p C_p |J|^{2-p} \quad (j = 1,2)$$

because $\rho(J, J_j) = 0$. Since $J^\wedge \subset [\pi/3, 2\pi/3]$ and $J_3 = \Gamma^-$, we see that $|e^{ix} - e^{iy}| \geqq 1$ for $e^{ix} \in J$ and $e^{iy} \in J_3$. Therefore

$$S(J, J_3) = \int\int_{J^\wedge \times J_3^\wedge} |e^{ix} - e^{iy}|^{-p} dxdy \leqq \int\int_{J^\wedge \times J_3^\wedge} dxdy$$
$$= |J| |J_3| = \pi |J| \leqq \pi(\pi/3)^{p-1} |J|^{2-p}$$

in view of $|J| \leqq \pi/3$. Hence we have

$$S(I, I^c) \leqq 6\{2(\pi/2)^p C_p |J|^{2-p} + (\pi^p/3^{p-1}) |J|^{2-p}\}$$
$$= 6\pi^p(2^{1-p}C_p + 3^{1-p}) |J|^{2-p} = 6\pi^p(2^{1-p}C_p + 3^{1-p})(|I|/6)^{2-p}$$
$$= 6^{p-1}(2^{1-p}C_p + 3^{1-p})\pi^p |I|^{2-p} = (2^{p-1} + 3^{p-1}C_p)\pi^p |I|^{2-p}. \qquad \square$$

For any set E in Γ we denote by 1_E the characteristic function of E on Γ so that $1_E(\xi) = 1$ for $\xi \in E$ and $1_E(\xi) = 0$ for $\xi \in \Gamma \setminus E$. We then have

39. PROPOSITION. *For any exponent $p \in (1,2)$ there exists a positive constant C depending only on p such that*

(40) $$\| 1_I ; \Lambda_p(\Gamma) \| \leqq |I|^{1/p} + C |I|^{(2-p)/p}$$

for every open or closed subarc I of Γ.

Proof. Recall that

$$\| 1_I ; \Lambda_p(\Gamma) \| = \| 1_I ; L_p(\Gamma) \| + \left(\int\int_{\Gamma \times \Gamma} \frac{|1_I(\xi) - 1_I(\eta)|^p}{|\xi - \eta|^p} ds_\xi ds_\eta \right)^{1/p}$$
$$= |I|^{1/p} + (S(I, I^c) + S(I^c, I))^{1/p}.$$

By the estimate 38 we see that

$$\| 1_I ; \Lambda_p(\Gamma) \| \leqq |I|^{1/p} + \{2(2^{p-1} + 3^{p-1}C_p)\pi^p\}^{1/p} |I|^{(2-p)/p}.$$

Hence it suffices to choose $C = \{2(2^{p-1} + 3^{p-1}C_p)\}^{1/p}\pi$. □

41. $\mathscr{A}$-harmonic measures of boundary sets

In this section we assume that $1 < p < 2$ and study the $\mathscr{A}$-harmonic measure $\omega(A, \Delta ; \mathscr{A})$ of the boundary set

$$A = A((a_n), (b_n)) = \bigcup_{n=1}^{N} A_n$$

where $A_n = \{e^{i\theta} : a_n < \theta < b_n\}$ $(1 \leqq n < N + 1 ; N \leqq \infty)$ is introduced in (12). Since 0 is a competing function in the definition (12), we see that $\omega(A, \Delta ; \mathscr{A}) \geqq 0$ on Δ. Since any competing function h in (12) satisfies $h \leqq 1$ on Δ, we see that $\omega(A, \Delta ; \mathscr{A}) \leqq 1$ on Δ. Thus we have

$$0 \leqq \omega(A, \Delta ; \mathscr{A})(x) \leqq 1 \quad (x \in \Delta). \tag{42}$$

As for the boundary behavior of $\omega(A, \Delta ; \mathscr{A})$ we have the following relation:

$$\begin{cases} \lim\limits_{x\in\Delta, x\to\xi} \omega(A, \Delta ; \mathscr{A})(x) = 1 & (\xi \in A), \\ \lim\limits_{x\in\Delta, x\to\xi} \omega(A, \Delta ; \mathscr{A})(x) = 0 & (\xi \in \Gamma \backslash \bar{A}). \end{cases} \tag{43}$$

In fact, suppose first that $\xi \in A$. There is a function $\varphi \in C_0^\infty(\mathbf{R}^2)$ such that $0 \leqq \varphi \leqq 1$ on $\mathbf{R}^2$, $\varphi(\xi) = 1$ and $\varphi = 0$ on $\Gamma \backslash A$. Since φ belongs to $C(\bar{\Delta}) \cap W_p^1(\Delta)$, $h = \pi_{\mathscr{A}}^{\Delta}\varphi$ is a competing function in (12) and we have

$$h(x) \leqq \omega(A, \Delta ; \mathscr{A})(x) \leqq 1 \quad (x \in \Delta).$$

Thus $h(x) \to 1$ $(x \in \Delta, x \to \xi)$ implies the first relation in (43). Next we assume $\xi \in \Gamma \backslash \bar{A}$. There is a function ψ in $C_0^\infty(\mathbf{R}^2)$ such that $0 \leqq \psi \leqq 1$ on $\mathbf{R}^2$, $\psi(\xi) = 1$ and $\psi = 0$ on $\bar{A}$. Then $\varphi = 1 - \psi$ belongs to $C(\bar{\Delta}) \cap W_p^1(\Delta)$ and $g = \pi_{\mathscr{A}}^{\Delta}\varphi$ is in $C(\bar{\Delta}) \cap H_{\mathscr{A}}(\Delta)$ such that $0 \leqq g \leqq 1$ on $\bar{\Delta}$, $g(\xi) = 0$ and $g = 1$ on $\bar{A}$. Let h be any competing function in (12). Since $h \leqq g$ on Γ, the comparison principle (cf. e.g. [2, p. 183]) implies that $h \leqq g$ on Δ. Thus

$$0 \leqq \omega(A, \Delta ; \mathscr{A})(x) \leqq g(x) \quad (x \in \Delta).$$

That $g(x) \to 0$ $(x \in \Delta, x \to \xi)$ implies the second relation in (43).

We are now ready to prove the following result announced in the introductory part. Only here we assume that $1 < p \leqq 2$.

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44. Proposition. *The function* $\omega(A, \Delta; \mathscr{A})$ *is an* $\mathscr{A}$*-harmonic measure in the sense of Heins.*

Proof. We denote by K the closure of the set consisting of points $e^{\iota a_n}$ and $e^{\iota b_n}$ $(1 \leqq n < N+1; N \leqq \infty)$. We can find a sequence $(K_m)_{1 \leqq m < \infty}$ of unions K_m of a finite number of mutually disjoint closed discs such that

$$K_m \supset K_{m+1} \supset K \quad (m = 1,2,\dots)$$

and

$$\bigcap_{m=1}^{\infty} K_m = K.$$

Choose an $R \in (1, \infty)$ such that $K_1 \subset G := \Delta(0, R)$. We can find an f_m in $C(\bar{G}) \cap W_p^1(G)$ such that $f_m | K_m = 1$ and $f_m | \partial G = 0$ for each $m = 1,2,\cdots$. Moreover, by the lattice property of $C(\bar{G}) \cap W_p^1(G)$, we can assume that $0 \leqq f_{m+1} \leqq f_m \leqq 1$ on $\bar{G}$ $(m = 1,2,\cdots)$. Since $G \setminus K_m$ is $\mathscr{A}$-regular, the function w_m defined by

$$w_m(x) := \begin{cases} (\pi_{\mathscr{A}}^{G \setminus K_m} f_m)(x) & (x \in G \setminus K_m), \\ f_m(x) & (x \in K_m \cup \partial G) \end{cases}$$

for each $m = 1,2,\cdots$ belongs to $C(\bar{G}) \cap H_{\mathscr{A}}(G \setminus K_m) \cap W_p^1(G)$, and satisfies $w_m | K_m = 1$ and $w_m | \partial G = 0$. The sequence $(w_m)_{1 \leqq m < \infty}$ is decreasing along with $(f_m)_{1 \leqq m < \infty}$. By the Harnack principle (cf. e.g. [2, p. 113]), $w = \lim_{m \to \infty} w_m$ is $\mathscr{A}$-harmonic on $G \setminus K$. Clearly $w \in C(\bar{G} \setminus K)$ and $w | \partial G = 0$.

Consider the p-capacity $\operatorname{cap}_p(K_m, G)$ of the condenser (K_m, G) given by

$$\operatorname{cap}_p(K_m, G) = \inf \int_{G \setminus K_m} |\nabla \varphi(x)|^p dx$$

where the infimum is taken with respect to φ in $C_0^\infty(G)$ with $\varphi \geqq 1$ on K_m. The p-capacity $\operatorname{cap}_p(K, G)$ is similarly defined. It is a fundamental property of the p-capacity (cf. e.g. [2, Chap. 2, in particular, p. 28]) that

$$\lim_{m \to \infty} \operatorname{cap}_p(K_m, G) = \operatorname{cap}_p(K, G)$$

since K_m and K are compact and $K_m \downarrow K$. Note that

$$K = \{e^{\iota a_n}\}_{1 \leqq n < N+1} \cup \{e^{\iota b_n}\}_{1 \leqq n < N+1} \cup X$$

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where X consists of only one point $\lim_{n\to\infty} e^{ia_n} = \lim_{n\to\infty} e^{ib_n}$ if $N = \infty$ and $X = \emptyset$ if $N < \infty$. By the subadditivity of the p-capacity and the vanishingness of the p-capacity for one point we see that

$$\mathrm{cap}_p(K, G) \leqq \sum_{n=1}^{N} \{\mathrm{cap}_p(\{e^{ia_n}\}, G) + \mathrm{cap}_p(\{e^{ib_n}\}, G)\} + \mathrm{cap}_p(X, G) = 0$$

and therefore we conclude that

$$\lim_{m\to\infty} \mathrm{cap}_p(K_m, G) = 0.$$

For any competing function $\varphi \in C_0^\infty(G)$ with $\varphi \geqq 1$ on K_m for the p-capacity $\mathrm{cap}_p(K_m, G)$ we set $\varphi_m = \max(\min(\varphi, 1), 0)$. Clearly

$$w_m = \pi_{\mathscr{A}}^{G\setminus K_m} f_m = \pi_{\mathscr{A}}^{G\setminus K_m} \varphi_m.$$

By (30) we see that

$$\int_G |\nabla w_m(x)|^p dx = \int_{G\setminus K_m} |\nabla w_m(x)|^p dx$$
$$\leqq \left(\frac{\beta}{\alpha}\right)^p \int_{G\setminus K_m} |\nabla \varphi_m(x)|^p dx \leqq \left(\frac{\beta}{\alpha}\right)^p \int_{G\setminus K_m} |\nabla \varphi(x)|^p dx.$$

Hence we have

$$\int_G |\nabla w_m(x)|^p dx \leqq \left(\frac{\beta}{\alpha}\right)^p \mathrm{cap}_p(K_m, G) \to 0 \quad (m \to \infty)$$

and therefore we can conclude that $\{\nabla w_m\}_{1\leqq m<\infty}$ converges to zero strongly in $L_p(G, \mathbf{R}^2)$ and hence converges to zero weakly in $L_p(G, \mathbf{R}^2)$. As the locally uniform limit of the decreasing sequence $\{w_m\}$ with $0 \leqq w_m \leqq 1$, the function w is bounded and continuous on $G\setminus K$. Hence we may view that $w \in L_p(G, \mathbf{R})$. Thus, by $w_m \downarrow w$ a.e. on G, we have

$$\int_G \nabla w(x) \cdot \Phi(x) dx = -\int_G w(x) \nabla \cdot \Phi(x) dx$$
$$= -\lim_{m\to\infty} \int_G w_m(x) \nabla \cdot \Phi(x) dx = \lim_{m\to\infty} \int_G \nabla w_m(x) \cdot \Phi(x) dx = 0$$

for every C^∞ vector field Φ on G with compact support. This means that $\nabla w(x) = 0$ on G and thus w is a constant on G. Hence $w \mid \partial G = 0$ implies that

(45) $$\lim_{m\to\infty} w_m(x) = 0 \quad (x \in G\setminus K).$$

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It is clear that $\omega(A, \Delta; \mathscr{A}) \geqq 0$ and $1 - \omega(A, \Delta; \mathscr{A}) \geqq 0$ on Δ. Take any $\mathscr{A}$-harmonic function h on Δ such that $\omega(A, \Delta; \mathscr{A}) \geqq h$ and $1 - \omega(A, \Delta; \mathscr{A}) \geqq h$ on Δ. By (43) we see that

$$\limsup_{x\in\Delta, x\to\eta} h(x) \leqq 0 \quad (\eta \in \Gamma \setminus K).$$

It is clear that $h \leqq 1$ and $w_m \geqq 0$ on Δ. Hence we see that

$$\limsup_{x\in\Delta\setminus K_m, x\to\eta} h(x) \leqq \limsup_{x\in\Delta\setminus K_m, x\to\eta} w_m(x) = \lim_{x\in\Delta\setminus K_m, x\to\eta} w_m(x)$$

for every η in $\partial(\Delta \setminus K_m)$. By the comparison principle (cf. e.g. [2, p.183]) we have $h \leqq w_m$ on $\Delta \setminus K_m$. On letting $m \uparrow \infty$, (45) yields $h \leqq 0$ on Δ. This proves the existence of the greatest $\mathscr{A}$-harmonic minorant $\omega(A, \Delta; \mathscr{A}) \wedge (1 - \omega(A, \Delta; \mathscr{A}))$ of $\omega(A, \Delta; \mathscr{A})$ and $1 - \omega(A, \Delta; \mathscr{A})$ on Δ and therefore we have

$$\omega(A, \Delta; \mathscr{A}) \wedge (1 - \omega(A, \Delta; \mathscr{A})) = 0$$

which is the defining property of $\mathscr{A}$-harmonic measure on Δ in the sense of Heins. □

We next study the $\mathscr{A}$-harmonic measure $\omega(A, \Delta; \mathscr{A})$ when $N < \infty$ so that A is the union of a finite number N of open arcs $A_n : A = \cup_{n=1}^{N} A_n$ $(N < \infty)$. Let $X = \cup_{n=1}^{N} X_n$ and $Y = \cup_{n=1}^{N} Y_n$ where X_n and Y_n are open arcs in Γ such that $\bar{X}_n \subset A_n \subset \bar{A}_n \subset Y_n \subset \bar{Y}_n \subset \Gamma^+$ $(n = 1, 2, \cdots, N)$ and $\bar{Y}_n \cap \bar{Y}_m = \emptyset$ $(n \neq m)$. Such an X will be referred to as being *admissible* for A. In view of Proposition 39 we see that

$$\| 1_X; \Lambda_p(\Gamma) \| \leqq \sum_{n=1}^{N} \| 1_{X_m}; \Lambda_p(\Gamma) \| \leqq \sum_{n=1}^{N} (|X_n|^{1/p} + C|X_n|^{(2-p)/p})$$

so that we have

$$(46) \qquad \| 1_X; \Lambda_p(\Gamma) \| \leqq C_N \text{ and similarly } \| 1_Y; \Lambda_p(\Gamma) \|, \| 1_A; \Lambda_p(\Gamma) \| \leqq C_N$$

where $C_N = N(\pi^{1/p} + C\pi^{(2-p)/p})$ is a constant depending only on N (and p). Therefore we can define $w_X = \tau 1_X$ and $w_Y = \tau 1_Y$. By Lemma 32, (43) and the comparison principle, we deduce

$$(47) \qquad w_X(x) \leqq \omega(A, \Delta; \mathscr{A})(x) \leqq w_Y(x) \quad (x \in \Delta).$$

By the comparison principle and the Harnack principle

$$\underline{w}_A = \lim_{X\uparrow A} w_X \quad \text{and} \quad \bar{w}_A = \lim_{Y\downarrow A} w_Y$$

are well defined and $\mathscr{A}$-harmonic on Δ. Similarly, (46) assures the possibility of defining $w_A = \tau 1_A$. We will show that

(48) $$\underline{w}_A(x) = \bar{w}_A(x) = w_A(x) \quad (x \in \Delta).$$

This with (47) implies that $\omega(A, \Delta; \mathscr{A}) = w_A$ on Δ. Thus we can conclude the following result.

49. PROPOSITION. *If $N < \infty$ and $1 < p < 2$, then $1_A \in \Lambda_p(\Gamma)$, the $\mathscr{A}$-harmonic measure $\omega(A, \Delta; \mathscr{A})$ is p-Dirichlet finite on Δ, and the trace $\gamma(\omega(A, \Delta; \mathscr{A})) = 1_A$ on Γ.*

Proof. We only have to show the relation (48). By (26) and (46) we see that

$$\inf_{\gamma f = 1_X} \| \nabla f ; L_p(\Delta) \| \leqq \inf_{\gamma f = 1_X} \| f ; W_p^1(\Delta) \| \leqq C \| 1_X ; \Lambda_p(\Gamma) \| \leqq CC_N.$$

By the quasi Dirichlet principle (30),

$$\| \nabla w_X ; L_p(\Delta) \| \leqq (\beta / \alpha) \| \nabla f ; L_p(\Delta) \|$$

for any f with $\gamma f = 1_X$ since $\pi f = w_X$. Hence we see that

$$\| \nabla w_X ; L_p(\Delta) \| \leqq C$$

where we denote by C the constant $(\beta / \alpha) CC_N$. Any bounded set in the reflexive Banach space $L_p(\Delta) = L_p(\Delta; \mathbf{R}^2)$ is weakly sequentially compact. Terefore we can find a countable sequence $(X(m))_{1 \leqq m < \infty}$ in the set $\{X\}$ of admissible X such that $X(m) \subset X(m+1)$,

$$\lim_{m \to \infty} w_{X(m)} = \underline{w}_A$$

locally uniformly on Δ, and $(\nabla w_{X(m)})_{1 \leqq m < \infty}$ is weakly convergent in $L_p(\Delta)$. Since $0 \leqq \underline{w}_A \leqq 1$ on Δ, $\underline{w}_A$ belongs to $L_p(\Delta)$ and

$$\int_\Delta \underline{w}_A(x) \nabla \cdot \Phi(x) \, dx = \lim_{m \to \infty} \int_\Delta w_{X(m)}(x) \nabla \cdot \Phi(x) \, dx$$
$$= - \lim_{m \to \infty} \int_\Delta \nabla w_{X(m)}(x) \cdot \Phi(x) \, dx = - \int_\Delta (\operatorname{weak} \lim_{m \to \infty} \nabla w_{X(m)}(x)) \cdot \Phi(x) \, dx$$

for every C^∞ vector field Φ with compact support in Δ. This means that the distributional gradient $\nabla \underline{w}_A = \operatorname{weak} \lim_{m \to \infty} \nabla w_{X(m)} \in L_p(\Delta)$ and therefore $\underline{w}_A \in W_p^1(\Delta)$. By (47), $w_X \leqq \underline{w}_A \leqq \omega(A, \Delta; \mathscr{A})$ on Δ for any admissible X. By (43) we see that

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$$1_X \leqq \gamma \underline{w}_A \leqq 1_A$$

a.e. on Γ for any admissible X. A fortiori we can conclude that $\gamma\underline{w}_A = 1_A$ in $L_p(\Gamma)$. Hence $\gamma\underline{w}_A = \gamma w_A = 1_A$ implies that $\underline{w}_A = w_A = \tau 1_A$. Similarly we can show that $\bar{w}_A = w_A = \tau 1_A$. The proof of (48) and hence that of Proposition 49 is thus complete. □

We turn to the study of the $\mathscr{A}$-harmonic measure $\omega(A, \Delta\,;\mathscr{A})$ when $N = \infty$ so that $A = \cup_{n=1}^{\infty} A_n$. We will base our reasoning upon the fact that $1_{\cup_{n=1}^{k} A_n}$ always belongs to $\Lambda_p(\Gamma)$ for every $k < \infty$ as was shown in Proposition 49. However $1_A = 1_{\cup_{n=1}^{\infty} A_n}$ may or may not belong to $\Lambda_p(\Gamma)$ in general.

50. PROPOSITION. *Suppose $N = \infty$ and $1 < p < 2$. The $\mathscr{A}$-harmonic measure $\omega(A, \Delta\,;\mathscr{A})$ is p-Dirichlet finite on Δ if and only if $1_A \in \Lambda_p(\Gamma)$ and in this case the trace $\gamma(\omega(A, \Delta\,;\mathscr{A})) = 1_A$ on Γ.*

Proof. Suppose $\omega(A, \Delta\,;\mathscr{A})$ is p-Dirichlet finite so that $\omega(A, \Delta\,;\mathscr{A})$ belongs to $W_p^1(\Delta)$. Then by (43) and (23) we see that $\gamma(\omega(A, \Delta\,;\mathscr{A})) = 1_A$ on Γ except for the boundary of $\cup_{n=1}^{\infty}(A_n \cup B_n)$ relative to Γ and hence a.e. on Γ. Therefore 1_A belongs to $\Lambda_p(\Gamma)$.

Convsersely assume that $1_A \in \Lambda_p(\Gamma)$. Then we can define $w_A = \tau 1_A$ so that $D_p(w_A) = \|\nabla w_A\,; L_p(\Delta)\|^p < \infty$. Let $a = \lim_{k\uparrow\infty} a_k$ which belongs to $(0, \pi]$. Set

$$r_k = |\, e^{ia_{k+1}} - e^{ia} \,| \quad (k = 1,2,\cdots)$$

and choose a function χ_k on $\bar{\Delta}(e^{ia}, 3) = \overline{\Delta(e^{ia}, 3)}$ such that χ_k is continuous on $\bar{\Delta}(e^{ia}, 3)$, p-harmonic on $\Delta(e^{ia},3)\backslash\bar{\Delta}(e^{ia}, r_k)$, $\chi_k \,|\, \bar{\Delta}(e^{ia}, r_k) = 0$ and $\chi_k \,|\, \partial\Delta(e^{ia}, 3) = 1$. Choose an arbitrary φ in $C_0^\infty(\Delta(e^{ia}, 3))$ with $\varphi \geqq 1$ on $\bar{\Delta}(e^{ia}, r_k)$ and set $\psi = \max(\min(\varphi, 1), 0)$. Observe that

$$1 - \chi_k = \pi_p^{\Delta(e^{ia},3)\backslash\bar{\Delta}(e^{ia},r_k)}\psi$$

on $\Delta(e^{ia}, 3)\backslash\bar{\Delta}(e^{ia},r_k)$ where $\pi_p = \pi_{\mathscr{A}}$ with $\mathscr{A}(x, h) = |\,h\,|^{p-2}h$. The quasi Dirichlet principle (30) is nothing but the Dirichlet principle in this case of $\alpha = \beta = 1$ for $\mathscr{A}(x, h) = |\,h\,|^{p-2}h$:

$$\begin{aligned}\|\nabla(1-\chi_k)\,; L_p(\Delta(e^{ia}, 3)\backslash\bar{\Delta}(e^{ia},r_k))\| &\leqq \|\nabla\psi : L_p(\Delta(e^{ia}, 3)\backslash\bar{\Delta}(e^{ia},r_k)\| \\ &\leqq \|\nabla\varphi\,; L_p(\Delta(e^{ia}, 3)\backslash\bar{\Delta}(e^{ia},r_k))\|.\end{aligned}$$

Since $\text{cap}_p(\bar{\Delta}(e^{ia},r_k)), \Delta(e^{ia}, 3))$ is the infimum of $\|\nabla\varphi\,; L_p(\Delta(e^{ia}, 3))\|^p$ for every $\varphi \in C_0^\infty(\Delta(e^{ie}, 3))$ with $\varphi \geqq 1$ on $\bar{\Delta}(e^{ie}, r_k)$, we see that

$$\|\nabla\chi_k\,;L_p(\Delta(e^{ia},3))\|^p \leqq \mathrm{cap}_p(\bar{\Delta}(e^{ia},r_k),\Delta(e^{ia},3))$$

(and actually we can replace $\leqq$ by $=$ in the above). Note that $\bar{\Delta}(e^{ia}, r_k)$ and $\{e^{ia}\}$ are compact and $\bar{\Delta}(e^{ia}, r_k) \downarrow \{e^{ia}\}$ as $k \to \infty$. This assures that

$$\mathrm{cap}_p(\bar{\Delta}(e^{ia},r_k),\Delta(e^{ia},3)) \downarrow \mathrm{cap}_p(\{e^{ia}\},\Delta(e^{ia},3))\ (k\uparrow\infty).$$

Since $\mathrm{cap}_p(\{e^{ia}\}, \Delta(e^{ia}, 3)) = 0$ (cf. e.g. [2, p. 35]), we see that

$$\lim_{k\uparrow\infty}\int_{\Delta(e^{ia},3)}|\nabla\chi_k(x)|^p dx = 0.$$

By the comparison principle and the Harnack principle, we see that $(\chi_k)_{1\leqq k<\infty}$ is increasing and converges to a p-harmonic function χ locally uniformly on $\Delta(e^{ia}, 3)\backslash\{e^{ia}\}$. Here $0 \leqq \chi \leqq 1$, $\chi \in C(\bar{\Delta}(e^{ia}, 3)\backslash\{e^{ia}\})$ and $\chi\,|\,\partial\Delta(e^{ia}, 3) = 1$, and in particular $\chi \in L_p(\Delta(e^{ia}, 3))$. Hence

$$\int_{\Delta(e^{ia},3)}\chi(x)\nabla\cdot\Phi(x) = \lim_{k\to\infty}\int_{\Delta(e^{ia},3)}\chi_k(x)\nabla\cdot\Phi(x)dx$$

$$= -\lim_{k\to\infty}\int_{\Delta(e^{ia},3)}\nabla\chi_k(x)\cdot\Phi(x)dx = 0$$

for every C^∞ vector field Φ with compact support in $\Delta(e^{ia}, 3)$. This proves that $\nabla\chi = 0$ on $\Delta(e^{ia}, 3)$. Thus χ is a constant, which must be 1. Therefore we see in particular that $\chi_k \uparrow 1$ $(k \uparrow \infty)$ locally uniformly on $\bar{\Delta}\backslash\{e^{ia}\}$ and $D_p(\chi_k) = \|\nabla\chi_k\,; L_p(\Delta)\|^p \downarrow 0$ $(k \uparrow \infty)$.

Next we consider the sequence $(\chi_k w_A)_{1\leqq k<\infty}$ in Δ. Clearly we see that $\chi_k w_A \uparrow w_A$ $(k \uparrow \infty)$ locally uniformly on Δ. We also have that $D_p(\chi_k w_A - w_A) \to 0$ $(k \uparrow \infty)$. In fact,

$$D_p(\chi_k w_A - w_A)^{1/p}$$

$$\leqq \left(\int_\Delta|\chi_k(x)-1|^p|\nabla w_A(x)|^p dx\right)^{1/p} + \left(\int_\Delta|w_A(x)|^p|\nabla\chi_k(x)|^p dx\right)^{1/p}$$

$$\leqq \left(\int_\Delta|\chi_k(x)-1|^p d\mu(x)\right)^{1/p} + \left(\int_\Delta|\nabla\chi_k(x)|^p dx\right)^{1/p},$$

where $d\mu(x) = |\nabla w_A(x)|^p dx$ is a finite measure on Δ. The second term of the rightmost side of the above is $D_p(\chi_k) \downarrow 0$ $(k \uparrow \infty)$. The first term of the rightmost side of the above tends to zero as $k \uparrow \infty$ by the Lebesgue dominated convergence theorem since $\chi_k \uparrow 1$ on Δ as $k \uparrow \infty$.

We now set

$$u_k = \pi^{\Delta}_{\mathscr{A}}(\chi_k w_A) = \tau^{\Delta}_{\mathscr{A}}((\gamma\chi_k)1_A) \leqq w_A$$

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on Δ. The last inequality comes from the monotoneity of $\pi_{\mathscr{A}}^{\Delta}$ and $\tau_{\mathscr{A}}^{\Delta}$. By the same reason, $(u_k)_{1\leqq k<\infty}$ is increasing on Δ. By the Harnack principle there exists an $\mathscr{A}$-harmonic function u on Δ such that $u_k \uparrow u \leqq w_A$ $(k \uparrow \infty)$ on Δ. By the quasi Dirichlet principle

$$D_p(u_k) \leqq (\beta/\alpha)^p D_p(\chi_k w_A) \to (\beta/\alpha)^p D_p(w_A) \quad (k \uparrow \infty).$$

Hence $(\nabla u_k)_{1\leqq k<\infty}$ is a bounded sequence in $L_p(\Delta)$ and we can extract a weakly convergent subsequence $(\nabla u_{k'})$. Then

$$\int_{\Delta} u(x)\,\nabla\cdot\Phi(x)\,dx = \lim_{k'\to\infty}\int_{\Delta} u_{k'}(x)\,\nabla\cdot\Phi(x)\,dx$$
$$= -\lim_{k'\to\infty}\int_{\Delta}\nabla u_{k'}(x)\cdot\Phi(x)\,dx = -\int_{\Delta}(\text{weak}\lim_{k'\to\infty}\nabla u_{k'}(x))\cdot\Phi(x)\,dx$$

for every C^∞ vector field Φ with compact support in Δ, which proves that the distributional $\nabla u = \text{weak}\lim_{k'\uparrow\infty}\nabla u_{k'}$ belongs to $L_p(\Delta)$. Hence $D_p(u) < \infty$ and $u \in W_p^1(\Delta)$. Therefore $\gamma u_k \leqq \gamma u \leqq \gamma w_A$ or $(\gamma\chi_k)1_A \leqq \gamma u \leqq 1_A$ a.e. on Γ. Since $\chi_k \uparrow 1$ $(k \uparrow \infty)$ locally uniformly on $\bar{\Delta}\setminus\{e^{ia}\}$ and thus $\gamma\chi_k \uparrow 1$ $(k \uparrow \infty)$ a.e. on Γ, we see that $\gamma u = 1_A$ so that $u = w_A$, i.e. $\lim_{k\uparrow\infty} u_k = w_A$ on Δ.

Observe that

$$(\gamma\chi_k)1_{\cup_{n=1}^k A_n} \leqq 1_A$$

so that we have $u_k \leqq w_{\cup_{n=1}^k A_n} \leqq w_A$ on Δ. By Proposition 49 we have

$$\omega(\bigcup_{n=1}^k A_n,\ \Delta\,;\mathscr{A}) = w_{\cup_{n=1}^k A_n}$$

on Δ. Hence we have

$$u_k \leqq \omega(\bigcup_{n=1}^k A_n,\ \Delta\,;\mathscr{A}) \leqq w_A$$

on Δ and by letting $k \uparrow \infty$ we conclude that

$$\lim_{k\uparrow\infty}\omega(\bigcup_{n=1}^k A_n,\ \Delta\,;\mathscr{A}) = w_A$$

on Δ. Since $O_k = \cup_{n=1}^k A_n$ is open in Γ, $O_k \subset O_{k+1}$ and

$$O = \bigcup_{k=1}^\infty O_k = \bigcup_{n=1}^\infty A_n = A$$

is again open, we can show (cf. e.g. [2, p. 29]) that

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$$\lim_{k\uparrow\infty}\omega(\bigcup_{n=1}^{k}A_n,\Delta;\mathscr{A})=\lim_{k\uparrow\infty}\omega(O_k,\Delta;\mathscr{A})=\omega(O,\Delta;\mathscr{A})=\omega(A,\Delta;\mathscr{A}).$$

Thus $\omega(A,\Delta;\mathscr{A})=w_A=\tau 1_A$ is p-Dirichlet finite and $\gamma(\omega(A,\Delta;\mathscr{A}))=1_A\in\Lambda_p(\Gamma)$. □

51. Proof of Main theorem

If $N<\infty$, then, by Proposition 49, $\omega(A,\Delta;\mathscr{A})$ is p-Dirichlet finite on Δ. Hence, hereafter in this proof, we assume that $N=\infty$ so that $A=A((a_n),(b_n))=\cup_{n=1}^{\infty}A_n$. Let

$$B_0=\Gamma\setminus\overline{\bigcup_{n=1}^{\infty}(A_n\cup B_n)}$$

and set $B=\cup_{n=0}^{\infty}B_n$.

We now start the essential part of this proof by showing that (15) implies the p-Dirichlet finiteness of $\omega(A,\Delta;\mathscr{A})$ on Δ. Suppose first that $\sum_{n=1}^{\infty}|B_n|^{2-p}<\infty$. Observe that

$$\|1_A;\Lambda_p(\Gamma)\|=\|1_A;L_p(\Gamma)\|+\left(\int\int_{\Gamma\times\Gamma}\frac{|1_A(\xi)-1_A(\eta)|^p}{|\xi-\eta|^p}ds_\xi ds_\eta\right)^{1/p}$$
$$=|A|^{1/p}+(2S(A^c,A))^{1/p}.$$

By the estimate 38, $S(B_n,B_n^c)\leqq C|B_n|^{2-p}$ where C is a constant independent of $n=1,2,\cdots$. Therefore we have

$$S(A^c,A)=S((\bar{A})^c,A)=S(\bigcup_{n=0}^{\infty}B_n,A)=\sum_{n=0}^{\infty}S(B_n,A)$$
$$\leqq\sum_{n=0}^{\infty}S(B_n,B_n^c)\leqq C\sum_{n=0}^{\infty}|B_n|^{2-p}<\infty.$$

Hence we see that $1_A\in\Lambda_p(\Gamma)$. Next suppose that $\sum_{n=1}^{\infty}|A_n|^{2-p}<\infty$. In the same fasion as above simply replacing the role of A and $(A_n)_1^{\infty}$ by B and $(B_n)_0^{\infty}$, we see that $1_B\in\Lambda_p(\Gamma)$. Clearly

$$1_A=1-1_{A^c}=1-1_{\bar{B}}=1-1_B$$

a.e. on Γ and thus $1_A\in\Lambda_p(\Gamma)$. Hence in any case the condition (15) implies that $1_A\in\Lambda_p(\Gamma)$. By Proposition 50 we can conclude that $\omega(A,\Delta;\mathscr{A})$ is p-Dirichlet finite.

We close this proof by showing that (16) implies that $\omega(A,\Delta;\mathscr{A})$ is p-Dirichlet infinite. We prove this by contradiction. Suppose, contrary to the

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assertion, that $\omega(A, \Delta; \mathscr{A})$ is p-Dirichlet finite. By Proposition 50 we must have $1_A \in \Lambda_p(\Gamma)$. Since A_n and B_n $(n \geqq 1)$ are in Γ^+, (36) implies that

$$T(A_n, B_n) \leqq S(A_n, B_n) \quad (n = 1, 2, \cdots).$$

Therefore we deduce that, for any fixed positive integer k,

$$\sum_{n=1}^{k} T(A_n, B_n) \leqq \sum_{n=1}^{k} S(A_n, B_n) \leqq \sum_{n=1}^{k}\left(\sum_{m=1}^{k} S(A_n, B_m)\right)$$
$$= S(\bigcup_{n=1}^{k} A_n, \bigcup_{m=1}^{k} B_m) \leqq 2S(\bigcup_{n=1}^{\infty} A_n, \bigcup_{m=0}^{\infty} B_m) = 2S(A, (\bar{A})^c)$$
$$= 2S(A, A^c) = \int\int_{\Gamma\times\Gamma} \frac{|1_A(\xi) - 1_A(\eta)|^p}{|\xi - \eta|^p} ds_\xi ds_\eta \leqq \| 1_A ; \Lambda_p(\Gamma) \|^p.$$

On letting $k \uparrow \infty$, we obtain

$$\sum_{n=1}^{\infty} T(A_n, B_n) \leqq \| 1_A ; \Lambda_p(\Gamma) \|^p. \tag{52}$$

By the identity 37 we have

$$T(A_n, B_n) = C_p(|A_n|^{2-p} + |B_n|^{2-p} - (|A_n| + |B_n|)^{2-p}).$$

Here we used the fact that the Riemannian distance $\rho = \rho(A_n, B_n) = 0$ considered in Γ since $\bar{A}_n \cap \bar{B}_n = \{e^{ib_n}\} \neq \emptyset$.

We pause here to observe the validity of the following simple and elementary inequality for $1 < p < 2$:

$$x^{2-p} + y^{2-p} - (x+y)^{2-p} \geqq a^{2-p} + b^{2-p} - (a+b)^{2-p} \quad (0 \leqq a \leqq x,\ 0 \leqq b \leqq y). \tag{53}$$

In fact, consider $f_y(x) = x^{2-p} + y^{2-p} - (x+y)^{2-p}$ as a function of $x \geqq 0$ for an arbitrary fixed $y \geqq 0$. Since

$$\frac{d}{dx} f_y(x) = (2-p)\{x^{1-p} - (x+y)^{1-p}\} \geqq 0 \quad (x > 0),$$

we see that $f_y(x)$ is increasing and hence $f_y(x) \geqq f_y(a)$ $(0 \leqq a \leqq x)$. By the symmetry $f_y(a) = f_a(y)$ we also see that $f_a(y) \geqq f_a(b)$ or $f_y(a) \geqq f_b(a)$. Thus $f_y(x) \geqq f_b(a)$ $(0 \leqq a \leqq x,\ 0 \leqq b \leqq y)$ which proves (53).

On setting $x = |A_n|$, $y = |B_n|$, and $a = b = \min(|A_n|, |B_n|)$ in (53), we obtain

$$|A_n|^{2-p} + |B_n|^{2-p} - (|A_n| + |B_n|)^{2-p}$$
$$\geqq 2(\min(|A_n|, |B_n|))^{2-p} - (2\min(|A_n|, |B_n|))^{2-p}.$$

Since the left hand side of the above is $C_p^{-1}T(A_n, B_n)$, we have

$$C_p C \cdot \min(|A_n|^{2-p}, |B_n|^{2-p}) \leqq T(A_n, B_n)$$

where $C = 2 - 2^{2-p} \in (0, 1)$. Hence by (52) and (16)

$$\infty = C_p C \sum_{n=1}^{\infty} \min(|A_n|^{2-p}, |B_n|^{2-p}) \leqq \| 1_A ; \Lambda_p(\Gamma) \| < \infty,$$

which is clearly a contradiction. □

54. Appendix: Nonlinearity of $\mathscr{A}_2(\mathbf{R}^2)$

The p-Laplace operator $\mathscr{A}(x, h) = |h|^{p-2}h$ is a typical example of $\mathscr{A} \in \mathscr{A}_p(\mathbf{R}^d)$ which makes the equation (6) nonlinear if $p \neq 2$. However it is important to recognize that $\mathscr{A}_2(\mathbf{R}^d)$ contains an $\mathscr{A}$ which produces a genuinely nonlinear equation (6) as was pointed out e.g. by Martio in [4]. Even in the borderline conformal case $p = d = 2$, the $\mathscr{A}$-harmonicity in general belongs in essence to the category of nonlinearity. In this appendix we will exhibit such an $\mathscr{A} \in \mathscr{A}_2(\mathbf{R}^d)$ for every dimension $d \geqq 2$. The author owes a lot to Professor Masaru Hara in constructing this example.

As a required $\mathscr{A} \in \mathscr{A}_2(\mathbf{R}^d)$ we only have to take the one of the form $\mathscr{A}(x, h) = A(h)$ independent of $x \in \mathbf{R}^d$ such that $A = A_d : \mathbf{R}^d \to \mathbf{R}^d$ $(d \geqq 2)$ is nonlinear.

Consider a closed surface Σ in $\mathbf{R}^d$ $(d \geqq 2)$ which is star-shaped and symmetric with respect to the origin 0 of $\mathbf{R}^d$ belonging to the region bounded by Σ. In terms of the polar coordinate expression $x = r\omega$ of $x \in \mathbf{R}^d \setminus \{0\}$ with $r = |x|$ and $\omega = x/|x|$ in ∂B^d, since Σ is star-shaped with respect to 0, we have the polar coordinate expression of Σ as follows:

$$\Sigma : r = g(\omega) \quad (\omega \in \partial B^d).$$

By the symmetry of Σ with respect to 0 we see that $g(-\omega) = g(\omega)$ for every $\omega \in \partial B^d$. Since the origin 0 is contained in the interior region bounded by Σ, we have

$$c_\Sigma := \inf\{g(\omega) : \omega \in \partial B^d\} > 0.$$

We then set

$$C_\Sigma := \sup\left(\frac{|g(\omega) - g(\bar{\omega})|}{|\omega - \bar{\omega}|} : \omega, \bar{\omega} \in \partial B^d, \omega \neq \bar{\omega}\right)$$

which lies in $(0, \infty]$ at the moment. As a candidate of the required A we now set

$$A(h) = \begin{cases} g(h/|h|)h & (h \neq 0), \\ 0 & (h = 0). \end{cases}$$

Then we have the following

55. FACT. *If the condition $C_\Sigma < \sqrt{2}\, c_\Sigma$ is satisfied, then A belongs to $\mathscr{A}_2(\mathbf{R}^d)$ $(d \geqq 2)$ and moreover A is not linear if and only if Σ is not a sphere with center* 0.

Proof. The continuity of $A(h)$ at $h \in \mathbf{R}^d \setminus \{0\}$ follows from that of g. Since $|A(h)| \leqq (\sup_{\partial B^d} g)\,|h|$ and $A(0) = 0$, $A(h)$ is continuous at $h = 0$. Thus A satisfies (1). Observe that

$$A(h)\cdot h = g(h/|h|)h\cdot h \geqq c_\Sigma |h|^2 \quad (h \neq 0)$$

which shows the validity of (2) for $p = 2$ by taking $\alpha = c_\Sigma$. Similarly

$$|A(h)| = |g(h/|h|)|\,|h| \leqq (\sup_{\partial B^d} g)\,|h|^{2-1} \quad (h \neq 0)$$

which assures (3) for $p = 2$ by taking $\beta = \sup_{\partial B^d} g$. In passing we observe that $0 < \alpha \leqq \beta < \infty$. Next we ascertain that (5) is valid for $p = 2$. If $\lambda > 0$, then

$$A(\lambda h) = g(\lambda h/\lambda|h|)\lambda h = \lambda A(h) \quad (h \neq 0).$$

If $\lambda < 0$, then, by $\lambda = -|\lambda|$ and $g(-\omega) = g(\omega)$, we see that

$$A(\lambda h) = A(|\lambda|(-h)) = |\lambda| A(-h) = |\lambda|\, g(-h/|-h|)(-h)$$
$$= -|\lambda|\, g(h/|h|)h = \lambda A(h) \quad (h \neq 0).$$

Therefore the proof of (4) only is nontrivial. We need to show that

$$(A(h) - A(\bar{h}))\cdot(h - \bar{h}) > 0 \quad (h \neq \bar{h}). \tag{56}$$

When one of h and $\bar{h}$ is 0, the other is nonzero and a fortiori (2) and $A(0) = 0$ trivially imply (56). Thus we assume that both of h and $\bar{h}$ are not 0. We can moreover assume that $|\bar{h}| = 1$ so that we may set $h = r\omega$ $(r = |h|,\ \omega \in \partial B^d)$, $\bar{h} = \bar{\omega}$ $(\bar{\omega} \in \partial B^d)$ and

$$\omega\cdot\bar{\omega} = \cos\theta \quad (\theta \in [0, \pi]).$$

Then $h \neq \bar{h}$ is equivalent to either $r \neq 1$ or $\omega \neq \bar{\omega}$ (or $\theta \neq 0$). Hence (56) is equivalent to

$$(g(\omega)r\omega - g(\bar{\omega})\bar{\omega})\cdot(r\omega - \bar{\omega}) > 0 \quad (r \neq 1 \text{ or } \omega \neq \bar{\omega}),$$

which can be restated as

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(57) $\quad Q := g(\omega)r^2 - \{(g(\omega) + g(\bar{\omega}))\cos\theta\}r + g(\bar{\omega}) > 0 \ \ (r \neq 1 \text{ or } \theta \neq 0).$

We thus have to prove (57). If $\theta = 0$, then $\omega = \bar{\omega}$ and $r \neq 1$ so that

$$Q = g(\omega)(r-1)^2 > 0$$

and (57) is certainly true. If $\theta \in [\pi/2, \pi]$, then $\cos\theta = -|\cos\theta|$ and hence we have

$$Q = g(\omega)r^2 + \{(g(\omega) + g(\bar{\omega}))|\cos\theta|\}r + g(\bar{\omega}) > 0$$

so that (57) is also true in this case. To prove (57) we thus only have to treat the case $\theta \in (0, \pi/2)$. Viewing Q as the quadratic form of r, it is sufficient to show that the discriminant of Q is negative:

$$(g(\omega) + g(\bar{\omega}))^2\cos^2\theta - 4g(\omega)g(\bar{\omega})$$
$$= (g(\omega) - g(\bar{\omega}))^2 - (g(\omega) + g(\bar{\omega}))^2\sin^2\theta < 0.$$

Since $|\omega - \bar{\omega}|^2 = 4\sin^2(\theta/2) > 0$, the above inequality is equivalent to

(58) $\quad D := \dfrac{(g(\omega) - g(\bar{\omega}))^2}{|\omega - \bar{\omega}|^2} - (g(\omega) + g(\bar{\omega}))^2\cos^2(\theta/2) < 0 \quad (0 < \theta < \pi/2).$

By virtue of $C_\Sigma < \sqrt{2}c_\Sigma$ we see that

$$D \leqq C_\Sigma^2 - 4c_\Sigma^2\cos^2(\pi/4) = C_\Sigma^2 - 2c_\Sigma^2 < 0,$$

i.e. (58) is valid. Therefore we have shown that $A \in \mathscr{A}_2(\mathbf{R}^d)$ if $C_\Sigma < \sqrt{2}c_\Sigma$.

Clearly A is linear if g is constant on ∂B^d or equivalently Σ is a sphere with center 0. Conversely assume that A is linear. Fix an arbitrary $\omega_0 \in \partial B^d$ and take any $\omega \in \partial B^d$ different from $\pm\omega_0$. Then $A(\omega) + A(\omega_0) = A(\omega + \omega_0)$ or

$$g(\omega)\omega + g(\omega_0)\omega_0 = g((\omega + \omega_0)/|\omega + \omega_0|)(\omega + \omega_0)$$

and the linear independence of $\{\omega, \omega_0\}$ implies $g(\omega) = g(\omega_0) = g((\omega + \omega_0)/|\omega + \omega_0|)$ so that $g \equiv g(\omega_0)$ on ∂B^d,, i.e. Σ is a sphere with center 0. □

59. Example. *Let Σ be a hyperellipsoid*

$$\sum_{i=1}^{d} \frac{(x^i)^2}{(a^i)^2} = 1 \quad (0 < a^1 \leqq a^2 \leqq \cdots \leqq a^d).$$

If $a^d - a^1$ is positive but enough close to zero, e.g. if

(60) $\quad a^1 < a^d < \sqrt{d/(d-1)}\,a^1,$

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then Σ induces a nonlinear $A \in \mathscr{A}_2(\mathbf{R}^d)$ $(d \geqq 2)$ as in the proof of Fact 55. *On the contrary, if $a^d - a^1$ is sufficiently large, e.g. if $a^d > 6a^1$, then $A \notin \mathscr{A}_2(\mathbf{R}^d)$ $(d \geqq 2)$.*

Proof. Assume (60). We express Σ as $r = g(\omega)$ $(\omega \in \partial B^d)$ by the polar coordinate (r, ω) :

$$g(\omega) = \left(\sum_{i=1}^{d-1} ((a^i)^{-2} - (a^d)^{-2})(\omega^i)^2 + (a^d)^{-2}\right)^{-1/2} \quad (\omega = (\omega^1, \cdots, \omega^d)).$$

Then clearly we have

$$c_\Sigma = \inf\{g(\omega) : \omega \in \partial B^d\} = a^1 > 0.$$

We see that

$$|\partial g / \partial \omega^i| = ((a^i)^{-2} - (a^d)^{-2}) |\omega^i| g(\omega)^3 \leqq ((a^1)^{-2} - (a^d)^{-2})(a^d)^3$$

$(i = 1, \cdots, d-1)$. Therefore we deduce

$$\begin{aligned} C_\Sigma &\leqq \sqrt{d-1}((a^1)^{-2} - (a^d)^{-2})(a^d)^3 = \sqrt{d-1}((a^d)^2 - (a^1)^2)a^d(a^1)^{-2} \\ &< \sqrt{d-1}((d/(d-1))(a^1)^2 - (a^1)^2)\sqrt{d/(d-1)}\,a^1(a^1)^{-2} \\ &= (\sqrt{d}/(d-1))a^1 \leqq \sqrt{2}a^1 = \sqrt{2}c_\Sigma, \end{aligned}$$

by which Fact 55 implies the first assertion.

We proceed to the proof of the second part. Observe that $A \in \mathscr{A}_2(\mathbf{R}^d)$ implies (58). Set $\omega = (1/4, 0, \cdots, 0, \sqrt{15}/4)$ and

$$\bar{\omega} = (1/4 + \varepsilon, 0, \cdots, 0, \sqrt{15}/4 - (\sqrt{15}/2 - \sqrt{15/4 - 4\varepsilon(1/2 + \varepsilon)})/2)$$

for sufficiently small $\varepsilon > 0$ in (58). On letting $\bar{\omega} \to \omega$ or equivalently $\varepsilon \downarrow 0$ or $\theta \to 0$ in (58) with the above choice of ω and $\bar{\omega}$ we deduce

$$\frac{15}{16}(((a^1)^{-2} - (a^d)^{-2})4^{-1}g(\omega)^3)^2 - 4g(\omega)^2 \leqq 0.$$

Since $1 < \sqrt{15} - 2$ and $\sqrt{15} + 30 < 36$, we have $(a^1)^{-2} < 36(a^d)^{-2}$ or $a^d < 6a^1$. Hence we must have $A \notin \mathscr{A}_2(\mathbf{R}^d)$ if $a^d > 6a^1$. □

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