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Strong orthogonality between the Möbius function, additive characters and Fourier coefficients of cusp forms

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Abstract

Let $\nu_f(n)$ be the nth normalized Fourier coefficient of a Hecke–Maass cusp form f for $\mathrm{SL}(2,\mathbb{Z})$ and let α be a real number. We prove strong oscillations of the argument of $\nu_f(n)\mu(n)\exp(2\pi i n\alpha)$ as n takes consecutive integral values.

1. Introduction

Fourier coefficients of cusp forms are mysterious objects and an interesting question for a fixed form is how its Fourier coefficients are distributed. There are many results from which the distribution appears to be highly random. For example, consider the following uniform bound on linear forms involving normalized Fourier coefficients $\nu_f(n)$ of a Maass cusp form f (see § 2 for the normalization) twisted by an additive character $e(\alpha) := \exp(2\pi i\alpha)$ (see [Iwa95, Theorem 8.1])

$$\sum_{|n|\leqslant N} \nu_f(n)e(n\alpha) \ll_f N^{1/2}\log 2N. \tag{1}$$

We emphasize that the implied constant here depends only on f and not on the real number α. The estimate (1) signifies an enormous number (square-root of the length of summation) of cancellations. This means that the Fourier coefficients are quite far from being aligned with the values of any fixed additive character and therefore, the bound (1) can be interpreted as manifestation of *non-correlation* or a kind of '*orthogonality*' between the Fourier coefficients of $(\nu_f(n))$ and the sequence $(e(n\alpha))$. Following [GT12, Sar], we say two sequences (x_n) and (y_n) of complex numbers are *asymptotically orthogonal* (in short, 'orthogonal') if

$$\sum_{1\leqslant n\leqslant N} x_n y_n = o\biggl(\biggl(\sum_{n\leqslant N}|x_n|^2\biggr)^{1/2}\biggl(\sum_{n\leqslant N}|y_n|^2\biggr)^{1/2}\biggr) \tag{2}$$

as $N \longrightarrow \infty$; and *strongly asymptotically orthogonal* (in short, 'strongly orthogonal') if

$$\sum_{1\leqslant n\leqslant N} x_n y_n = O_A\biggl((\log N)^{-A}\sum_{n\leqslant N}|x_n y_n|\biggr) \tag{3}$$

for every $A \geqslant 0$, uniformly for $N \geqslant 2$. The bound (1) shows that the two sequences $(\nu_f(n))$ and $(e(n\alpha))$ are strongly orthogonal. The question we seek to answer is whether strong orthogonality

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is manifested if, instead of the sum in (1), we consider the corresponding sum over primes

$$\mathcal{P}_f(X,\alpha) := \sum_{\substack{p\leqslant X\\ p\,\mathrm{prime}}} \nu_f(p)e(p\alpha). \tag{4}$$

Another interesting question is to ask whether the sequences $(\nu_f(n)e(n\alpha))$ and $(\mu(n))$ are strongly orthogonal. The *Möbius randomness law* (see [IK04, § 13.1]) asserts that the sequence $(\mu(n))$ should be orthogonal to any '*reasonable*' sequence. Sarnak has recently posed a more precise conjecture in this direction and we refer the reader to [BSZ13, CS13, Sar, SU11] for recent developments on this theme. In particular, [Sar, Conjecture 4] proposes to replace the condition '*reasonable*' by '*bounded with zero topological entropy*'. The sequences $(\nu_f(n))$ or more generally $(\nu_f(n)e(n\alpha))$ do not fit immediately in this context, because they are unbounded. One can nevertheless expect that a suitable reformulation would apply to these sequences and it would be interesting to know if they have entropy zero (e.g. does the sequence of signs of $\nu_f(n)$ have entropy zero?)

This question leads us to investigate cancellations in the sum *dual* to (1) (in the sense of Dirichlet convolution)

$$\mathcal{M}_f(X,\alpha) := \sum_{1\leqslant n\leqslant X} \mu(n)\nu_f(n)e(n\alpha). \tag{5}$$

Using classical techniques from analytic number theory and a recent impressive result due to Miller [Mil06], we establish bounds for both (4) and (5) that go beyond strong orthogonality, at least when f is a Maass cusp form for the full modular group $\mathrm{SL}(2,\mathbb{Z})$ (of arbitrary weight and Laplace eigenvalue). Here our definition of Maass form is general enough to include holomorphic modular forms. Our main theorem is as follows.

Theorem 1.1. *There exists an effective absolute $c_0>0$ such that, for any Maass cusp form f for the group $\mathrm{SL}(2,\mathbb{Z})$, of arbitrary weight and Laplace eigenvalue, there exists an effective constant $C_0(f)>0$ such that one has the inequalities*

$$|\mathcal{P}_f(X,\alpha)| \leqslant C_0(f)X\exp(-c_0\sqrt{\log X}), \tag{6}$$

and

$$|\mathcal{M}_f(X,\alpha)| \leqslant C_0(f)X\exp(-c_0\sqrt{\log X}), \tag{7}$$

for every $\alpha\in\mathbb{R}$ and $X\geqslant 2$.

The strong orthogonality we mentioned above now follows from the lower bound given in Proposition 3.1. In particular, (7) says that the Möbius randomness law is true in the case of the function $n\mapsto \nu_f(n)e(n\alpha)$ in a strong sense. Theorem 1.1 can also be interpreted as the prime number theorem (denoted henceforth by PNT) for Fourier coefficients of cusp forms with additive twists. In fact, (5) is the GL(2) analogue of a result of Davenport (see [Dav37] or [IK04, § 13.5]) which says that for any real number α, $X\geqslant 2$ and $A>0$, we have the bound

$$\sum_{n\leqslant X} \mu(n)e(n\alpha) \ll_A X(\log X)^{-A}. \tag{8}$$

The weaker bound here is a reflection of the exceptional zero (see [IK04, ch. 5]) which is not yet ruled out in the GL(1) situation. By contrast, Hoffstein and Ramakrishnan [HR95] have shown that there are no exceptional zeros for L-functions on GL(2) that are not associated to grossencharacters of quadratic fields.

As soon as α has a sufficiently good approximation by rationals, for example, if we have suitable control over the infinite continued fraction expansion of α, then the upper bound (6) is highly improved and we obtain a power saving. The most typical case is the *golden ratio* $\alpha = \rho = (1+\sqrt{5})/2$. In that particular case, we know that for every $X > 2$, there is a fraction a/q, $(a,q)=1$, satisfying (92) and the inequality $\sqrt{X} < q < 2\sqrt{X}$. The formula (119) then directly leads to the following corollary.

Corollary 1.1. *We have the bound*

$$\mathcal{M}_f(X,\rho) \ll X^{(59/60)+\varepsilon}.$$

Theorem 1.1 is suitable for invoking the circle method. For instance, reserving the letter p to denote primes, we have the following corollary. The proof follows directly from the basic identity of the circle method and the Parseval formula.

Corollary 1.2. *There exists an effective absolute $c_0 > 0$, such that for any Maass cusp form f for the group $\mathrm{SL}(2,\mathbb{Z})$ there exists an effective constant $C_0(f)$ such that one has the inequality*

$$\left|\sum\sum\sum_{N=p+a+b} \nu_f(p)\alpha_a\beta_b\right| \leqslant C_0(f) N \exp(-c_0\sqrt{\log N}) \|\alpha_N\| \, \|\beta_N\|, \tag{9}$$

for every $N \geqslant 4$, for every sequence of complex numbers $(\alpha_a)_{a\geqslant 1}$ and $(\beta_b)_{b\geqslant 1}$ where we denote $\|\alpha_N\|^2 = \sum_{1\leqslant a\leqslant N} |\alpha_a|^2$ and $\|\beta_N\|^2 = \sum_{1\leqslant b\leqslant N} |\beta_b|^2$. In particular, for the Ramanujan τ-function and for $N \geqslant 6$, one has the inequality

$$\left|\sum\sum\sum_{N=p_1+p_2+p_3} \tau(p_1)\right| \leqslant C_0 N^{15/2} \exp(-c_0\sqrt{\log N}), \tag{10}$$

where C_0 and c_0 are some positive constants, both effectively computable.

To see the interest of (9), suppose that the sequences (α_a) and (β_b) are the characteristic functions of sequences of positive integers $\mathcal{A}$ and $\mathcal{B}$, with counting functions $A(N)$ and $B(N)$, up to N. If f is holomorphic, Deligne's bound (18) implies the trivial bound

$$\left|\sum\sum\sum_{N=p+a+b} \nu_f(p)\alpha_a\beta_b\right| \ll A(N)B(N).$$

Hence, (9) is interesting as soon as the sequences $\mathcal{A}$ and $\mathcal{B}$ are dense enough, which means the condition $A(N)B(N) \gg N^2\exp(-2c_0\sqrt{\log N})$ is satisfied for sufficiently large N; for instance, when $\mathcal{A}$ and $\mathcal{B}$ are the sequence of primes or certain sequences of smooth numbers: $\mathcal{A} = \mathcal{B} = \{n : p \mid n \Rightarrow p < \exp(\log^\theta n)\}$, where θ is any fixed real number satisfying $\theta > 1/2$. Note that (10) is trivial if N is even; but if $N \geqslant 7$ is odd, the famous Vinogradov's theorem gives the lower bound

$$\sum\sum\sum_{N=p_1+p_2+p_3} 1 \gg N^2(\log N)^{-3}.$$

In other words, (10) shows a lot of oscillations of the coefficient $\tau(p_1)$ in the expression of N of the form $N = p_1+p_2+p_3$. The same is true for the coefficient $\tau(p_1)\tau(p_2)\tau(p_3)$.

Our proof is along the lines of Davenport's [Dav37] and it follows different paths depending on the diophantine nature of α: whether or not it is near a rational number with denominator sufficiently small. In the first case, i.e. when α belongs to the so called *major arcs*, we can use a suitable PNT for automorphic L-functions. The formulas (6) and (7), though apparently not equivalent, are recognized to have the same depth. We only prove the bound (7) since the proof of (7) is more delicate than the proof of (6). One reason for this is that we need to prove the required PNT Theorem 4.1 from scratch.

For *minor arcs*, i.e., when α cannot be approximated by rationals with small denominators, we apply Vinogradov's method for exponential sum via Vaughan's identity. Thus we are led to the so called sums of type I and type II. In estimating the type II sum, the more difficult one, we encounter a sum which is naturally related to the symmetric square lift of the Maass form f. A result of Miller (see [Mil06, Theorem 1.1]) suitably adapted to our requirement (see Lemma 6.4) is crucial here. Miller's theorem, which is a consequence of Voronoi's summation formula for GL(3) (see [MS06] and also [GL06]), says the following: for a cusp form on $\mathrm{GL}(3,\mathbb{Z})\backslash\mathrm{GL}(3,\mathbb{R})$ with Fourier coefficients $a_{r,n}$, one has

$$\sum_{n\leqslant T} a_{r,n} e(n\alpha) \ll T^{(3/4)+\varepsilon}, \tag{11}$$

where the implied constant depends only on the form, the integer r and ε. This is why we confine ourselves to the level one situation as the analogous result in the case of a general level, though expected, is not yet available.

However, in certain ranges of the variables (11) gives trivial bounds and we need to appeal to the oscillations of the additive character $n \mapsto e(\alpha n)$. Here the condition that α belongs to the minor arcs becomes important (see the classical Lemma 7.1 below).

This brings us to another difference between the proofs of (6) and (7). This is due to the difference between the combinatorial structures of Λ and μ. It is more difficult in this context to apply the Vaughan identity (89) for the Möbius function than its classical analogue for the von Mangoldt function. The reason is that one needs to control the greatest common divisors of the variables of summations in the case of the Möbius function whereas this problem disappears completely in the case of the von Mangoldt function (as two distinct primes are coprime). This problem is amplified by the fact that $n \mapsto \lambda_f(n)$ is not completely multiplicative (see Lemma 5.1). To circumvent this, we introduce a function λ^* (see (65)) to average out the chaotic behavior of the function λ_f (see (65)). Then the average behavior of the function λ^* is controlled thanks to the recent result of Lau and Lü [LL11] on higher moments of Fourier coefficients of Maass cusp forms. In the case where f is holomorphic, the proof is highly shortened due to Deligne's bound.

1.1 Some remarks

Remark 1. We expect both the sums (4) and (5) to be quite small, at least on average. Indeed, it is relatively easy to see that *square-root cancellations* take place in both the sums in the mean-square sense. By the Parseval formula and the Rankin–Selberg estimate (see (20)) it readily follows that

$$\int_0^1 |\mathcal{M}_f(X,\alpha)|^2\,d\alpha \leqslant \sum_{1\leqslant n\leqslant X} |\nu_f(n)|^2 \ll_f X,$$

and similarly for $\mathcal{P}_f(X,\alpha)$. Using a simple observation of Oesterlé (see [MS02, § 1]) we can even get the pointwise bound

$$\mathcal{M}_f(X,\alpha), \quad \mathcal{P}_f(X,\alpha) \ll_{\alpha,\epsilon,f} X^{(1/2)+\varepsilon}$$

for any $\varepsilon > 0$, for almost all α (in the sense of Lebesgue measure). Recall the famous theorem of Carleson [Car66] which says that if (c_n) is a sequence of complex numbers satisfying $\sum_{n=1}^{\infty} |c_n|^2 < \infty$, then the Fourier series $\sum_{n=1}^{\infty} c_n e(n\alpha)$ converges for almost all real α. Now the Rankin–Selberg estimate (20) and partial summation allows us to apply the theorem to the

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sequence $c_n = \nu_f(n)/n^{1/2+\varepsilon}$, where $\varepsilon > 0$ is arbitrary, and draw the desired conclusion. Of course, this line of arguments does not give any non-trivial bound for any specific value of α.

Remark 2. Regarding the sum appearing in (5), it turns out that proving mere orthogonality between $(\mu(n))$ and the sequence $(\nu_f(n)e(n\alpha))$ is not very difficult. Indeed, bounds of the type

$$\sum_{1 \leqslant n \leqslant X} |\lambda_f(n)| \ll_f X(\log X)^{-\delta}$$

for some $0 < \delta \leqslant 1$ for normalized Hecke eigenvalues $\lambda_f(n)$ of holomorphic forms f have been known for quite some time. See, for example, [EMS84, Mur85, Ran85]. For Maass forms also, one can easily conclude that

$$\sum_{1 \leqslant n \leqslant X} |\lambda_f(n)| = o(X)$$

as $X \longrightarrow \infty$ from [Hol09, (66)] and [Ell80, Theorem 2]. Orthogonality follows from this bound and (20). However, as the lower bound (25) shows, it is not possible to save an arbitrary large power of logarithm in the above sum. The situation is exactly similar for the sum over primes.

1.2 Notation and conventions

We follow the well-known notation and conventions described below:

- $d(n)$ denotes the number of divisors of the integer n, $d_3(n)$ is the number of ways of writing $n = n_1n_2n_3$, where the n_i are positive integers. The number of prime divisors of n is $\omega(n)$ and $\varphi(n)$ denotes the number of moduli coprime to n;
- (m, n) and $[m, n]$ denote the g.c.d and the l.c.m. of integers m and n;
- ε denotes a positive unspecified real number, different in different occurrences;
- in asymptotic formulae of the form $A(X) = B(X) + O_\beta(C(X))$ or $A(X) \ll_\beta B(X)$ the suffix β signifies the dependence of the implied constant on some parameter β which is fixed with respect to the variable X. However, dependence of various parameters will sometimes be suppressed when it is either not important for our purpose or is clear from the context;
- $w \sim W$ denotes $W < w \leqslant 2W$.

2. Background on Maass forms

2.1 Maass forms

This section contains a very brief account of the theory of Maass forms based primarily on [DFI02, §§ 4–6]. See also [Bum98, § 2.1]. One of our aims is to explain the embedding of the holomorphic modular forms in the space of Maass forms so that we can give a unified proof of our result. Although we shall work only with forms of level one, we consider a general level q in this section.

Let k be an integer, q a positive integer, and χ, a Dirichlet character modulo q that satisfies the consistency condition $\chi(-1) = (-1)^k$. Such a character gives rise to a character of the Hecke congruence group $\Gamma_0(q)$ by declaring $\chi(\gamma) = \chi(d)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q)$. For $z \in \mathbb{H}$, the upper half plane, we set

$$j_\gamma(z) := (cz + d)|cz + d|^{-1} = e^{i \arg(cz+d)}.$$

A function $f : \mathbb{H} \longrightarrow \mathbb{C}$ that satisfies the condition

$$f(\gamma z) = \chi(\gamma) j_\gamma(z)^k f(z)$$

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for all $\gamma \in \Gamma_0(q)$ is called an *automorphic function* of weight k, level q, and character (also called *nebentypus*) χ. The Laplace operator of weight k is defined by

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x},$$

and a smooth automorphic function f as above that is also an eigenfunction of the Laplace operator; i.e., $(\Delta_k + \lambda)f = 0$ for some complex number λ, is called a *Maass form* of corresponding weight, level, character, and Laplace eigenvalue λ. One writes $\lambda(s) = s(1-s)$ and $s = 1/2 + ir$, with $r, s \in \mathbb{C}$, r being known as the *spectral parameter*. It is related to the Laplace eigenvalue λ by the equation

$$\lambda = \tfrac{1}{4} + r^2. \tag{12}$$

Beware that some authors define 'Maass forms' to be what are Maass forms of weight zero in our setting. One can show that $\lambda(|k|/2)$ is the lowest eigenvalue of $-\Delta_k$ and if $k \geqslant 0$ (respectively $k \leqslant 0$) and f is a Maass form with this lowest eigenvalue, then the Cauchy–Riemann equation shows that $y^{-k/2}f(z)$ (respectively $y^{k/2}\overline{f(z)}$) is a holomorphic function. These holomorphic functions are actually the classical *modular forms* (see [DFI02, § 4]). A fact that we require is that the Laplace eigenvalue $\lambda(s) = s(1-s)$ of a Maass cusp form which is not induced from a holomorphic form must satisfy (see [DFI02, Corollary 4.4])

$$\Re s = \tfrac{1}{2} \quad \text{or} \quad 0 < s < 1. \tag{13}$$

However, the *Selberg eigenvalue conjecture* asserts that the latter case never occurs (see § 2.4 also).

2.2 Normalizations of Fourier coefficients

Given a holomorphic cusp form F with a Fourier expansion at the cusp at ∞ of the form

$$F(z) = \sum_{n\geqslant 1} a_F(n)e(nz),$$

we define the *normalized Fourier coefficients* of a holomorphic cusp form F to be

$$\psi_F(n) = a_F(n)/n^{(k-1)/2}, \tag{14}$$

where k is the weight of F.

Now we come to Maass forms. We consider Maass cusp forms only. See [DFI02, § 4] for the definition. We shall denote the space of Maass forms of level q, weight k, and character $\chi \pmod q$ by $\mathcal{C}_k(q,\chi)$. A form in this space admits Fourier expansion at the cusp at ∞ in terms of Whittaker functions $W_{\alpha,\beta}$ as follows (see [DFI02, (5.1)])

$$f(z) = \sum_{n\neq 0} \rho_f(n)W_{kn/2|n|,ir}(4\pi|n|y)e(nx),$$

where r is the spectral parameter. When we speak of Maass cusp forms, we shall always assume that they have norm one, i.e., $\langle f, f\rangle = 1$ (see [DFI02, (4.37)]). We define the *normalized Fourier coefficients* of a Maass cusp form f (see [Iwa95, ch. 8]) by

$$\nu_f(n) := \left(\frac{4\pi|n|}{\cosh \pi r}\right)^{1/2}\rho_f(n) \tag{15}$$

provided f is not induced from a holomorphic form, i.e., the Laplace eigenvalue of f is not $\lambda(|k|/2)$. Note that if f is such a Maass cusp form, then by (13), the spectral parameter r

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satisfies $r \in \mathbb{R}$ or $0 < \frac{1}{2} + ir < 1$, and therefore,

$$\pi^{-1} \cosh \pi r = \Gamma(1/2 + ir)^{-1} \Gamma(1/2 - ir)^{-1} \neq 0.$$

Now we consider Maass cusp forms which are induced from the holomorphic modular forms. Let F be a holomorphic form of weight $k \geqslant 0$. The Fourier coefficients of F are related to the coefficients $\rho_f(n)$ where f is the Maass cusp form associated to F in the following way

$$f(z) = y^{k/2} F(z) \quad \text{or} \quad f(z) = y^{k/2} \overline{F}(z).$$

In the first case, the weight of the induced Maass form is k and in the second, it is $-k$. We know that in both cases the Laplace eigenvalue is $\lambda(k/2)$ and thus the spectral parameter is given by $r = -i(k-1)/2$. Now the Whittaker function has the property (see [DFI02, (4.21)]) that $W_{\alpha,\alpha-1/2}(y) = y^{\alpha} e^{-y/2}$. Using this fact, we infer (see (14)) that

$$f(z) = y^{k/2} F(z), \quad \rho_f(n) = \frac{a_F(n)}{(4\pi n)^{k/2}} = \frac{\psi_F(n)}{n^{1/2}(4\pi)^{k/2}}$$

for $n \geqslant 1$, and $\rho_f(n) = 0$ for $n \leqslant 0$. Similarly, when $f(z) = y^{k/2}\overline{F}(z)$, we have $\rho_f(n) = \overline{a_F(n)}/(4\pi n)^{k/2} = \overline{\psi_F(n)}/n^{1/2}(4\pi)^{k/2}$ for $n \geqslant 1$, and $\rho_f(n) = 0$ for $n \leqslant 0$. Accordingly, for $f(z) = y^{k/2}F(z)$ (respectively $f(z) = y^{k/2}\overline{F(z)}$) where F is a holomorphic cusp form, we define $\nu_f(n) = \psi_F(n)/(4\pi)^{(k-1)/2}$ (respectively $\overline{\psi_F(n)}/(4\pi)^{(k-1)/2}$) for $n \geqslant 1$ and $\nu_f(n) = 0$ otherwise.

2.3 Hecke operators

The definition of the nth Hecke operator $T_{n,\chi}$, $n \geqslant 1$ acting on the space of modular forms of level q, weight k, and character $\chi \pmod q$ is given by

$$T_{n,\chi} : F(z) \mapsto (T_{n,\chi}F)(z) = \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{b \pmod d} F\left(\frac{az+b}{d}\right).$$

For an eigenfunction F of T_n, we shall denote the eigenvalue by $\lambda_F(n)$. If F is a *primitive form* (i.e., newform) then its Fourier coefficients $a_F(n)$ are related to the eigenvalues $\lambda_F(n)$ by

$$a_F(n) = a_F(1)\lambda_F(n), \tag{16}$$

and, moreover, $a_F(1) \neq 0$. Hence, the Fourier coefficients and the Hecke eigenvalues coincide up to a multiplicative factor that depends only on the form F. We define the action of the nth Hecke operator $T'_{n,\chi}$ on $\mathcal{C}_k(q,\chi)$ by (see [DFI02, ch. 6])

$$T'_{n,\chi} : f(z) \mapsto (T'_{n,\chi}f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \chi(a) \sum_{b \pmod d} f\left(\frac{az+b}{d}\right).$$

Note that this definition is independent of the weight k. The Hecke theory for Maass forms is parallel to the theory for modular forms and an important fact is that there is an orthonormal basis (called Hecke basis) of Maass cusp forms consisting of forms that are common eigenfunctions of the Hecke operators $T'_{n,\chi}$ with $(n,q) = 1$. The forms in a Hecke basis will be called *Hecke–Maass cusp forms*. A Hecke–Maass cusp form in the new subspace (consisting of forms that are not linear combinations of forms induced from lower levels) is called a *newform* or a *primitive form*. Note that a Hecke–Maass cusp form of level one is trivially a primitive form. The Hecke eigenvalue $\lambda_f(n)$ and the normalized Fourier coefficient $\nu_f(n)$ of a Hecke–Maass cusp form are related by

$$\nu_f(\pm n) = \nu_f(\pm 1)\lambda_f(n); \quad n \geqslant 1. \tag{17}$$

Moreover, for a Hecke–Maass cusp form f which is not induced from a holomorphic form, we have the relation $\nu_f(-1) = \varepsilon_f \nu_f(1)$, where $\varepsilon_f = 1$ or -1 and the form f is accordingly called *even* or *odd*. The following proposition is easy to check.

Proposition 2.1. *Suppose F is a holomorphic cusp form of weight k, level q and character $\chi \,(\mathrm{mod}\, q)$ and let $f(z) = y^{k/2}F(z)$ (respectively $f(z) = y^{k/2}\overline{F}(z)$) be the associated Maass cusp form in $\mathcal{C}_k(q,\chi)$ (respectively $\mathcal{C}_{-k}(q,\chi)$) with Laplace eigenvalue $\lambda(k/2)$. Then F is an eigenfunction of the nth Hecke operator if and only if f is. Moreover, the nth Hecke eigenvalues $\lambda_F(n)$ and $\lambda_f(n)$ of F and f respectively are related by*

$$\lambda_f(n) = \frac{\lambda_F(n)}{n^{(k-1)/2}} \quad \biggl(\text{respectively } \frac{\overline{\lambda_F(n)}}{n^{(k-1)/2}}\biggr).$$

By the above proposition, (16) and (17), for any primitive Maass cusp form f, whether or not it is induced from a holomorphic form, we have that

$$\nu_f(n) = \nu_f(1)\lambda_f(n)$$

for $n \geqslant 1$ and $\nu_f(1) \neq 0$. Hence, for any fixed primitive Maass cusp form f, the normalized Fourier coefficients $\nu_f(n)$ for $n \geqslant 1$ and the Hecke eigenvalues $\lambda_f(n)$ are the same up to multiplication by a nonzero constant. *From now on, whenever we talk of primitive forms we mean primitive Maass cusp forms with the understanding that holomorphic modular forms are included in them.*

2.4 The Ramanujan conjecture

The general Ramanujan conjecture asserts that for a primitive Maass cusp form $f \in \mathcal{C}_k(q,\chi)$ and a prime p, $p \nmid q$, the Hecke eigenvalue $\lambda_f(p)$ satisfies the bound

$$|\lambda_f(p)| \leqslant 2. \tag{18}$$

Although this conjecture is wide open, we know from the works of Kim and Shahidi, Kim, and Kim and Sarnak [Kim03, KS02a, KS02b] that

$$|\lambda_f(p)| \leqslant 2p^{7/64}. \tag{19}$$

For forms induced from holomorphic forms, the Ramanujan conjecture is a famous theorem due to Deligne. A related conjecture concerns the size of the Laplace eigenvalues λ. Indeed, the Selberg eigenvalue conjecture, which says that for Maass cusp forms of weight zero, the spectral parameter r should always be real (see (12)), can be interpreted as the Ramanujan conjecture for the infinite prime. If Selberg's conjecture is true, then we must have $\lambda \geqslant 1/4$. If this is not the case, then (12) implies that r is purely imaginary with $|r| < 1/2$. Even though we do not know the truth of the Selberg conjecture, the work of Kim and Sarnak cited above also gives the bound $|r| \leqslant 7/64$ if such exceptional eigenvalues $\lambda < 1/4$ do actually occur.

3. Moments of Hecke eigenvalues

For a fixed Hecke–Maass cusp form f, we require bounds for sums of the type $\sum_{1\leqslant n\leqslant X} |\lambda_f(n)|^{2j}$. Rankin [Ran39] and Selberg [Sel40] had independently treated similar sums in the case of holomorphic forms for $j = 1$. We can use standard tools of analytic number theory coupled with knowledge of analytic properties of higher degree L-functions to bound such moments. Works of Gelbart and Jacquet [GJ78], and of Kim and Shahidi [KS02a, KS02b] are sufficient to prove the following theorem.

Theorem A. *Let f be a Hecke–Maass cusp form for the group* $\mathrm{SL}(2,\mathbb{Z})$*. We have, for any $X \geqslant 1$, the equality*

$$\sum_{1\leqslant n\leqslant X} |\lambda_f(n)|^2 = C_f X + O_f(X^{3/5}), \tag{20}$$

where $C_f > 0$ is a constant that depends only on the form f and the same is true for the implied constant. For $j = 2, 3$, and 4, we have,

$$\sum_{1\leqslant n\leqslant X} |\lambda_f(n)|^{2j} = X P_{f,j}(\log X) + O_f(X^{c_j+\varepsilon}) \tag{21}$$

for any $\varepsilon > 0$. Here the c_j are explicit constants strictly smaller than one and the $P_{f,j}$ are polynomials of degree $1, 4$, and 13 respectively and their coefficients depend on f.

The first one is the well-known Rankin–Selberg estimate and a detailed proof of (21) with explicit numerical constants appears in [LL11]. See, in particular, [LL11, Remark 1.7] and its proof at the end of that paper. Note that they only consider what is defined as a weight zero Maass cusp form here but their proof works for general Hecke–Maass forms on $\mathrm{SL}(2,\mathbb{Z})$ of any weight. This can be seen by noting that the shape of the L-function and the Gamma factors remain the same (see [DFI02, (8.17)]) if we take the more general definition of Maass form as considered here. We note the following obvious corollary of (20) which will be required later. It can be improved slightly (by a fractional exponent of $\log X$) as mentioned in Remark 2 in the introduction.

Corollary 3.1. *For any Hecke–Maass cusp form f for the group* $\mathrm{SL}(2,\mathbb{Z})$*, and any $X \geqslant 1$, we have*

$$\sum_{1\leqslant n\leqslant X} |\lambda_f(n)| \ll_f X, \tag{22}$$

where the implied constant depends only on f.

3.1 Moments of Hecke eigenvalues at primes

The following bound on the second moment of the Hecke eigenvalues at primes is a consequence of PNT for the Rankin–Selberg L-function $L(s, f\otimes f)$. See, for example, [LY05, LY07], [LWY05, Corollary 1.2 and Lemma 5.1]. Similar results were obtained by Rankin [Ran73] and Perelli [Per82] in the context of holomorphic forms.

Theorem B. *For a Hecke–Maass cusp form f for the group* $\mathrm{SL}(2,\mathbb{Z})$*, we have the bound*

$$\sum_{1\leqslant n\leqslant X} \Lambda(n)|\lambda_f(n)|^2 \ll_f X,$$

for any $X \geqslant 2$.

Note that if f was a holomorphic form then the theorem would follow trivially from PNT and Deligne's bound on Hecke eigenvalues.

From the above theorem, we deduce the following result.

Corollary 3.2. *For a Hecke–Maass cusp form f for the group* $\mathrm{SL}(2,\mathbb{Z})$*, we have the estimates*

$$\sum_{1\leqslant p\leqslant X} |\lambda_f(p)|\log p \ll_f X, \tag{23}$$

and

$$\sum_{1 \leqslant p \leqslant X} |\lambda_f(p)| \ll_f X/\log X, \tag{24}$$

for any $X \geqslant 2$.

We also need a lower bound for the above sum and we follow the approach of Holowinsky [Hol09, § 4.1] in proving the following proposition. See [Ran85, Wu09, WX] for more precise results in this direction.

Proposition 3.1. *For a Hecke–Maass cusp form* f *of level one we have the bound*

$$\sum_{1 \leqslant p \leqslant X} |\lambda_f(p)| \gg_f X/\log X, \tag{25}$$

for all X *sufficiently large.*

Proof. We start with a polynomial of the form

$$f(x) = c_0 + c_1(x^2 - 1) + c_2(x^4 - 2) + c_3(x^6 - 5),$$

where the c_i are real, $c_0 > 0$, and $f(x)$ satisfies $f(x) \leqslant |x|$ for all real values of x. For example, one can check that the polynomial

$$f(x) = 0.01 + (0.09)(x^2 - 1) + (0.1)(x^4 - 2) - (0.05)(x^6 - 5)$$

satisfies all the conditions. Now, for each prime p, we put $x = \lambda_f(p)$ and then sum over them. The following relations are consequences of Hecke's formula (63).

For any prime p, we have

$$\begin{aligned}
&\lambda_f(p)^2 - 1 = \lambda_f(p^2),\\
&\lambda_f(p)^4 - 2 = \lambda_f(p^4) + 3\lambda_f(p^2),\\
&\lambda_f(p)^6 - 5 = \lambda_f(p^6) + 5\lambda_f(p^4) + 9\lambda_f(p^2).
\end{aligned}$$

Now note that $\lambda_f(p^j)$ is the pth coefficient of the jth symmetric power L-function $L(s, \mathrm{sym}^j f)$. By facts known about symmetric power L-functions, it follows (see, for example, [Bru03, (2.23)]) that

$$\sum_{p \leqslant X} \lambda_f(p^j) = o(X/\log X)$$

as $X \longrightarrow \infty$ for $1 \leqslant j \leqslant 8$. Therefore, by the above comments and PNT, we have the bound (25). ◻

4. The prime number theorem

4.1 Statements of the theorems

Our goal in this section is to obtain non-trivial bounds for the sums $\sum_{p \leqslant X} \lambda_f(p)\chi(p)$ and $\sum_{n \leqslant X} \mu(n)\lambda_f(n)\chi(n)$, where χ is a Dirichlet character modulo q and f is a Hecke–Maass cusp form of level one. This will play an important role in the proof of the main theorem (see § 7.1). Recall that $\sum_{n=1}^{\infty} \lambda_f(n)\chi(n)e(nz)$ is a primitive cusp form of level q^2, provided $\chi \pmod q$ is primitive (see [Iwa97, § 7.3], [Li75, Theorem 9], and [CI00, § 4, Remarks]). To see that the twisted form is an eigenfunction of the Laplace operator, one notes that the Laplace operator commutes with the *slash operator* (see [DFI02, § 4]). It is natural at this point to apply PNT for L-functions on GL(2) to estimate the above sums. A famous result due to Hoffstein and Ramakrishnan [HR95, Theorem C, part (3)] says the following.

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Theorem C. *There is an effectively computable absolute constant $c > 0$ such that for any primitive form f of some level q, spectral parameter r, and weight k, the L-function $L(s, f)$ does not vanish in the region*

$$\sigma \geqslant 1 - \frac{c}{\log(q(|t| + |r| + 2))}.$$

Now [IK04, Theorem. 5.13], more specifically formula (5.52), leads to the following, taking into account the absence of the exceptional zero.

Theorem D. *Let f be a primitive Maass cusp form of some level q, spectral parameter r, and weight k. For any $X \geqslant 2$, we have*

$$\sum_{p \leqslant X} \lambda_f(p) \log p \ll \sqrt{q(|r| + 3)}X \exp\bigg(-\frac{c}{2}\sqrt{\log X}\bigg), \tag{26}$$

where the implied constant is absolute and c is as in the previous theorem.

If f is a Hecke–Maass cusp form on $\mathrm{SL}(2, \mathbb{Z})$ and $\chi \pmod q$ is a primitive Dirichlet character, then applying the above theorem to the twisted form $f \otimes \chi$ we get the estimate

$$\sum_{p \leqslant X} \lambda_f(p)\chi(p) \log p \ll q\sqrt{(|r| + 3)}X \exp\bigg(-\frac{c}{2}\sqrt{\log X}\bigg), \tag{27}$$

where the implied constant is absolute. Apparently, it is not possible to deduce from (27) a similar bound for the sum $\sum_{1 \leqslant n \leqslant X} \lambda_f(n)\mu(n)\chi(n)$ by the combinatorial device presented in the proof of [IK04, Corollary 5.29]. So we shall prove from scratch the following theorem.

Theorem 4.1. *Let f be any Hecke–Maass cusp form for the full modular group and let $\chi \pmod q$ be any Dirichlet character. Let $X \geqslant 2$. Then we have,*

$$\sum_{p \leqslant X} \lambda_f(p)\chi(p) \log p \ll_f \sqrt{q}X \exp(-c_1\sqrt{\log X}) \tag{28}$$

and

$$\sum_{n \leqslant X} \lambda_f(n)\mu(n)\chi(n) \ll_f \sqrt{q}X \exp(-c_1\sqrt{\log X}), \tag{29}$$

where the implied constant depends only on the form f and $c_1 = \sqrt{c}/10$, where c is the same absolute constant that appears in Theorem C.

4.2 Idea of the proof

We prove the second bound (29) only as this is the harder one and we follow the classical method using the Perron formula and Dirichlet series. To prove it, we need to give a good bound for the associated Dirichlet series $M(s, f \otimes \chi)$ (see (52)) in the zero-free region. This is the content of Lemma 4.5. To obtain this bound, we first relate it to the reciprocal of the L-function $L(s, f \otimes \chi)$ (see (54)). Now a suitable bound for the reciprocal of the L-function follows from a similar bound for the logarithmic derivative of the L-function and this is done in the proof of Lemma 4.3. Thus we are reduced to bounding the logarithmic derivative of the L-function which is done in the proof of Lemma 4.1 using standard techniques from complex analysis. The proof of Lemma 4.1 also requires a uniform lower bound of the Euler factors and this is the content of Lemma 4.4.

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It is clear that the proof of (28) will be similar and the only difference will be that instead of $M(s, f \otimes \chi)$, we shall have to work with the logarithmic derivative of $L(s, f \otimes \chi)$, the required bound of which is established in Lemma 4.1. We prove the lemmas mentioned above in the next subsection. First we introduce some notation valid for this section only. We shall write Ω to denote the region in the complex plane given by

$$\Omega = \left\{\sigma + it : \sigma \geqslant 1 - \frac{c}{6\mathcal{L}}\right\},$$

and $\mathcal{L}$ to denote

$$\mathcal{L} := \log(q(|t| + |r| + 2)). \tag{30}$$

4.3 Preparatory lemmas

First we start by estimating the logarithmic derivative of the L-function.

LEMMA 4.1. *Let f and χ be as in Theorem 4.1. Let c be the constant appearing in Theorem C. Then, for every $s \in \Omega$, we have*

$$\frac{L'(s, f \otimes \chi)}{L(s, f \otimes \chi)} \ll_f \mathcal{L}, \tag{31}$$

where the implied constant depends only on the form f.

To prove this lemma, we first recall a consequence of the Borel–Carathéodory theorem (see [Tit86, § 3.9, Lemma α]).

LEMMA 4.2. *Let $s_0 \in \mathbb{C}$, $r > 0$ and U an open set containing the disk $\{s : |s - s_0| \leqslant r\}$. Let $M \geqslant 1$ and h a holomorphic function on U, satisfying $h(s_0) \neq 0$ and the inequality*

$$\left|\frac{h(s)}{h(s_0)}\right| \leqslant e^M,$$

in the disk $|s - s_0| \leqslant r$. Then, for every s satisfying the inequality $|s - s_0| \leqslant r/4$, one has the inequality

$$\left|\frac{h'(s)}{h(s)} - \sum_{\substack{\rho : h(\rho) = 0 \\ |s_0 - \rho| \leqslant r/2}} \frac{1}{s - \rho}\right| \leqslant 48\frac{M}{r}.$$

Now we prove Lemma 4.1.

Proof. We consider two cases separately: χ is primitive and otherwise.

Case 1: *χ is a primitive character.* We first suppose that χ is a primitive character modulo q. Then we know that $f \otimes \chi$ is a primitive Maass cusp form of level q^2. The L-function attached to f is

$$L(s, f) = \sum_n \frac{\lambda_f(n)}{n^s} = \prod_p (1 - \lambda_f(p)p^{-s} + p^{-2s})^{-1},$$

and the L-function attached to the twisted form $f \otimes \chi$ is

$$L(s, f \otimes \chi) = \sum_n \frac{\lambda_f(n)\chi(n)}{n^s} = \prod_p L_p(s, f \otimes \chi)^{-1} \tag{32}$$

where the local factor is

$$L_p(s, f \otimes \chi) = (1 - \lambda_f(p)\chi(p)p^{-s} + \chi^2(p)p^{-2s}). \tag{33}$$

By (22), the infinite series and the Euler product appearing in (32) are absolutely convergent for $\sigma > 1$. We know from the theory of automorphic L-functions that the function $L(s, f \otimes \chi)$ has an analytic continuation to the whole complex plane and satisfies a functional equation relating the values at s and $1-s$ and has a polynomial growth in the critical strip, i.e., for some absolute constant A, one has the bound

$$L(s, f \otimes \chi) \leqslant e^{A\mathcal{L}}, \tag{34}$$

uniformly for $\sigma \geqslant 1/2$ (see [IK04, (5.20)]). Taking the logarithmic derivatives of (32), we have for $\sigma > 1$ the equality

$$-\frac{L'(s, f \otimes \chi)}{L(s, f \otimes \chi)} = \sum_p \frac{\lambda_f(p)\chi(p)(\log p)p^{-s} - 2\chi^2(p)(\log p)p^{-2s}}{1 - \lambda_f(p)\chi(p)p^{-s} + \chi^2(p)p^{-2s}}. \tag{35}$$

We take a point $s = \sigma + it$ in the region Ω. We shall consider t as fixed and develop different arguments according to the value of σ. We first assume that

$$\Re s = \sigma \geqslant 1 + \frac{1}{100}. \tag{36}$$

Then, the inequality (22) (with $q = 1$) combined with (35) easily shows

$$\frac{L'(s, f \otimes \chi)}{L(s, f \otimes \chi)} \ll_f 1,$$

uniformly for s satisfying (36). We now suppose that s satisfies

$$1 + \frac{c}{10\mathcal{L}} \leqslant \sigma \leqslant \frac{101}{100}. \tag{37}$$

Since $L'(s, f \otimes \chi)/L(s, f \otimes \chi)$ converges absolutely in the region $\Re s > 1$ (see (22) and (35)) we expand it in Dirichlet series

$$-\frac{L'(s, f \otimes \chi)}{L(s, f \otimes \chi)} = \sum_{n \geqslant 1} \frac{\Lambda_{f \otimes \chi}(n)}{n^s} \tag{38}$$

(see [IK04, (5.25)]). The support of the function $\Lambda_{f\otimes\chi}$ is included in the set of powers of primes. We deduce the inequality

$$\left| -\frac{L'(s, f \otimes \chi)}{L(s, f \otimes \chi)} \right| \leqslant \sum_p \frac{|\lambda_f(p)| \log p}{p^\sigma} + O(1),$$

the contribution from the higher powers of primes being absorbed in the $O(1)$ term thanks to the Kim–Sarnak bound (19). Applying (23) to the above sum via partial summation, we get the inequalities

$$\begin{aligned} -\frac{L'(s, f \otimes \chi)}{L(s, f \otimes \chi)} &\ll_f \frac{1}{\sigma - 1} + 1 \\ &\ll_f \mathcal{L}, \end{aligned} \tag{39}$$

uniformly for s satisfying (37) and thus the bound (31) for s in that region.

The imaginary part t being fixed all the time, we consider the three points

$$s = \sigma + it, \quad s_1 = 1 + \frac{c}{10\mathcal{L}} + it, \quad s_0 = \frac{101}{100} + it, \tag{40}$$

where σ satisfies

$$1 - \frac{c}{6\mathcal{L}} \leqslant \sigma < 1 + \frac{c}{10\mathcal{L}} := \sigma_1. \tag{41}$$

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We plan to apply Lemma 4.2 twice to the function $h(s) = L(s, f \otimes \chi)$ at the point s_0 and $r = 1/2$. Note that, uniformly over t, one has $h(s_0) \asymp 1$ by the Dirichlet series and the Euler product expression (32). By (34), we can choose $M \ll \mathcal{L}$, where the implied constant is absolute. So we can write the two equalities

$$\frac{L'(s, f \otimes \chi)}{L(s, f \otimes \chi)} = \sum_{\substack{|s_0 - \rho| < 1/4 \\ L(\rho, f \otimes \chi) = 0}} \frac{1}{s - \rho} + O(\mathcal{L}), \tag{42}$$

and

$$\frac{L'(s_1, f \otimes \chi)}{L(s_1, f \otimes \chi)} = \sum_{\substack{|s_0 - \rho| < 1/4 \\ L(\rho, f \otimes \chi) = 0}} \frac{1}{s_1 - \rho} + O(\mathcal{L}), \tag{43}$$

since we have $|s_1 - s_0| \leqslant |s - s_0| \leqslant 1/8$. Subtracting (42) from (43) and using (39) (at the point s_1) we deduce the equality

$$\frac{L'(s, f \otimes \chi)}{L(s, f \otimes \chi)} = \sum_{\substack{|s_0 - \rho| < 1/4 \\ L(\rho, f \otimes \chi) = 0}} \frac{1}{s - \rho} - \sum_{\substack{|s_0 - \rho| < 1/4 \\ L(\rho, f \otimes \chi) = 0}} \frac{1}{s_1 - \rho} + O_f(\mathcal{L}).$$

Moreover, we have the inequalities

$$\begin{aligned} \frac{1}{s - \rho} - \frac{1}{s_1 - \rho} &\ll \frac{|s - s_1|}{|s - \rho|^2} \\ &\ll \frac{1}{\mathcal{L}|s - \rho|^2} \\ &\ll \Re \frac{1}{s_1 - \rho}, \end{aligned}$$

since, by Theorem C and the definitions (40), we have the inequalities

$$|s - \rho| \gg |s_1 - \rho| \quad \text{and} \quad \Re(s_1 - \rho) \gg \mathcal{L}^{-1},$$

valid uniformly. Now we sum over the zeros ρ of $L(s, f \otimes \chi)$ with $|s_0 - \rho| < 1/4$ and apply (43) and (39) again to obtain

$$\frac{L'(s, f \otimes \chi)}{L(s, f \otimes \chi)} \ll_f \mathcal{L}. \tag{44}$$

This gives (31) when χ is primitive.

Case 2: χ *is not primitive.* We suppose that the Dirichlet character χ modulo q is induced by a primitive character χ^* modulo q^*. From the equality

$$L(s, f \otimes \chi) = L(s, f \otimes \chi^*) \prod_{p \mid q, p \nmid q^*} L_p(s, f \otimes \chi^*),$$

we deduce the following equality between logarithmic derivatives

$$-\frac{L'(s, f \otimes \chi)}{L(s, f \otimes \chi)} = -\frac{L'(s, f \otimes \chi^*)}{L(s, f \otimes \chi^*)} + O\bigg(\sum_{p \mid q, p \nmid q^*} \frac{|\lambda_f(p)| \log p}{p^\sigma} \bigg) + O(1),$$

where, for the second term on the right-hand side, we use a uniform lower bound for $|L_p(s, f \otimes \chi^*)|$ for $\sigma \geqslant 99/100$ and this will be proved in Lemma 4.4 below. Using (19) once more, we have

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the equality

$$-\frac{L'(s,f\otimes\chi)}{L(s,f\otimes\chi)} = -\frac{L'(s,f\otimes\chi^*)}{L(s,f\otimes\chi^*)} + O\biggl(\sum_{p\mid q} p^{-3/4}\biggr) + O(1)$$
$$= -\frac{L'(s,f\otimes\chi^*)}{L(s,f\otimes\chi^*)} + O(\log^{1/4}(q+1)),$$

uniformly for $\sigma \geqslant 99/100$. Combining with (44), we complete the proof of Lemma 4.1 in all the cases. □

4.4 Bounds for L and L^{-1} inside Ω

From Lemma 4.1, we now deduce upper bounds for L, L^{-1} and some allied functions inside Ω.

Lemma 4.3. *Under the conditions of Lemma 4.1, we have the uniform bound*

$$L(s,f\otimes\chi) \quad \text{and} \quad L^{-1}(s,f\otimes\chi) \ll_f \mathcal{L},$$

for all $s\in\Omega$ where the implied constant depends only on f.

Proof. Let s and s_1 be as in (40) and we first suppose that σ satisfies (41). Integrating the bound given by Lemma 4.1 between s_1 and s, we obtain the inequality

$$\log L(\sigma_1+it,f\otimes\chi) - \log L(\sigma+it,f\otimes\chi) \ll_f 1. \tag{45}$$

To bound $|L(s_1,f\otimes\chi)|$ from above, we use the Dirichlet series expression (32) to write

$$|L(s_1,f\otimes\chi)| \leqslant \sum_{n\geqslant 1}\frac{|\lambda_f(n)|}{n^{\sigma_1}}$$
$$\ll_f \mathcal{L}, \tag{46}$$

using the estimate (22) and partial summation.

To bound $|L(s_1,f\otimes\chi)|^{-1}$ from above we introduce local factors M_p defined by

$$M_p(s,f\otimes\chi) := 1 - \frac{\lambda_f(p)\chi(p)}{p^s} \tag{47}$$

for each prime p. If $\Re s \geqslant 99/100$ and $p\geqslant 3$, it easily follows from (19) that

$$M_p(s,f\otimes\chi)\neq 0 \quad \text{and} \quad L_p(s,f\otimes\chi)\neq 0. \tag{48}$$

However, we shall obtain a more precise statement concerning L_p below; namely, Lemma 4.4.

Write the function L^{-1} as

$$L^{-1}(s) = L_2(s)\biggl(\prod_{p\geqslant 3} M_p(s)\biggr)G_{\geqslant 3}(s), \tag{49}$$

with

$$G_{\geqslant 3}(s) := \prod_{p\geqslant 3}(L_p(s)/M_p(s))$$

where we voluntarily dropped the symbol $f\otimes\chi$. Computing each of the local factors, we see that the function $G_{\geqslant 3}(s)$ has an expression as an infinite product absolutely convergent for $\Re s\geqslant 99/100$; and hence $G_{\geqslant 3}$ is uniformly bounded in that region. In other words, uniformly over characters χ and for $\Re s \geqslant 99/100$, we have

$$G_{\geqslant 3}(s) \quad \text{and} \quad G_{\geqslant 3}^{-1}(s) \ll 1. \tag{50}$$

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For the second term in the right-hand side of (49), we may write

$$\Big|\prod_{p\geqslant 3} M_p(s_1)\Big| = \Big|\sum_{2\nmid n} \frac{\mu(n)\chi(n)\lambda_f(n)}{n^{s_1}}\Big| \leqslant \sum_{n\geqslant 1} \frac{|\lambda_f(n)|}{n^{\sigma_1}} \ll_f \mathcal{L}$$

by the multiplicativity of $\lambda_f(n)$ on squarefree integers and (46). Furthermore, we have $|L_2(s_1)| \leqslant 3$. Gathering all these remarks into (49), we deduce the inequality

$$|L^{-1}(s_1, f\otimes\chi)| \ll_f \mathcal{L}. \tag{51}$$

Now (46) and (51) yield

$$|\log L(\sigma_1 + it, f\otimes\chi)| \leqslant \log\mathcal{L} + O_f(1).$$

Combining this with (45) we complete the proof of Lemma 4.3 when σ satisfies (41). In the remaining case, when $\sigma > \sigma_1$, instead of using (45), we merely adapt the proof of (46) and (51) as we are in the region of absolute convergence. □

4.5 Extension to the M-function

The Dirichlet series attached to the arithmetical function appearing in the second part of Theorem 4.1 is

$$M(s, f\otimes\chi) := \sum_{n} \frac{\mu(n)\lambda_f(n)\chi(n)}{n^s}. \tag{52}$$

By (22), we know that this series converges for $\Re s > 1$. In that region, it admits an Euler product expansion

$$M(s, f\otimes\chi) = \prod_p M_p(s, f\otimes\chi), \tag{53}$$

where $M_p(s, f\otimes\chi)$ is defined in (47). The Dirichlet series $M(s, f\otimes\chi)$ is not far from $L^{-1}(s)$. More precisely, from (49) and from (53), we deduce the equality which is true for every $s\in\Omega$

$$M(s, f\otimes\chi) = L_2(s)^{-1}L^{-1}(s)M_2(s)G^{-1}_{\geqslant 3}(s). \tag{54}$$

By (50) and Lemma 4.3 we control all the terms but the first one in the region $s\in\Omega$. Now none of the local Euler factors L_p defined in (33) vanishes in the half plane $\{s : \Re s > 1\}$, otherwise the global L-function would have a pole in this region, which it does not by the general theory of automorphic L-functions. We shall now prove a uniform lower bound for these functions $|L_p|$, in particular, for $p = 2$. We have the following lemma.

Lemma 4.4. *There is an absolute constant $C_0 > 0$ such that for any Hecke–Maass cusp form f for the full modular group, any Dirichlet character $\chi \pmod q$ for any integer $q\geqslant 1$, any prime $p\geqslant 2$, and for every s such that $\Re s \geqslant 99/100$, the bound*

$$|L_p(s, f\otimes\chi)| \geqslant C_0$$

holds.

Proof. When $p\geqslant 3$, one has the inequality

$$\begin{aligned} |L_p(s, f\otimes\chi)| &\geqslant 1 - \frac{2\cdot p^{7/64}}{p^{99/100}} - \frac{1}{p^{99/50}} \\ &\geqslant 1 - 2\cdot 3^{-1409/1600} - 3^{-99/50} \\ &> 1/8 \end{aligned}$$

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by a direct application of the definition (33) and of the inequality (19). The prime 2 requires a more careful analysis. We write $z := \chi(2)/2^{\sigma+it}$, $u := \frac{1}{2}\lambda_f(2)$ and

$$L_2(s, f \otimes \chi) = 1 - 2uz + z^2 := G(u, z).$$

By self-adjointness of Hecke operators, we know that the Hecke eigenvalues, in particular, $\lambda_f(2)$ and hence u, are real. The existence of $C_0 > 0$ such that $|L_2(s, f \otimes \chi)| \geqslant C_0$ for all s with $\Re s \geqslant 99/100$ is a consequence of the inequality

$$|G(u, z)| \geqslant C_0, \tag{55}$$

for all (u, z) belonging to the set

$$\mathcal{K} := \{(u, z) \in \mathbb{R} \times \mathbb{C} : |u| \leqslant 2^{7/64}, |z| \leqslant 2^{-99/100}\},$$

(by an application of (19)). Since $\mathcal{K}$ is compact and G is a continuous function, the proof of (55) is reduced to the proof of the non-vanishing of $G(u, z)$ on $\mathcal{K}$.

Let $(u_0, z_0) \in \mathcal{K}$, satisfying $G(u_0, z_0) = 0$. We then have

$$u_0 = \frac{1}{2}\left(z_0 + \frac{1}{z_0}\right) = \frac{1}{2}\left(z_0 + \frac{\overline{z_0}}{|z_0|^2}\right).$$

This implies that z_0 is necessarily real, since $|z_0| \neq 1$.

Finally, for z real such that $|z| \leqslant 2^{-99/100}$, we have

$$\left|\frac{1}{2}\left(z + \frac{1}{z}\right)\right| \geqslant \frac{1}{2}(2^{99/100} + 2^{-99/100}) = 1.24\cdots > 2^{7/64} = 1.07\cdots.$$

This gives a contradiction. Hence G cannot vanish on $\mathcal{K}$ and (55) is proved. The proof of Lemma 4.4 is now complete. □

It remains to gather in (54) the upper bounds contained in the Lemmas 4.3 and 4.4, in formula (50), and the bound $|M_2(s)| \leqslant 3$ for $s \in \Omega$ to obtain the following lemma.

Lemma 4.5. *Under the conditions of Theorem 4.1, we have the bound*

$$M(s, f \otimes \chi) \ll_f \mathcal{L},$$

uniformly for $s \in \Omega$.

We now have all the tools to give a sketch of the proof of Theorem 4.1.

4.6 Proof of Theorem 4.1

The idea of the proof is quite standard (see for instance [IK04, Theorem 5.13]). We apply the Perron formula (see [Tit86, Lemma 3.12]) to the Dirichlet series $M(s, f \otimes \chi)$ defined in (52) and move the contour inside the zero-free region where we can give a good estimate of the function M. We use a smoothed version of the classical Perron formula using Mellin inversion. To this end, we consider a function ϕ with support on $[0, X+Y]$, such that $0 \leqslant \phi(x) \leqslant 1$ for $0 \leqslant x \leqslant X+Y$ and $\phi(x) = 0$ for $x \geqslant X+Y$. Here, Y $(1 \leqslant Y \leqslant X/2)$ is a parameter to be chosen later. To be specific, we take

$$\phi(x) = \min\left(\frac{x}{Y}, 1, 1 + \frac{X - x}{Y}\right)$$

for $0 \leqslant x \leqslant X+Y$ and

$$\phi(x) = 0$$

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elsewhere. Then the Mellin transform of ϕ satisfies (see [IK04, p. 111])

$$\hat{\phi}(s) \ll \frac{X^\sigma}{|s|}\min\left(1, \frac{X}{|s|Y}\right) \tag{56}$$

for $1/2 \leqslant \Re s \leqslant 2$. After these preliminaries, we now give the proof of the theorem.

Proof. We have

$$\begin{aligned}\sum_{n\leqslant X}\lambda_f(n)\chi(n)\mu(n) &= \sum_{n\geqslant 1}\lambda_f(n)\chi(n)\mu(n)\phi(n)\\ &\quad + O\biggl(\sum_{0<n\leqslant Y}|\lambda_f(n)|\biggr) + O\biggl(\sum_{X<n\leqslant X+Y}|\lambda_f(n)|\biggr),\end{aligned} \tag{57}$$

and also

$$\sum_{0<n\leqslant Y}|\lambda_f(n)|, \qquad \sum_{X<n\leqslant X+Y}|\lambda_f(n)| \ll_f Y \tag{58}$$

by Cauchy's inequality and the asymptotic formula (20), provided $Y \geqslant X^{3/5}$. By the Mellin inversion formula, we can write

$$\sum_{n\geqslant 1}\lambda_f(n)\chi(n)\mu(n)\phi(n) = \frac{1}{2\pi i}\int_{(2)} M(s, f\otimes\chi)\hat{\phi}(s)\, ds. \tag{59}$$

Now we move the contour of the integral to the left and deform it so that it coincides with the boundary of the region Ω. Since Ω is wholly contained in the zero-free region for $L(s, f\otimes\chi)$, we do not encounter any pole of M and thus it remains to estimate the integral over the left edge $\partial\Omega$ of Ω. We assume that q is not very large, namely

$$q \leqslant \exp(2c_1\sqrt{\log X}), \tag{60}$$

otherwise (29) is a trivial consequence of (22). Let us write $T := X/Y$, a parameter to be chosen later subject to $2 \leqslant T \leqslant X^{1/4}$. By (56), (59) and Lemma 4.5, we deduce the inequalities

$$\begin{aligned}\sum_{n\geqslant 1}\lambda_f(n)\chi(n)\mu(n)\phi(n) &\ll_f \int_{\partial\Omega}\left|\mathcal{L}\cdot\frac{X^\sigma}{|s|}\min\left(1, \frac{X}{|s|Y}\right)\right| d|s|\\ &\ll_f \left\{\int_1^{T^2}\mathcal{L}\frac{X^{\sigma(t)}}{t}\, dt + \int_{T^2}^\infty \mathcal{L}\frac{X^2}{Y}\cdot\frac{1}{t^2}\, dt\right\}\\ &\ll_f (X^{\sigma(T^2)} + Y)\log^2(q(T+|r|+2)),\end{aligned} \tag{61}$$

with

$$\sigma(t) := 1 - \frac{c}{6\mathcal{L}}$$

for t real. For the definition of $\mathcal{L}$ see (30). It remains to put this in (57), to use (58), to choose

$$T := \exp(2c_1\sqrt{\log X}),$$

and to recall the assumption (60) to finally write the inequalities

$$\begin{aligned}\sum_{n\leqslant X}\lambda_f(n)\chi(n)\mu(n) &\ll_f \left(X^{\sigma(T^2)}+\frac{X}{T}\right)\log^2(q(T+|r|+2))\\ &\ll_f X\left\{\exp\left(-\frac{c\log X}{6\log(\exp(7c_1\sqrt{\log X}))}\right)\right.\\ &\qquad\left.+\exp\left(-2c_1\sqrt{\log X}\right)\right\}\log^2(q(T+|r|+2))\\ &\ll_f X\exp(-c_1\sqrt{\log X}),\end{aligned}$$

by the definition of c_1. This completes the proof of Theorem 4.1. □

5. Hecke multiplicative functions

5.1 Hecke relation

The following relation satisfied by Hecke eigenvalues is well known. See [IK04, ch. 14] and [Iwa95, ch. 8], for instance.

Lemma 5.1. *For every m and $n \geqslant 1$, we have*

$$\lambda_f(m)\lambda_f(n)=\sum_{d\mid(m,n)}\lambda_f\left(\frac{mn}{d^2}\right). \tag{62}$$

Definition 1. We call a function $\lambda:\mathbb{N}\longrightarrow\mathbb{R}$ Hecke multiplicative if $\lambda(1)=1$ and λ satisfies the relation

$$\lambda(m)\lambda(n)=\sum_{d\mid(m,n)}\lambda\left(\frac{mn}{d^2}\right). \tag{63}$$

Here we restrict ourselves to real valued functions as this is enough for our purpose and in what follows we need positivity of λ^2. Soundararajan [Sou10] had introduced a similar definition in the context of his work on the *quantum unique ergodicity conjecture*. Note that a Hecke multiplicative function is automatically multiplicative. From (63), we can easily deduce the dual formula

$$\lambda(mn)=\sum_{d\mid(m,n)}\mu(d)\lambda\left(\frac{m}{d}\right)\lambda\left(\frac{n}{d}\right). \tag{64}$$

5.2 The $\boldsymbol{\lambda^*}$ function

Given a Hecke multiplicative function λ, we introduce a new function λ^* which can be thought of as an analogue of (the square-root of) the divisor function.

Definition 2. Let $\lambda:\mathbb{N}\to\mathbb{R}$ be an arithmetic function. We define the arithmetical function λ^* by declaring

$$\lambda^*(n)=\left(\sum_{d\mid n}\lambda^2(d)\right)^{1/2}\quad\text{for } n\geqslant 1. \tag{65}$$

Note that in the trivial case $\lambda\equiv 1$ then we have $\lambda^*(n)=\sqrt{d(n)}$ where $d(n)$ is the number of positive integers of the integer n. When λ is a Hecke multiplicative function, the associated λ^* inherits some regularity properties which justify its introduction. Here are some of these.

Lemma 5.2. *Let λ be a Hecke multiplicative function. Let m and n be any positive integers. Then the following hold:*

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(a) $\lambda^*(n) \geqslant 1$;
(b) $|\lambda(m)| \leqslant \lambda^*(m)$;
(c) *if* $m|n$, *then* $\lambda^*(m) \leqslant \lambda^*(n)$;
(d) *if* $(m,n)=1$ *then* $\lambda^*(mn) = \lambda^*(m)\lambda^*(n)$;
(e) $|\lambda(mn)| \leqslant \lambda^*(m)\lambda^*(n)$;
(f) $\lambda^*(mn) \leqslant d^{1/2}(m)d^{1/2}(n)\lambda^*(m)\lambda^*(n)$;
(g) $|\lambda(m)\lambda(n)| \leqslant d^{1/2}((m,n))\lambda^*(mn)$.

Proof. The first three assertions are trivial since $\lambda(1)=1$ and $\lambda^2(d) \geqslant 0$, for all d. The part (d) is a consequence of the fact that if $d|mn$, then d can be uniquely written as $d = d_1 d_2$ where d_1 and d_2 respectively divide m and n. We also use the relation $\lambda(ab) = \lambda(a)\lambda(b)$, when a and b are coprime. For the part (e), we use (64) to write

$$|\lambda(mn)| \leqslant \sum_{d|(m,n)} \left|\lambda\left(\frac{m}{d}\right)\lambda\left(\frac{n}{d}\right)\right| \leqslant \left(\sum_{d|(m,n)} \lambda^2\left(\frac{m}{d}\right)\right)^{1/2} \cdot \left(\sum_{d|(m,n)} \lambda^2\left(\frac{n}{d}\right)\right)^{1/2},$$

hence the result by extending summation. In the case of (f), we write

$$\begin{aligned}
\lambda^{*2}(mn) &= \sum_{d|mn} \lambda^2\left(\frac{mn}{d}\right) \\
&\leqslant \sum_{d_1|m}\sum_{d_2|n} \lambda^2\left(\frac{m}{d_1}\cdot\frac{n}{d_2}\right) \\
&\leqslant \sum_{d_1|m}\sum_{d_2|n} \lambda^{*2}\left(\frac{m}{d_1}\right)\cdot\lambda^{*2}\left(\frac{n}{d_2}\right) \\
&\leqslant \sum_{d_1|m} \lambda^{*2}(m) \cdot \sum_{d_2|n} \lambda^{*2}(n) \\
&\leqslant d(m)d(n)\lambda^{*2}(m)\lambda^{*2}(n),
\end{aligned}$$

by (e) and (c). For (g), we write by (63), the inequalities

$$\begin{aligned}
|\lambda(m)\lambda(n)| &\leqslant \sum_{d|(m,n)} \left|\lambda\left(\frac{mn}{d^2}\right)\right| \\
&\leqslant \left(\sum_{d|(m,n)} 1\right)^{1/2} \cdot \left(\sum_{d|(m,n)} \lambda^2\left(\frac{mn}{d^2}\right)\right)^{1/2} \\
&\leqslant d^{1/2}((m,n))\lambda^*(mn),
\end{aligned}$$

by the Cauchy–Schwarz inequality and extending summation. □

5.3 Moments of $\lambda^*(n)$

The divisor function $d(n)$ satisfies nice bounds if we sum its powers over an interval. Indeed, for any positive integer A, we have, for $X \geqslant 1$,

$$\sum_{n \leqslant X} d^A(n) \ll_A X(\log X)^{2^A - 1}. \tag{66}$$

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For this classical bound see [MV07, p. 61] for instance. The function λ^* also displays similar regularity and it is reasonable to expect that moments of λ^* should be of the same size as corresponding moments of λ (up to log factors). With a specific application in mind, we prove a particular case of this regularity.

Proposition 5.1. *Suppose a Hecke multiplicative function λ satisfies the bound*

$$\sum_{m\leqslant M} \lambda^6(m) \ll_\lambda M(\log M)^4 \tag{67}$$

uniformly for all $M \geqslant 2$. Then for any positive integer A, there is some integer $A_1 = A_1(A)$, such that, uniformly for $X \geqslant 2$, one has the estimate

$$\sum_{m\leqslant X} d^A(m)\lambda^{*4}(m) \ll X(\log X)^{A_1}, \tag{68}$$

where the implied constant depends only on λ and A.

Proof. Throughout the proof we denote by A_1 some unspecified but effective function of A. The value of A_1 may be different in different occurrences. By the definition (65) of the function λ^*, one has the equality

$$\begin{aligned}\sum_{m\leqslant X} d^A(m)\,\lambda^{*4}(m) &= \sum_{m\leqslant X} d^A(m)\Big(\sum_{d\mid m}\lambda^2(d)\Big)^2\\ &= \sum_{d_1,d_2}\sum \lambda^2(d_1)\lambda^2(d_2) \sum_{\substack{m\leqslant X\\ [d_1,d_2]\mid m}} d^A(m),\end{aligned} \tag{69}$$

where $[d_1, d_2]$ is the least common multiple of d_1 and d_2. Using the inequality

$$d(ab) \leqslant d(a)d(b), \tag{70}$$

and (66), we transform (69) into

$$\sum_{m\leqslant X} d^A(m)\lambda^{*4}(m) \ll \mathcal{L}^{A_1}\sum_{d_1,d_2}\sum d^A(d_1)d^A(d_2)\lambda^2(d_1)\lambda^2(d_2)\frac{X}{[d_1,d_2]},$$

where $\mathcal{L}$ now has the meaning

$$\mathcal{L} := \log 2X.$$

Since $[d_1, d_2] = d_1d_2(d_1, d_2)^{-1}$ we extend the summation over all the divisors δ of d_1 and d_2, to obtain the series of inequalities

$$\begin{aligned}\sum_{m\leqslant X} d^A(m)\lambda^{*4}(m) &\ll X\mathcal{L}^{A_1}\sum_{\delta\leqslant X}\delta\Big(\sum_{\delta\mid d_1\leqslant X} d^A(d_1)\frac{\lambda^2(d_1)}{d_1}\Big)^2\\ &\ll X\mathcal{L}^{A_1}\sum_{\delta\leqslant X}\delta\Big(\sum_{\delta\mid d_1\leqslant X}\frac{\lambda^4(d_1)}{d_1}\Big)\cdot\Big(\sum_{\delta\mid d_1\leqslant X}\frac{d^{2A}(d_1)}{d_1}\Big)\\ &\leqslant X\mathcal{L}^{A_1}\sum_{\delta\leqslant X} d^{2A}(\delta)\Big(\sum_{\delta\mid d_1\leqslant X}\frac{\lambda^4(d_1)}{d_1}\Big)\end{aligned}$$

$$\begin{aligned}&\leqslant X\mathcal{L}^{A_1}\Big(\sum_{d_1\leqslant X} d^{2A+1}(d_1)\frac{\lambda^4(d_1)}{d_1}\Big)\\ &\leqslant X\mathcal{L}^{A_1}\Big(\sum_{d_1\leqslant X}\frac{\lambda^6(d_1)}{d_1}\Big)^{2/3}\Big(\sum_{d_1\leqslant X}\frac{d^{6A+3}(d_1)}{d_1}\Big)^{1/3}\\ &\ll X\mathcal{L}^{A_1},\end{aligned}$$

where we used the Cauchy–Schwarz inequality, the inequalities (70) and (66), Hölder's inequality, and finally the assumption (67) combined with Abel summation. □

6. Additive twists and Miller's theorem

6.1 GL(2)

For later applications in the estimation of Type I sums we prove the following lemma.

Lemma 6.1. *Let f be a cusp form on $\mathrm{SL}(2,\mathbb{Z})$. Then uniformly for N an integer $\geqslant 1$, for $X \geqslant 1$ and for $\alpha\in\mathbb{R}$ one has the inequality*

$$\sum_{n\leqslant X}\lambda_f(Nn)e(\alpha n)\ll_f \sqrt{X}\log(2X)\,d(N)^{1/2}\lambda_f^*(N).$$

Proof. We use (64) and (1) to write

$$\begin{aligned}\sum_{n\leqslant X}\lambda_f(Nn)e(\alpha n) &= \sum_{d\mid N}\mu(d)\lambda_f(N/d)\sum_{k\leqslant X/d}\lambda_f(k)e(\alpha\, dk)\\ &\ll \sqrt{X}(\log 2X)\sum_{d\mid N}\mu^2(d)|\lambda_f(N/d)|d^{-1/2}.\end{aligned}$$

It remains to apply the Cauchy–Schwarz inequality and to refer to the definition (65) to conclude the proof. □

6.2 GL(3)

We recall the main theorem in [Mil06] already mentioned in (11) above. Miller's theorem depends crucially on the Voronoi summation formula for GL(3) which was first established by Miller and Schmidt [MS06] (see also [GL06] for a different treatment). A concrete introduction to the theory of higher degree automorphic forms is the book [Gol06].

Theorem E. *Let $a_{r,n}$ denote the Fourier coefficients of a cusp form f on*

$$\mathrm{GL}(3,\mathbb{Z})\backslash\mathrm{GL}(3,\mathbb{R}).$$

Then for every $\varepsilon>0$, for every integer r, and for every $T\geqslant 1$, one has the inequality

$$\sum_{n\leqslant T}a_{r,n}e(n\alpha)\ll_{f,r,\varepsilon} T^{(3/4)+\varepsilon},$$

where the implied constant depends only on the form f, r, and ε.

Applying this theorem to the symmetric square lift of a Hecke–Maass cusp form f of level one and noting that we can write the coefficients of $L(s,\mathrm{sym}^2 f)$ as convolutions from the expression

$$L(s,\mathrm{sym}^2 f)=\zeta(2s)\sum_{n=1}^{\infty}\frac{\lambda_f(n^2)}{n^s},$$

we obtain the following corollary. See [MS04, pp. 434–435] for details.

Corollary 6.1. *For every Hecke–Maass cusp form f of level one and for every $\varepsilon > 0$, there exists a function $C(f, \varepsilon)$ such that, for every $T \geqslant 1$ one has the inequality*

$$\left|\sum_{n\leqslant T}\bigg(\sum_{n=md^2}\lambda_f(m^2)\bigg)e(n\alpha)\right| \leqslant C(f,\varepsilon)T^{(3/4)+\varepsilon}.$$

6.3 Application of Miller's theorem

We have the following lemma.

Lemma 6.2. *For every Hecke–Maass cusp form f of level one and for every positive ε, we have*

$$\sum_{n\leqslant T}\lambda_f(n^2)e(n\alpha) \ll_{\varepsilon,f} T^{(3/4)+\varepsilon},$$

uniformly for $T \geqslant 1$.

Proof. Let

$$M(T,\alpha) := \sum_{n\leqslant T}\bigg(\sum_{n=md^2}\lambda_f(m^2)\bigg)e(n\alpha),$$

and let

$$S(T,\alpha) := \sum_{n\leqslant T}\lambda_f(n^2)e(n\alpha).$$

We claim the equality

$$S(T,\alpha) = \sum_{r\leqslant\sqrt{T}}\mu(r)M\bigg(\frac{T}{r^2}, r^2\alpha\bigg), \tag{71}$$

and Lemma 6.2 directly follows from Proposition 6.1 after a summation over r. To prove (71), we write

$$\begin{aligned} S(T,\alpha) &= \sum_{m\leqslant\sqrt{T}}\bigg(\sum_{r\mid m}\mu(r)\bigg)S\bigg(\frac{T}{m^2}, m^2\alpha\bigg) \\ &= \sum_{r\leqslant\sqrt{T}}\mu(r)\sum_{\ell\leqslant\sqrt{T}/r}\sum_{k\leqslant T/(\ell^2r^2)}\lambda_f(k^2)e(kr^2\ell^2\alpha). \end{aligned}$$

The proof now follows by making the change of variables $n = k\ell^2$. □

Let us denote, for a positive integer A,

$$S(T,A,\alpha) := \sum_{n\leqslant T}\lambda_f(An^2)e(n\alpha).$$

By (64) and the observation that for a squarefree ℓ, $\ell \mid n^2$ if and only if $\ell \mid n$, we have

$$S(T,A,\alpha) = \sum_{\ell\mid A}\mu(\ell)\lambda_f(A/\ell)S(T/\ell,\ell,\ell\alpha), \tag{72}$$

for any integer A. Now we prove a key lemma.

Lemma 6.3. *Let f be a Hecke–Maass cusp form of level one and let ε be any positive real number. Then we have the bound*

$$S(T,A,\alpha) \ll_{\varepsilon,f} (1+\omega(A))d_3(A)|\lambda_f(A)|T^{(3/4)+\varepsilon}, \tag{73}$$

uniformly for $T \geqslant 1$, for squarefree $A \geqslant 1$ and for real α.

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Proof. We shall prove this lemma for every squarefree A by induction on T, with the same implicit constant as the one contained in the statement of Lemma 6.2. For $T_0 \leqslant 1$, formula (73) is correct for any A. Similarly, (73) is correct for any T when $A = 1$, with the same constant as in Lemma 6.2. Suppose now that there exists $T_0 \geqslant 1$, such that (73) is true for any $T \leqslant T_0$ and any A squarefree. We now prove that the same holds for any $T \leqslant 2T_0$.

We start with the relation (72). The first term corresponding to $\ell = 1$ is $\lambda_f(A)S(T, 1, \alpha)$ and $S(T, 1, \alpha) = O_\varepsilon(T^{(3/4)+\varepsilon})$ by Lemma 6.2. For $\ell > 1$, we use the induction hypothesis. Since A is squarefree, for $\ell|A$ we have $(\ell, A/\ell) = 1$ (hence $\lambda_f(A) = \lambda_f(A/\ell)\lambda_f(\ell)$) and also $1 + \omega(\ell) \leqslant \omega(A)$ for $\ell \neq A$. Thus we have,

$$S(T, A, \alpha) \ll |\lambda_f(A)|T^{(3/4)+\varepsilon}\Big\{1 + \omega(A) \sum_{\substack{\ell|A \\ 1<\ell<A}} |\mu(\ell)| \frac{d_3(\ell)}{\ell^{(3/4)+\varepsilon}} + \frac{d_3(A)(1 + \omega(A))}{A^{(3/4)+\varepsilon}}\Big\}$$
$$\leqslant |\lambda_f(A)|T^{(3/4)+\varepsilon}\Big\{1 + \omega(A) \prod_{p|A}\Big(1 + \frac{3}{p^{(3/4)+\varepsilon}}\Big) + \frac{d_3(A)}{A^{(3/4)+\varepsilon}}\Big\}.$$

Now we note that
$$\prod_{p|A}\Big(1 + \frac{3}{p^{3/4}}\Big) < d_3(A),$$
as $3/p^{3/4} < 2$ for all primes p. Since we also have $1 + d_3(A)/A^{3/4} < d_3(A)$ for all $A \geqslant 2$, we deduce
$$S(T, A, \alpha) \ll \lambda_f(A)T^{(3/4)+\varepsilon}\Big\{1 + \omega(A)d_3(A) + \frac{d_3(A)}{A^{3/4}}\Big\}$$
$$\ll (1 + \omega(A))d_3(A)\lambda_f(A)T^{(3/4)+\varepsilon}.$$ □

Now we generalize this to all integers A, squarefree or not, by using the function λ_f^* defined in (65).

Lemma 6.4. *Let f be a Hecke–Maass cusp form of level one and let $\varepsilon > 0$ be any real number. Then we have the inequality*
$$S(T, A, \alpha) \ll_{\varepsilon,f} (1 + \omega(A))d_3(A){\lambda_f^*}^2(A)T^{(3/4)+\varepsilon},$$
uniformly for $T \geqslant 1$, for $A \geqslant 1$ and for real α.

Proof. We start from (72). Applying (73), we obtain
$$S(T, A, \alpha) \ll (1 + \omega(A))d_3(A)T^{(3/4)+\varepsilon} \sum_{\ell|A} |\lambda_f(\ell)|\, |\lambda_f(A/\ell)|$$
$$\ll (1 + \omega(A))d_3(A){\lambda_f^*}^2(A)T^{(3/4)+\varepsilon},$$
by the Cauchy–Schwarz inequality and the definition (65). □

7. The proof of Theorem 1.1

We assume throughout the rest of the paper that f is a Hecke–Maass cusp form of level one. There is no loss of generality in doing so as the space of Maass cusp forms is spanned by the Hecke forms. Recall that for such a form, the Fourier coefficients $\nu_f(n)$ and the Hecke eigenvalues $\lambda_f(n)$

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coincide up to multiplication by the non-zero constant $\nu_f(1)$. We prove only the bound (7) for the sum involving the Möbius function. The proof of the bound (6) is structurally identical and, in fact, simpler as explained in the introduction. Throughout the rest of the paper, f denotes a Hecke–Maass cusp form for the group $\mathrm{SL}(2,\mathbb{Z})$.

7.1 Initial steps

Let us write

$$T(X,\alpha)=\sum_{1\leqslant n\leqslant X}\lambda_f(n)\mu(n)e(n\alpha).$$

We fix a parameter Q to be optimized later. Now, Dirichlet's theorem on Diophantine approximation ensures that given any $\alpha\in[0,1)$, there is always a rational number a/q, $(a,q)=1$ such that

$$1\leqslant q\leqslant Q \quad\text{and}\quad \left|\alpha-\frac{a}{q}\right|\leqslant\frac{1}{qQ}. \tag{74}$$

By partial summation, we have

$$|T(X,\alpha)|\ll\left|T\left(X,\frac{a}{q}\right)\right|+\int_1^X\left|\left(\alpha-\frac{a}{q}\right)T\left(x,\frac{a}{q}\right)\right|dx+1. \tag{75}$$

We now plan a general study of the sum $T(x,a/q)$. We first write the equality

$$T\left(x,\frac{a}{q}\right)=\sum_{b(\mathrm{mod}\ q)}e\left(\frac{ab}{q}\right)\sum_{\substack{n\equiv b(\mathrm{mod}\ q)\\ n\leqslant x}}\lambda_f(n)\mu(n). \tag{76}$$

To detect the congruence $n\equiv b \bmod q$ by Dirichlet characters, we must first ensure the coprimality of the class and the modulus. So we introduce

$$d=(b,q),\quad b_1=b/d,\quad q_1=q/d,\ \text{and}\ \chi_d,\quad \text{the principal character modulo } d.$$

This gives the equalities

$$\begin{aligned}\sum_{\substack{n\equiv b(\mathrm{mod}\ q)\\ n\leqslant x}}\lambda_f(n)\mu(n)&=\sum_{\substack{n_1\equiv b_1(\mathrm{mod}\ q_1)\\ n_1\leqslant x/d}}\lambda_f(dn_1)\mu(dn_1)\\ &=\lambda_f(d)\mu(d)\sum_{\substack{n_1\equiv b_1(\mathrm{mod}\ q_1)\\ n_1\leqslant x/d}}\lambda_f(n_1)\mu(n_1)\chi_d(n_1)\\ &=\frac{\lambda_f(d)\mu(d)}{\varphi(q_1)}\sum_{\chi(\mathrm{mod}\ q_1)}\overline{\chi}(b_1)\sum_{n_1\leqslant x/d}\lambda_f(n_1)\mu(n_1)(\chi\chi_d)(n_1).\end{aligned} \tag{77}$$

Since $\chi\chi_d$ is a character of modulus dq_1, we can apply Theorem 4.1 with $q:=dq_1$ to the inner sum. This gives

$$\sum_{n_1\leqslant x/d}\lambda_f(n_1)\mu(n_1)(\chi\chi_d)(n_1)\ll\sqrt{dq_1}\frac{X}{d}\exp(-c_1\sqrt{\log(X/d)}).$$

Bounding $\lambda_f(d)$ by (19), we finally have

$$T\left(x,\frac{a}{q}\right)\ll q^{3/2}X\exp(-c_1\sqrt{\log(X/q)}). \tag{78}$$

Now the proof will proceed differently depending on the size of q compared to X.

7.2 Major arcs

By (78), (75) and (74), we have

$$T(X,\alpha) \ll q^{3/2} X \exp(-c_1\sqrt{\log(X/q)})\left(1 + \left|\alpha - \frac{a}{q}\right| X\right)$$
$$\ll \sqrt{q} X \exp(-c_1\sqrt{\log(X/q)})\left(q + \frac{X}{Q}\right). \tag{79}$$

Now we choose

$$Q = X \exp\left(-\frac{c_1}{3}\sqrt{\log X}\right). \tag{80}$$

If

$$q \leqslant \frac{X}{Q} = \exp\left(\frac{c_1}{3}\sqrt{\log X}\right), \tag{81}$$

then by (79),

$$T(X,\alpha) \ll X \exp\left(-\frac{c_1}{10}\sqrt{\log X}\right).$$

Therefore, we have proved Theorem 1.1 if α admits a good enough rational approximation a/q, $(a,q)=1$, satisfying (74) with Q as above, and q satisfies the bound (81). On the other hand, if α is such that (81) is true for no rational number a/q, $(a,q)=1$, satisfying (74), then this method does not work and we apply the Vinogradov method as explained in the next few subsections.

7.3 Minor arcs

After the pioneering work of Vinogradov, Gallagher, Vaughan and others, we know how to quickly enter into the combinatorial structure of the functions Λ and μ. In our situation we use the following proposition (see [IK04, Proposition 13.5] for instance).

Proposition 7.1. *Let $y,z \geqslant 1$. Then for any $m > \max\{y,z\}$, we have*

$$\mu(m) = -\sum_{\substack{bc\mid m\\ b\leqslant y, c\leqslant z}} \mu(b)\mu(c) + \sum_{\substack{bc\mid m\\ b>y, c>z}} \mu(b)\mu(c). \tag{82}$$

Accordingly, we decompose the sum $T(X,\alpha)$ as

$$T(X,\alpha) = -T_1(X,\alpha) + T_2(X,\alpha) + O(y+z), \tag{83}$$

where

$$T_1(X,\alpha) = \sum_{b\leqslant y}\mu(b)\sum_{c\leqslant z}\mu(c)\sum_{k\leqslant X/bc}\lambda_f(kbc)e(kbc\alpha), \tag{84}$$

and

$$T_2(X,\alpha) = \sum_{b>y}\mu(b)\sum_{c>z}\mu(c)\sum_{k\leqslant X/bc}\lambda_f(kbc)e(kbc\alpha) \tag{85}$$

are called *sums of type I and type II* respectively. The parameters $y \geqslant 1$ and $z \geqslant 1$ will be chosen later optimally (they will be of size about $O(X^{1/5})$). The error term in (83) comes from the contribution of the $m \leqslant \max\{y,z\}$ and is handled with the inequality (22).

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7.4 Type I sums

A direct application of Lemma 6.1 to the inner sum of (84) leads to the upper bound

$$\sum_{k\leqslant X/bc}\lambda_f(kbc)e(kbc\alpha)\ll_\varepsilon (bc)^{2\varepsilon}\lambda_f^*(bc)\left(\frac{X}{bc}\right)^{(1/2)+\varepsilon},$$

after using standard bounds for the arithmetical functions involved. Inserting this bound in (84) and writing $m:=bc$ we obtain

$$\begin{aligned}T_1(X,\alpha)&\ll_\varepsilon X^{(1/2)+\varepsilon}(yz)^\varepsilon\sum_{m\leqslant yz}\frac{d(m)\lambda_f^*(m)}{m^{1/2}}\\&\ll_\varepsilon (Xyz)^{(1/2)+\varepsilon}\end{aligned}\tag{86}$$

by Theorem A, Proposition 5.1, and partial summation.

7.5 Type II sum

Now we come to the most delicate part of the proof which is the estimation of the type II sum. Introducing the notation

$$\beta_\ell:=\sum_{\substack{b\mid\ell\\ b>y}}\mu(b),$$

we see that

$$T_2(X,\alpha)=\sum_\ell\beta_\ell\sum_{\substack{c>z\\ \ell c\leqslant X}}\mu(c)\lambda_f(c\ell)e(\alpha c\ell)$$

and β_ℓ satisfies the bound

$$|\beta_\ell|\leqslant d(\ell).\tag{87}$$

Now we introduce two parameters L and C which will be chosen later subject to

$$L>y,\quad C>z\quad \textit{and}\quad LC\leqslant X.\tag{88}$$

We split the sum $T_2(X,\alpha)$ into $O((\log X)^2)$ many dyadic pieces of the form

$$T_2(C,L,\alpha)=\sum_{\ell\sim L}\beta_\ell\sum_{c\sim C}\mu(c)\lambda_f(c\ell)e(\alpha c\ell),$$

where the variables ℓ and c satisfy the extra condition

$$c\ell\leqslant X.\tag{89}$$

This extra condition is sometimes superfluous but allows us to suppress the dependence on X in the notation. By the Cauchy–Schwarz inequality we have

$$|T_2(C,L,\alpha)|^2\leqslant\left(\sum_{\ell\sim L}|\beta_\ell|^2\right)A(C,L,\alpha),\tag{90}$$

where

$$A(C,L,\alpha):=\sum_{\ell\sim L}\left|\sum_{c\sim C}\mu(c)\lambda_f(c\ell)e(\alpha c\ell)\right|^2,$$

with the extra constraint (89). By (87) and (66),

$$\sum_{\ell\sim L}|\beta_\ell|^2\ll L(\log 2L)^3,\tag{91}$$

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where the implied constant is absolute. Now it remains to estimate $A(C, L, \alpha)$ and we can give a non-trivial bound as long as α is not close to rationals with small denominators. Precisely, we prove the following theorem.

Theorem 7.1. *Let f be a Hecke–Maass cusp form of level one. Suppose α is a real number and a/q is any rational number written as a reduced fraction such that*

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{q^2}. \tag{92}$$

Then there are absolute constants K and $K' > 0$, and for all $\varepsilon > 0$ a constant $C(\varepsilon)$, such that

$$A(C, L, \alpha) \leqslant C(\varepsilon)C^2L^{5/6}(CL)^{\varepsilon} + K'(C^{3/2}L + C^2Lq^{-1/2} + C^{3/2}L^{1/2}q^{1/2})(\log(2CL))^K,$$

uniformly for all C, L and $X > 1$.

Remark. To test the strength of Theorem 7.1, we first give a trivial bound of $A(C, L, \alpha)$. By $\mathcal{L}$, we now denote

$$\mathcal{L} := \log 2CL (\ll \log 2X).$$

We have

$$\begin{aligned} A(C, L, \alpha) &\leqslant C \sum_{\ell \sim L} \sum_{c \sim C} |\lambda_f(c\ell)|^2 \\ &\ll C \sum_{CL < m \leqslant 4CL} d(m)\lambda_f^2(m) \\ &\ll_f C^2L\mathcal{L}^2, \end{aligned}$$

by Cauchy's inequality, (21) and (66). Hence the theorem is useful if we have C, L and q satisfying the inequalities: C and $L \geqslant (CL)^{\varepsilon}$ and $\mathcal{L}^{K_1} \leqslant q \leqslant (CL)\mathcal{L}^{-K_1}$, where K_1 is an explicit constant. Now we give a proof of Theorem 7.1.

Proof. Throughout the proof, K will denote an unspecified but effective constant the value of which may change in different occurrences. Expanding squares and inverting summations, we can write

$$A(C, L, \alpha) = \sum_{c_1, c_2 \sim C}\sum \mu(c_1)\mu(c_2) \sum_{\ell \sim L} \lambda_f(c_1\ell)\lambda_f(c_2\ell)e(\alpha(c_1 - c_2)\ell), \tag{93}$$

where ℓ satisfies the extra inequality

$$\ell \leqslant \min\{X/c_1, X/c_2\}. \tag{94}$$

We first consider the diagonal $A^{\mathrm{diag}}(C, L, \alpha)$ corresponding to the contribution of the terms satisfying $c_1 = c_2$ in (93). The argument in the above remark shows that there exists an absolute and positive constant K such that

$$A^{\mathrm{diag}}(C, L, \alpha) \ll CL\mathcal{L}^K, \tag{95}$$

uniformly for α real, C, L and $X \geqslant 1$.

The off-diagonal part of the sum $A(C, L, \alpha)$ (see (93)) is given by

$$A^{\mathrm{offdiag}}(C, L, \alpha) := \sum_{\substack{c_1, c_2 \sim C \\ c_1 \neq c_2}}\sum \mu(c_1)\mu(c_2) \sum_{\ell \sim L} \lambda_f(\ell c_1)\lambda_f(\ell c_2)e(\alpha(c_1 - c_2)\ell),$$

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where ℓ satisfies (94). Let $\gamma=(c_1,c_2)$. We apply (63) with the choice $m=c_1\ell$, $n=c_2\ell$. This gives the equality

$$A^{\mathrm{offdiag}}(C,L,\alpha)=\sum_{\gamma\leqslant 2C}\ \sum_{\substack{c_1,c_2\sim C\\ c_1\neq c_2\\ (c_1,c_2)=\gamma}}\sum \mu(c_1)\mu(c_2)\sum_{\ell\sim L}\sum_{d\mid\ell\gamma}\lambda_f(\ell^2c_1c_2/d^2)e(\alpha(c_1-c_2)\ell),$$

where ℓ satisfies (94). Let us further factorize the variables by introducing

$$c_1'=c_1\gamma^{-1}\quad\text{and}\quad c_2'=c_2\gamma^{-1},$$

and

$$(\gamma,d):=\delta,\quad d:=\delta d'\quad\text{and}\quad \ell:=d'\nu. \tag{96}$$

Note also the equivalences

$$d\mid\ell\gamma\iff\frac{d}{(\gamma,d)}\left|\frac{\gamma}{(\gamma,d)}\cdot\ell\iff\frac{d}{(\gamma,d)}\right|\ell.$$

Thus we have,

$$\begin{aligned}A^{\mathrm{offdiag}}(C,L,\alpha)=\sum_{\gamma}\mu^2(\gamma)&\sum_{\substack{1<c_1',c_2'\sim C\gamma^{-1}\\ (\gamma,c_1'c_2')=(c_1',c_2')=1}}\sum\mu(c_1'c_2')\\ &\times\sum_{\delta\mid\gamma}\ \sum_{(d',\gamma\delta^{-1})=1}\ \sum_{\nu\sim Ld'^{-1}}\lambda_f\left(\frac{c_1'c_2'\gamma^2}{\delta^2}\cdot\nu^2\right)e(\alpha\gamma d'(c_1'-c_2')\nu),\end{aligned} \tag{97}$$

where ν satisfies the inequality

$$\nu\leqslant\min\{X/(\gamma c_1'd'),X/(\gamma c_2'\,d')\}. \tag{98}$$

Let $D'=D'(C,L)(<L)$ be a parameter to be fixed later. We split the sum $A^{\mathrm{offdiag}}(C,L,\alpha)$ into

$$A^{\mathrm{offdiag}}(C,L,\alpha)=A^{\mathrm{offdiag}}_{<D'}(C,L,\alpha)+A^{\mathrm{offdiag}}_{\geqslant D'}(C,L,\alpha), \tag{99}$$

according to whether $d'<D'$ or $d'\geqslant D'$ in the sum (97). By Lemma 6.4, we obtain the upper bound

$$\begin{aligned}A^{\mathrm{offdiag}}_{<D'}(C,L,\alpha)\ll(CL)^{\varepsilon}\sum_{\gamma}\mu^2(\gamma)&\sum_{\substack{1<c_1',c_2'\sim C\gamma^{-1}\\ (\gamma,c_1'c_2')=(c_1',c_2')=1}}\sum\mu^2(c_1'c_2')\\ &\times\sum_{\delta\mid\gamma}\ \sum_{\substack{(d',\gamma\delta^{-1})=1\\ d'<D'}}\lambda_f^{*2}\left(\frac{c_1'c_2'\gamma^2}{\delta^2}\right)(L/d')^{3/4+\varepsilon}.\end{aligned} \tag{100}$$

By Lemma 5.2(c), we know that $\lambda_f^{*2}(c_1'c_2'\gamma^2/\delta^2)\leqslant\lambda_f^{*2}(c_1'c_2'\gamma^2)$. Furthermore, each $c\leqslant 4C^2$ has $O(C^{\varepsilon})$ ways of being written as $c=c_1'c_2'\gamma^2$, with c_1', c_2' squarefree and coprime. Using these remarks, we simplify (100) into

$$A^{\mathrm{offdiag}}_{<D'}(C,L,\alpha)\ll {D'}^{1/4}L^{3/4}(CL)^{\varepsilon}\sum_{c\leqslant 4C^2}d(c)\lambda_f^{*2}(c). \tag{101}$$

It remains to note the inequality $\lambda_f^{*2}(m)\leqslant\lambda_f^{*4}(m)$ to apply (68) to finally deduce the following bound valid for every $\varepsilon>0$,

$$A^{\mathrm{offdiag}}_{<D'}(C,L,\alpha)\ll_{\varepsilon}C^2{D'}^{1/4}L^{3/4}(CL)^{\varepsilon}, \tag{102}$$

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uniformly for C, D', L and $X \geqslant 1$. The above bound is useful when D' is small. When D' is very close to L, we recover the trivial bound $A^{\mathrm{offdiag}}(C, L, \alpha) \ll C^2 L \mathcal{L}^K$. In that situation we will benefit from the cancellation of additive characters in a long sum over the variable d'.

The goal now is to give an upper bound for $A^{\mathrm{offdiag}}_{\geqslant D'}(C, L, \alpha)$. We start from the expressions (97) and (99) and rewrite as

$$A^{\mathrm{offdiag}}_{\geqslant D'}(C, L, \alpha) = \sum_{\gamma} \mu^2(\gamma) \sum_{\substack{1 < c_1', c_2' \sim C\gamma^{-1} \\ (\gamma, c_1' c_2') = (c_1', c_2') = 1}} \sum \mu(c_1' c_2') \times \sum_{\delta \mid \gamma} \sum_{\nu} \lambda_f\left(\frac{c_1' c_2' \gamma^2}{\delta^2} \cdot \nu^2\right) \sum_{\substack{(d', \gamma\delta^{-1}) = 1 \\ d' \geqslant D', d' \sim L/\nu}} e(\alpha\gamma\, d'(c_1' - c_2')\nu) \tag{103}$$

where d' now verifies the extra condition (see (89))

$$d' \leqslant \min\{X/(\gamma\nu c_1'), X/(\gamma\nu c_2')\}. \tag{104}$$

In the expression (103), the variable d' is not smooth completely, because of the coprimality condition $(d', \gamma\delta^{-1}) = 1$. Capturing the coprimality condition by the Möbius function, we write (103) as

$$A^{\mathrm{offdiag}}_{\geqslant D'}(C, L, \alpha) = \sum_{\gamma} \mu^2(\gamma) \sum_{\substack{1 < c_1', c_2' \sim C\gamma^{-1} \\ (\gamma, c_1' c_2') = (c_1', c_2') = 1}} \sum \mu(c_1' c_2') \times \sum_{\delta \mid \gamma} \sum_{\nu} \lambda_f\left(\frac{c_1' c_2' \gamma^2}{\delta^2} \cdot \nu^2\right) \sum_{u \mid \gamma\delta^{-1}} \mu(u) \sum_{\substack{d'' \geqslant D'/u \\ d'' \sim L/\nu u}} e(\alpha\gamma\nu(c_1' - c_2') u d''), \tag{105}$$

where now (104) is replaced by

$$d'' \leqslant \min\{X/(\gamma\nu c_1' u), X/(\gamma\nu c_2' u)\}. \tag{106}$$

Taking absolute values, extending the summation over all $u \mid \gamma$ and changing $\delta \mapsto \gamma\delta^{-1}$, we deduce from (105) the inequality

$$|A^{\mathrm{offdiag}}_{\geqslant D'}(C, L, \alpha)| \leqslant \sum_{\gamma} \mu^2(\gamma) \sum_{\substack{1 < c_1', c_2' \sim C\gamma^{-1} \\ (\gamma, c_1' c_2') = (c_1', c_2') = 1}} \sum \mu^2(c_1' c_2') \sum_{u \mid \gamma} \mu^2(u) \times \sum_{\delta \mid \gamma} \sum_{\nu} |\lambda_f(c_1' c_2' \delta^2 \nu^2)| \left| \sum_{\substack{d'' \geqslant D'/u \\ d'' \sim L/\nu u}} e(\alpha\gamma\nu(c_1' - c_2') u d'') \right|, \tag{107}$$

with the constraint (106) for the variable d''. We now split the ranges of variations of the variables γ, c_1', c_2', u and ν in the right-hand side of (107) into dyadic segments,

$$\gamma \sim \Gamma, \quad c_1' \sim C_1', \quad c_2' \sim C_2', \quad u \sim U \quad \text{and} \quad \nu \sim \mathcal{N}. \tag{108}$$

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We denote by $A(\Gamma, C_1', C_2', U, \mathcal{N})$ the corresponding contribution. The number of these subsums is $O(\mathcal{L}^5)$. Note that we have

$$\Gamma C_1' \asymp \Gamma C_2' \asymp C, \quad U \leqslant \Gamma, \quad L/\mathcal{N} > D'/2. \tag{109}$$

To condense the notation, we define

$$m := \gamma\nu u(c_1' - c_2'). \tag{110}$$

Using the well-known bound for sums of additive characters, we have

$$A(\Gamma, C_1', C_2', U, \mathcal{N}) \ll \sum_{1 \leqslant |m| \leqslant M} g(m) \min\Big(\frac{L}{\mathcal{N}U}, \|\alpha m\|^{-1}\Big), \tag{111}$$

where

$$M = 16\Gamma C_1' U\mathcal{N}, \quad (\asymp CU\mathcal{N}), \tag{112}$$

and $g(m)$ is the weight function

$$g(m) := \sum_{\gamma}\sum_{c_1'}\sum_{c_2'} \mu^2(c_1'c_2'\gamma) \sum_{u|\gamma}\sum_{\delta|\gamma}\sum_{\nu} |\lambda_f(c_1'c_2'\delta^2\nu^2)|, \tag{113}$$

where the variables $(\gamma, c_1', c_2', u, \nu)$ also satisfy (108) and (110). Now we recall the following classical lemma (see [IK04, p. 346], for instance).

Lemma 7.1. *The inequality*

$$\sum_{|m| \leqslant M} \min(N, \|\alpha m\|^{-1}) \ll (M + N + MNq^{-1} + q)\log 2q$$

holds uniformly for M and $N \geqslant 1$, α real, and any rational number a/q satisfying (92).

To apply Lemma 7.1 to (111), we first apply the Cauchy–Schwarz inequality with the view to taking advantage of the fact that although the size of the coefficients $g(m)$ may be difficult to control, the $\|\cdot\|_2$-norm of this sequence can still be estimated by the results of § 5. By the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} &A(\Gamma, C_1', C_2', U, \mathcal{N}) \\ &\quad \ll \Big(\frac{L}{\mathcal{N}U}\Big)^{1/2} \cdot \Big(\sum_{1 \leqslant |m| \leqslant M} g^2(m)\Big)^{1/2} \cdot \Big(M + \frac{L}{\mathcal{N}U} + \frac{LM}{q\mathcal{N}U} + q\Big)^{1/2} (\log(2q))^{1/2}. \end{aligned} \tag{114}$$

By Lemma 5.2 and the coprimality conditions of the variables c_1' and c_2', we get the inequalities

$$\begin{aligned} |\lambda_f(c_1'c_2'\delta^2\nu^2)| &\leqslant \lambda_f^*(c_1'c_2')\lambda_f^*(\delta^2\nu^2) \\ &\leqslant \lambda_f^*(c_1')\lambda_f^*(c_2')\lambda_f^*(\gamma^2\nu^2) \\ &\leqslant d(\gamma\nu)\lambda_f^*(c_1')\lambda_f^*(c_2'){\lambda_f^*}^2(\gamma\nu). \end{aligned}$$

Inserting this bound into the definition (113), we obtain the inequality

$$g(m) \leqslant \sum\sum_{\substack{u,\gamma,\nu \\ u\gamma\nu | m}}\sum d(\gamma\nu){\lambda_f^*}^2(\gamma\nu) \sum_{c_1'} \lambda_f^*(c_1')\lambda_f^*(c_1' + m/(u\gamma\nu)).$$

By the Cauchy–Schwarz inequality applied to the sum in c_1' (recall that we have $|m/(u\gamma\nu)| \ll C_2'$) and by (68), we get the upper bound

$$\begin{aligned} g(m) &\ll C_1'\mathcal{L}^K \sum\sum_{\substack{u,\gamma,\nu \\ u\gamma\nu|m}}\sum d(\gamma\nu)\lambda_f^{*2}(\gamma\nu) \\ &\ll \lambda_f^{*2}(m) C_1'\mathcal{L}^K \sum\sum_{\substack{u,\gamma,\nu \\ u\gamma\nu|m}}\sum d(\gamma\nu) \\ &\ll d^5(m)\lambda_f^{*2}(m) C_1'\mathcal{L}^K, \end{aligned}$$

by using Lemma 5.2(c) and trivial bound on the divisor functions. By the above inequality, we have

$$\begin{aligned} \sum_{1\leqslant |m|\leqslant M} g^2(m) &\ll C_1'^2\mathcal{L}^K \sum_{1\leqslant |m|\leqslant M} d^{10}(m)\lambda_f^{*4}(m) \\ &\ll C_1'^2 M\mathcal{L}^K \\ &\ll C_1'^3 U\Gamma\mathcal{N}\mathcal{L}^K, \end{aligned} \tag{115}$$

the last lines being consequences of Proposition 5.1 and the definition (112) of M. Inserting (115) in (114), we get the inequality

$$A(\Gamma, C_1', C_2', U, \mathcal{N}) \ll C_1'^{3/2} L^{1/2}\Gamma^{1/2}\cdot\left(M + \frac{L}{\mathcal{N}U} + \frac{LM}{q\mathcal{N}U} + q\right)^{1/2}\mathcal{L}^K. \tag{116}$$

We must take the supremum of the right-hand side of (116) under the constraints (109) and (112). We easily obtain

$$\begin{aligned} A(\Gamma, C_1', C_2', U, \mathcal{N}) &\ll (C^3 L\Gamma^{-2})^{1/2}\left(CU\mathcal{N} + L + \frac{CL}{q} + q\right)^{1/2}\mathcal{L}^K \\ &\ll (C^3 L\Gamma^{-2})^{1/2}\left(CLD'^{-1}\Gamma + L + \frac{CL}{q} + q\right)^{1/2}\mathcal{L}^K. \end{aligned}$$

By summing over all these subsums we have that if α satisfies (92), then there exists an absolute constant $K > 0$ such that

$$A^{\mathrm{offdiag}}_{\geqslant D'}(C, L, \alpha) \ll (C^2 LD'^{-1/2} + C^{3/2}L + C^2 Lq^{-1/2} + C^{3/2}L^{1/2}q^{1/2})\mathcal{L}^K, \tag{117}$$

uniformly for C, $L \geqslant 1$ and $1 \leqslant D' \leqslant L$.

Recall that we had divided the sum $A(C, L, \alpha)$ into

$$A(C, L, \alpha) = A^{\mathrm{diag}}(C, L, \alpha) + A^{\mathrm{offdiag}}_{<D'}(C, L, \alpha) + A^{\mathrm{offdiag}}_{\geqslant D'}(C, L, \alpha). \tag{118}$$

Using (95), (102) and (117) in (118) and giving the value $L^{1/3}$ to the parameter D' we complete the proof of Theorem 7.1. □

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7.6 The finishing touches

By (90), (91), and Theorem 7.1, we have the inequalities

$$\begin{aligned}|T_2(C,L,\alpha)|^2 &\ll L\mathcal{L}^3 A(C,L,\alpha)\\ &\ll_\varepsilon C^2L^{11/6}(CL)^\varepsilon + (C^{3/2}L^2 + C^2L^2q^{-1/2} + C^{3/2}L^{3/2}q^{1/2})\mathcal{L}^K,\end{aligned}$$

for any $\varepsilon > 0$ and for some absolute constant $K > 0$. Therefore, we have the inequality

$$T_2(X,\alpha) \ll_\varepsilon y^{-1/12}X^{1+\varepsilon} + (z^{-1/4}X + q^{-1/4}X + q^{1/4}X^{3/4})(\log X)^K, \tag{119}$$

by summing over the dyadic segments (see (88)). Recall that here we are considering only those α for which any rationals a/q, $(a,q) = 1$ satisfying (74), also satisfy $X/Q < q \leqslant Q$, where $Q = X\exp(-(c_1/3)\sqrt{\log X})$. To be precise, we make the choices $y = z = X^{1/5}$, and this gives the upper bound

$$T_2(X,\alpha) \ll X\exp\biggl(-\frac{c_1}{13}\sqrt{\log X}\biggr),$$

where the implied constant is absolute. This, together with (83) and (86), proves Theorem 1.1.

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