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ON THE EXISTENCE OF POSITIVE SOLUTIONS ON THE HALF-LINE TO NONLINEAR TWO-DIMENSIONAL DELAY DIFFERENTIAL SYSTEMS

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Dedicated to Professor Yiannis G. Sficas on the occasion of his 65th birthday

Abstract This paper is concerned with a boundary-value problem on the half-line for nonlinear two-dimensional delay differential systems with positive delays. A theorem is established, which provides sufficient conditions for the existence of positive solutions. The application of this theorem to the special case of second-order nonlinear delay differential equations is given. Also, the application of the theorem to two-dimensional Emden–Fowler-type delay differential systems with constant delays is presented. Moreover, some general examples demonstrating the applicability of the theorem are included.

Keywords: delay differential system; boundary-value problem on the half-line; positive solution

2000 *Mathematics subject classification:* Primary 34K10; 34B18; 34B40

1. Introduction

Recently, the author [**31**] established sufficient conditions for the existence of positive increasing solutions of a boundary-value problem on the half-line to second-order nonlinear delay differential equations *with positive delays*. The assumption that the delays are positive is essential to the approach in [**31**], and hence the results given in [**31**] cannot be applied to the corresponding boundary-value problem for second-order nonlinear *ordinary* differential equations. An old idea that appeared in [**30**] plays a crucial role in the study in [**31**]. (The seeds of this idea were presented in [**19**].)

Also, recently, the author [**32**] studied the problem of the existence of solutions and of the existence and uniqueness of solutions of a boundary-value problem on the half-line to nonlinear two-dimensional delay differential systems. The methods applied in [**32**] are based on the use of the Schauder–Tikhonov theorem and the Banach contraction principle. The results obtained in [**32**] include, as special cases, those given by Mavridis *et al*. [**20**] for second-order nonlinear delay differential equations.

The work in [**31**, **32**] is closely related to the work in [**20**, **21**] and, in a sense, to the work in [**2**].

We are essentially motivated by the recent work in [**31**] (and, in a sense, by the recent work in [**32**]). In this paper, a boundary-value problem on the half-line for nonlinear two-dimensional delay differential systems *with positive delays* is considered, and sufficient conditions are given that guarantee the existence of positive solutions. The results obtained are not applicable to the corresponding boundary-value problem for nonlinear two-dimensional *ordinary* differential systems. The results given in [**31**] can be derived, as special consequences, from those established here by reducing a second-order nonlinear delay differential equation to a nonlinear two-dimensional delay differential system of a special form. The techniques applied in this paper originate in [**31**]; also, some elements of the methods used in [**32**] are successfully employed here.

Consider the nonlinear two-dimensional delay differential system

$$x'(t) = g(t, y(t)), \qquad y'(t) = -f(t, x(t - T_1(t)), \dots, x(t - T_m(t))), \tag{1.1}$$

where m is a positive integer, g is a continuous real-valued function on $[0, \infty) \times \mathbb{R}$, f is a continuous real-valued function on $[0, \infty) \times \mathbb{R}^m$ and T_j, $j = 1, \dots, m$, are positive continuous real-valued functions on the interval $[0, \infty)$ with

$$\lim_{t\to\infty} (t - T_j(t)) = \infty, \quad j = 1, \dots, m.$$

Let us consider the positive real number τ defined by

$$\tau = -\min_{j=1,\dots,m} \min_{t\geqslant 0} (t - T_j(t)).$$

Our interest will be concentrated on *global* solutions of the delay differential system (1.1), i.e. on solutions of (1.1) on the *whole* interval $[0, \infty)$. By a *solution on* $[0, \infty)$ of (1.1), we mean a pair of two continuous real-valued functions x and y defined on the intervals $[-\tau, \infty)$ and $[0, \infty)$, respectively, which are continuously differentiable on $[0, \infty)$ and satisfy (1.1) for all $t \geqslant 0$.

Together with the delay differential system (1.1), we specify an initial condition of the form

$$x(t) = \phi(t) \quad \text{for } -\tau \leqslant t \leqslant 0, \tag{1.2}$$

where the initial function ϕ is a given continuous real-valued function on the interval $[-\tau, 0]$. Throughout the paper, it will be assumed that

$$\phi(0) = 0.$$

Moreover, along with (1.1), we impose the condition

$$\lim_{t\to\infty} y(t) = 0. \tag{1.3}$$

The delay differential system (1.1) together with the conditions (1.2) and (1.3) constitute a boundary-value problem (BVP) on the half-line. A solution on $[0, \infty)$ of (1.1) satisfying (1.2) and (1.3) is said to be a solution of the boundary-value problem (1.1)–(1.3) or, more briefly, a solution of the BVP (1.1)–(1.3).

Proposition 1.1 provides a useful integral representation of the BVP (1.1)–(1.3), which will be used in proving the main result of the paper (and in proving a basic lemma needed in the proof of our main result). This proposition was established in [**32**] for a more general boundary-value problem on the half-line for more general nonlinear two-dimensional delay differential systems, in which, however, the delays are assumed to be bounded. But, as it is easy to see, the restriction of the boundedness of the delays is not needed for the validity of the proposition.

Proposition 1.1. *Let x and y be two continuous real-valued functions defined on the intervals $[-\tau, \infty)$ and $[0, \infty)$, respectively. Then (x, y) is a solution of the BVP (1.1)–(1.3) if and only if*

$$x(t) = \begin{cases} \phi(t), & \text{for } -\tau \leqslant t \leqslant 0, \\ \displaystyle\int_0^t g(s, y(s))\,\mathrm{d}s, & \text{for } t \geqslant 0, \end{cases} \tag{1.4}$$

and

$$y(t) = \int_t^\infty f(s, x(s - T_1(s)), \dots, x(s - T_m(s)))\,\mathrm{d}s \quad \text{for } t \geqslant 0. \tag{1.5}$$

We are interested in studying the problem of the existence of solutions (x, y) of the BVP (1.1)–(1.3) with x being positive on $[-\tau, \infty) - \{0\}$. Therefore, in addition to the assumption that $\phi(0) = 0$ posed previously, without mentioning it any further, it will be supposed that

$$\phi(t) > 0 \quad \text{for } -\tau \leqslant t < 0.$$

The main result of this paper is Theorem 3.1, which will be stated and proved in § 3. This theorem provides sufficient conditions for the BVP (1.1)–(1.3) to have at least one solution (x, y) such that x is positive on $(0, \infty)$ and y is positive on $[0, \infty)$. Lemma 2.1, which will be established in § 2, plays a crucial role in proving Theorem 3.1; this lemma gives useful information about the solutions (x, y) of the BVP (1.1)–(1.3) with x being non-negative on $(0, \infty)$. Section 4 is devoted to the application of Theorem 3.1 (and of Lemma 2.1) to the special case of second-order nonlinear delay differential equations. Section 5 shows the application of Theorem 3.1 to (nonlinear) two-dimensional Emden–Fowler-type differential systems with constant delays. Some general examples, which demonstrate the applicability of our main result, will be presented in § 5.

The problem studied in the present paper is closely related to the general problem of deriving sufficient conditions for the existence of solutions with prescribed asymptotic behaviour to second- (or arbitrary-) order nonlinear ordinary and delay differential equations. Among numerous articles dealing with this general problem, we refer the reader to the most recent papers [**1–3**, **6–9**, **17**, **18**, **20–22**, **24–29**, **31**, **33–37**, **39**] (we also refer to [**13**, **14**, **40**]).

On the other hand, several articles have appeared in the literature which are concerned with the asymptotic behaviour of solutions of nonlinear ordinary differential systems (see,

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for example, [**12**, **15**, **16**, **38**]; in particular, see the monograph by Mirzov [**23**] and the references cited therein).

For the basic theory of delay differential equations and systems, the reader is referred to [**4**, **5**, **10**, **11**].

2. A basic lemma

Here, we will establish the following basic lemma.

Lemma 2.1. *Assume that the function g is positive on $[0,\infty)\times(0,\infty)$, i.e.*

$$g(t,z)>0 \quad \textit{for all } t\geqslant 0 \textit{ and } z>0. \tag{2.1}$$

Also, assume that the function f is positive on $[0,\infty)\times(0,\infty)^m$, i.e.

$$f(t,w_1,\dots,w_m)>0 \quad \textit{for all } t\geqslant 0 \textit{ and } w_1>0,\dots,w_m>0. \tag{2.2}$$

Let (x,y) be a solution of the BVP (1.1)–(1.3) with x being non-negative on the interval $(0,\infty)$. Then x is always positive on $(0,\infty)$; moreover, y is positive on $[0,\infty)$.

We notice here that, because of the continuity of g on $[0,\infty)\times[0,\infty)$, the hypothesis that g is positive on $[0,\infty)\times(0,\infty)$, i.e. that (2.1) holds, implies that the function g is non-negative on $[0,\infty)\times[0,\infty)$, i.e.

$$g(t,z)\geqslant 0 \quad \text{for all } t\geqslant 0 \text{ and } z\geqslant 0. \tag{2.3}$$

Similarly, as f is continuous on $[0,\infty)\times[0,\infty)^m$, the hypothesis that f is positive on $[0,\infty)\times(0,\infty)^m$, i.e. that (2.2) holds, guarantees that the function f is non-negative on $[0,\infty)\times[0,\infty)^m$, i.e.

$$f(t,w_1,\dots,w_m)\geqslant 0 \quad \text{for all } t\geqslant 0 \text{ and } w_1\geqslant 0,\dots,w_m\geqslant 0. \tag{2.4}$$

Now, we shall present an observation. Assume that (2.1) holds, and let (x,y) be a solution of the BVP (1.1)–(1.3) such that y is positive on the interval $[0,\infty)$. Then, from the first equation of (1.1), it follows that

$$x'(t)>0 \quad \text{for every } t\geqslant 0$$

and so x is strictly increasing on $[0,\infty)$. Hence, as $x(0)=\phi(0)=0$, x is positive on $(0,\infty)$.

Proof of Lemma 2.1. The proof is accomplished by proving that y is positive on the interval $[0,\infty)$.

First of all, we see that x is non-negative on the whole interval $[-\tau,\infty)$ and so we must have $x(t-T_j(t))\geqslant 0$ for $t\geqslant 0$, $j=1,\dots,m$. Consequently, by (2.4),

$$f(t,x(t-T_1(t)),\dots,x(t-T_m(t)))\geqslant 0 \quad \text{for every } t\geqslant 0. \tag{2.5}$$

Moreover, we observe that, by Proposition 1.1, the solution (x,y) satisfies (1.4) and (1.5).

Now we show that $y(0) > 0$. To this end, by applying (1.5) for $t = 0$, we get

$$y(0) = \int_0^\infty f(s, x(s - T_1(s)), \dots, x(s - T_m(s)))\,\mathrm{d}s. \tag{2.6}$$

As $-\tau \leqslant -T_j(0) < 0$, $j = 1, \dots, m$, we have $x(-T_j(0)) = \phi(-T_j(0)) > 0$, $j = 1, \dots, m$. Thus, by (2.2), we must have

$$f(0, x(-T_1(0)), \dots, x(-T_m(0))) > 0,$$

i.e.

$$f(t, x(t - T_1(t)), \dots, x(t - T_m(t)))|_{t=0} > 0.$$

In view of (2.5) and the above inequality, it follows from (2.6) that $y(0)$ is always positive.

Next, we prove that y is positive on the interval $(0, \infty)$. Assume, for the sake of contradiction, that y is not necessarily positive on $(0, \infty)$. Then, as $y(0) > 0$, we see that y always has zeros in the interval $(0, \infty)$. Let $t_0 > 0$ be the first zero of y in $(0, \infty)$; i.e. y is positive on $[0, t_0)$, and $y(t_0) = 0$. In view of (2.1) and (2.3), it follows from the first equation of (1.1) that $x'(t) > 0$ for $t \in [0, t_0)$, and $x'(t_0) \geqslant 0$. Hence, x is strictly increasing on $[0, t_0)$ and x is increasing on $[0, t_0]$. Thus, as $x(0) = \phi(0) = 0$, we see that x is always positive on $(0, t_0]$. Furthermore, by taking into account the fact that $y(t_0) = 0$ and applying (1.5) for $t = t_0$, we obtain

$$\int_{t_0}^\infty f(s, x(s - T_1(s)), \dots, x(s - T_m(s)))\,\mathrm{d}s = 0.$$

So, because of (2.5), we must have

$$f(t, x(t - T_1(t)), \dots, x(t - T_m(t))) = 0 \quad \text{for every } t \geqslant t_0. \tag{2.7}$$

By (2.7), the second equation of (1.1) gives $y'(t) = 0$ for all $t \geqslant t_0$, which means that y is constant on $[t_0, \infty)$. Hence, since $y(t_0) = 0$, we have $y(t) = 0$ for every $t \geqslant t_0$. So, by taking into account (2.3), from (1.4) we obtain, for each $t \geqslant t_0$,

$$x(t) = \int_0^{t_0} g(s, y(s))\,\mathrm{d}s + \int_{t_0}^t g(s, y(s))\,\mathrm{d}s = x(t_0) + \int_{t_0}^t g(s, 0)\,\mathrm{d}s \geqslant x(t_0).$$

Thus, as $x(t_0) > 0$, we have $x(t) > 0$ for every $t \geqslant t_0$. Consequently, x is always positive on the interval $(0, \infty)$. Finally, by the assumption that $\lim_{t\to\infty}(t - T_j(t)) = \infty$, $j = 1, \dots, m$, we can consider a point $t_1 > 0$ so that $t - T_j(t) > 0$ for all $t \geqslant t_1$, $j = 1, \dots, m$. Then, as x is positive on $(0, \infty)$, we have $x(t - T_j(t)) > 0$ for every $t \geqslant t_1$, $j = 1, \dots, m$. Therefore, by using (2.2), we find that

$$f(t, x(t - T_1(t)), \dots, x(t - T_m(t))) > 0 \quad \text{for all } t \geqslant t_1,$$

which contradicts (2.7).

The proof of the lemma is complete. $\square$

3. The main result

Our main result is the following theorem.

Theorem 3.1. *Let the assumptions of Lemma 2.1 be satisfied, i.e. (2.1) and (2.2) hold. Moreover, assume that, for each $t \geqslant 0$, the function $g(t,\cdot)$ is increasing on $[0,\infty)$ in the sense that $g(t,z_1) \leqslant g(t,z_2)$ for any z_1, z_2 in $[0,\infty)$ with $z_1 \leqslant z_2$. Also, assume that, for each $t \geqslant 0$, the function $f(t,\cdot,\dots,\cdot)$ is increasing on $[0,\infty)^m$ in the sense that $f(t,w_1,\dots,w_m) \leqslant f(t,v_1,\dots,v_m)$ for any $(w_1,\dots,w_m)$, $(v_1,\dots,v_m)$ in $[0,\infty)^m$ with $w_1 \leqslant v_1,\dots,w_m \leqslant v_m$.*

Let there exist a real number $c > 0$ such that

$$\int_0^{\infty} f(t,\rho_1(t),\dots,\rho_m(t))\,\mathrm{d}t \leqslant c, \tag{3.1}$$

where, for each $j \in \{1,\dots,m\}$, the function ρ_j depends on ϕ, c, g and is defined by

$$\rho_j(t) = \begin{cases} \phi(t - T_j(t)), & \text{if } 0 \leqslant t \leqslant T_j(t), \\ \displaystyle\int_0^{t-T_j(t)} g(s,c)\,\mathrm{d}s, & \text{if } t \geqslant T_j(t). \end{cases} \tag{3.2}$$

(Clearly, ρ_j, $j = 1,\dots,m$, are non-negative continuous real-valued functions on the interval $[0,\infty)$.) Then the BVP (1.1)–(1.3) has at least one solution (x,y) such that

$$0 < x(t) \leqslant \int_0^t g(s,c)\,\mathrm{d}s \quad \text{for every } t > 0 \tag{3.3}$$

and

$$0 < y(t) \leqslant c \qquad \text{for every } t \geqslant 0. \tag{3.4}$$

Proof. Let Y be the set of all continuous real-valued functions y defined on the interval $[0,\infty)$ that satisfy

$$0 \leqslant y(t) \leqslant c \quad \text{for every } t \geqslant 0. \tag{3.5}$$

For any function y in Y, let x denote the continuous real-valued function on the interval $[-\tau,\infty)$ defined by (1.4). (Note that $\phi(0) = 0$.)

Consider an arbitrary function y in Y. Then, in view of (3.5), we can use (2.3) as well as the assumption that, for each $t \geqslant 0$, the function $g(t,\cdot)$ is increasing on $[0,\infty)$ to obtain

$$0 \leqslant g(t,y(t)) \leqslant g(t,c) \quad \text{for } t \geqslant 0.$$

This gives

$$0 \leqslant \int_0^t g(s,y(s))\,\mathrm{d}s \leqslant \int_0^t g(s,c)\,\mathrm{d}s \quad \text{for } t \geqslant 0,$$

which, by the definition of x by (1.4), may be written as

$$0 \leqslant x(t) \leqslant \int_0^t g(s,c)\,\mathrm{d}s \quad \text{for every } t \geqslant 0. \tag{3.6}$$

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From (1.4) and (3.6) it follows that, for any $j \in \{1, \dots, m\}$ and every $t \geqslant 0$,

$$0 \leqslant x(t - T_j(t)) \begin{cases} = \phi(t - T_j(t)), & \text{if } 0 \leqslant t \leqslant T_j(t), \\ \leqslant \displaystyle\int_0^{t-T_j(t)} g(s, c)\,\mathrm{d}s, & \text{if } t \geqslant T_j(t). \end{cases}$$

(Note that $x(0) = \phi(0) = 0$.) Thus, by virtue of (3.2), we have

$$0 \leqslant x(t - T_j(t)) \leqslant \rho_j(t) \quad \text{for every } t \geqslant 0,\ j = 1, \dots, m.$$

Hence, by using (2.4) as well as the assumption that, for each $t \geqslant 0$, the function $f(t, \cdot, \dots, \cdot)$ is increasing on $[0, \infty)^m$, we find that

$$0 \leqslant f(t, x(t - T_1(t)), \dots, x(t - T_m(t))) \leqslant f(t, \rho_1(t), \dots, \rho_m(t)) \quad \text{for } t \geqslant 0. \tag{3.7}$$

Taking into account (3.7), for $t \geqslant 0$, we obtain

$$\begin{aligned} 0 &\leqslant \int_t^\infty f(s, x(s - T_1(s)), \dots, x(s - T_m(s)))\,\mathrm{d}s \\ &\leqslant \int_t^\infty f(s, \rho_1(s), \dots, \rho_m(s))\,\mathrm{d}s \\ &\leqslant \int_0^\infty f(s, \rho_1(s), \dots, \rho_m(s))\,\mathrm{d}s \end{aligned}$$

and consequently, because of hypothesis (3.1),

$$0 \leqslant \int_t^\infty f(s, x(s - T_1(s)), \dots, x(s - T_m(s)))\,\mathrm{d}s \leqslant c \quad \text{for every } t \geqslant 0. \tag{3.8}$$

As (3.8) holds true for all functions y in Y, we see that the formula

$$(My)(t) = \int_t^\infty f(s, x(s - T_1(s)), \dots, x(s - T_m(s)))\,\mathrm{d}s \quad \text{for } t \geqslant 0$$

makes sense for any y in Y, and defines a mapping M of Y into itself. We will show that the mapping M is increasing with respect to the usual pointwise ordering in Y. To this end, let us consider two arbitrary functions y and $\tilde{y}$ in Y with $y \leqslant \tilde{y}$, i.e. with $y(t) \leqslant \tilde{y}(t)$ for $t \geqslant 0$. Let $\tilde{x}$ denote the continuous real-valued function on $[-\tau, \infty)$ defined by (1.4) with $\tilde{x}$ instead of x and $\tilde{y}$ in place of y, i.e.

$$\tilde{x}(t) = \begin{cases} \phi(t), & \text{for } -\tau \leqslant t \leqslant 0, \\ \displaystyle\int_0^t g(s, \tilde{y}(s))\,\mathrm{d}s, & \text{for } t \geqslant 0. \end{cases} \tag{3.9}$$

As $0 \leqslant y(t) \leqslant \tilde{y}(t)$ for $t \geqslant 0$, by using (2.3) as well as the assumption that, for each $t \geqslant 0$, the function $g(t, \cdot)$ is increasing on $[0, \infty)$, we get

$$0 \leqslant \int_0^t g(s, y(s))\,\mathrm{d}s \leqslant \int_0^t g(s, \tilde{y}(s))\,\mathrm{d}s \quad \text{for } t \geqslant 0.$$

So, by taking into account the definitions of x and $\tilde{x}$ by (1.4) and (3.9), respectively, we have

$$0 \leqslant x(t) \leqslant \tilde{x}(t) \quad \text{for every } t \geqslant 0.$$

Thus, bearing in mind (1.4) and (3.9), we obtain, for each $j \in \{1, \dots, m\}$ and every $t \geqslant 0$,

$$0 \leqslant x(t - T_j(t)) \begin{cases} = \phi(t - T_j(t)) = \tilde{x}(t - T_j(t)), & \text{if } 0 \leqslant t \leqslant T_j(t), \\ \leqslant \tilde{x}(t - T_j(t)), & \text{if } t \geqslant T_j(t). \end{cases}$$

Hence, by the assumption that, for each $t \geqslant 0$, the function $f(t, \cdot, \dots, \cdot)$ is increasing on $[0, \infty)^m$, we derive

$$f(t, x(t - T_1(t)), \dots, x(t - T_m(t))) \leqslant f(t, \tilde{x}(t - T_1(t)), \dots, \tilde{x}(t - T_m(t)))$$

for all $t \geqslant 0$. This immediately gives

$$(My)(t) \leqslant (M\tilde{y})(t) \quad \text{for every } t \geqslant 0,$$

i.e. $My \leqslant M\tilde{y}$. Consequently, the mapping M is increasing.

Now, we define

$$y_0(t) = c \quad \text{for } t \geqslant 0$$

and

$$y_{n+1} = My_n, \quad n = 0, 1, \dots .$$

As M is an increasing mapping of Y into itself, it is not difficult to see that $(y_n)_{n=0,1,\dots}$ is a decreasing sequence of functions in Y. Set

$$y = \lim_{n \to \infty} y_n \quad \text{pointwise on } [0, \infty).$$

Let x be defined by (1.4). Moreover, for any integer $n \geqslant 0$, let x_n denote the continuous real-valued function on $[-\tau, \infty)$ defined by (1.4) with x_n in place of x and y_n instead of y, i.e.

$$x_n(t) = \begin{cases} \phi(t), & \text{for } -\tau \leqslant t \leqslant 0, \\ \displaystyle\int_0^t g(s, y_n(s))\, \mathrm{d}s, & \text{for } t \geqslant 0. \end{cases}$$

Then

$$x = \lim_{n \to \infty} x_n \quad \text{pointwise on } [-\tau, \infty).$$

By (3.7), we have

$$0 \leqslant f(t, x_n(t - T_1(t)), \dots, x_n(t - T_m(t))) \leqslant f(t, \rho_1(t), \dots, \rho_m(t))$$

for every $t \geqslant 0$ and all non-negative integers n. As hypothesis (3.1) implies, in particular, that

$$\int_0^\infty f(t, \rho_1(t), \dots, \rho_m(t))\, \mathrm{d}t < \infty,$$

we can apply the Lebesgue dominated convergence theorem to obtain, for every $t \geqslant 0$,

$$\lim_{n\to\infty}\int_t^\infty f(s, x_n(s-T_1(s)),\dots,x_n(s-T_m(s)))\,\mathrm{d}s = \int_t^\infty f(s, x(s-T_1(s)),\dots,x(s-T_m(s)))\,\mathrm{d}s.$$

Thus, we conclude that

$$\lim_{n\to\infty}(My_n)(t) = (My)(t) \quad \text{for every } t \geqslant 0.$$

Hence, we have

$$y(t) = \lim_{n\to\infty} y_{n+1}(t) = \lim_{n\to\infty}(My_n)(t) = (My)(t) \quad \text{for } t \geqslant 0$$

and consequently $y = My$, i.e. (1.5) holds. Also, (1.4) is satisfied. Therefore, by Proposition 1.1, (x, y) is a solution of the BVP (1.1)–(1.3). As $y \in Y$, (3.5) and (3.6) are satisfied. By (3.6), x is non-negative on the interval $(0,\infty)$; hence, Lemma 2.1 guarantees that x is always positive on $(0,\infty)$ and, in addition, that y is necessarily positive on $[0,\infty)$. Thus, the solution (x, y) satisfies (3.3) and (3.4).

The proof of the theorem is complete. □

It is evident that Lemma 2.1 plays a crucial role in proving Theorem 3.1. Moreover, one may easily see that the proof of Lemma 2.1 is essentially based on the use of the hypothesis that the initial function ϕ is positive on the interval $[-\tau, 0)$ (as well as on assumptions (2.1) and (2.2)). This hypothesis is fundamental, because of the fact that $\tau > 0$, which is a consequence of the fact that the delays T_j, $j = 1,\dots,m$, are positive on the interval $[0,\infty)$. It is clear that such a hypothesis cannot be posed in the case of the nonlinear two-dimensional ordinary differential systems, and hence Lemma 2.1 (and, consequently, Theorem 3.1) cannot be applied to the corresponding ordinary boundary-value problem. More precisely, let us consider the nonlinear two-dimensional delay differential system

$$x'(t) = g(t, y(t)), \qquad y'(t) = -f_0(t, x(t-\tau)), \tag{3.10}$$

where f_0 is a continuous real-valued function on $[0,\infty)\times\mathbb{R}$, and τ is a positive real constant. For $\tau = 0$, system (3.10) reduces to nonlinear two-dimensional ordinary differential system

$$x'(t) = g(t, y(t)), \qquad y'(t) = -f_0(t, x(t)), \tag{3.11}$$

and the initial condition (1.2) becomes

$$x(0) = 0. \tag{3.12}$$

That is, when $\tau = 0$, the BVP (3.10), (1.2), (1.3) reduces to the BVP (3.11), (3.12), (1.3). Lemma 2.1 and Theorem 3.1 can be applied to the delay BVP (3.10), (1.2), (1.3), while these results are not applicable to the ordinary BVP (3.11), (3.12), (1.3).

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4. Application to second-order nonlinear delay differential equations

Consider the second-order nonlinear delay differential equation

$$[r(t)x'(t)]' + f(t, x(t - T_1(t)), \dots, x(t - T_m(t))) = 0, \tag{4.1}$$

where r is a positive continuous real-valued function on the interval $[0, \infty)$. We are interested in solutions of (4.1) on the *whole* interval $[0, \infty)$. By a *solution on* $[0, \infty)$ of (4.1), we mean a continuous real-valued function x defined on the interval $[-\tau, \infty)$, which is continuously differentiable on $[0, \infty)$ and such that rx' is continuously differentiable on $[0, \infty)$ and (4.1) is satisfied for all $t \geqslant 0$. We associate with the delay differential equation (4.1) the initial condition (1.2) as well as the condition

$$\lim_{t\to\infty} r(t)x'(t) = 0. \tag{4.2}$$

Equations (4.1), (1.2), (4.2) constitute a BVP on the half-line. A solution of the BVP (4.1), (1.2), (4.2) is a solution on $[0, \infty)$ of (4.1) that satisfies the conditions (1.2) and (4.2).

The substitution $rx' = y$ transforms the second-order nonlinear delay differential equation (4.1) into the equivalent nonlinear two-dimensional delay differential system

$$x'(t) = \frac{1}{r(t)}y(t), \qquad y'(t) = -f(t, x(t - T_1(t)), \dots, x(t - T_m(t))). \tag{4.3}$$

By this substitution, the BVP (4.1), (1.2), (4.2) is transformed into the equivalent BVP (4.3), (1.2), (1.3), which is a special case of the BVP (1.1)–(1.3).

For convenience, we introduce some notation. By R we will denote the continuous real-valued function on the interval $[0, \infty)$ defined by the formula

$$R(t) = \int_0^t \frac{\mathrm{d}s}{r(s)} \quad \text{for } t \geqslant 0.$$

Clearly, $R(0) = 0$, and R is positive on $(0, \infty)$.

By applying Theorem 3.1 to the BVP (4.3), (1.2), (1.3), we are led to the following result concerning the BVP (4.1), (1.2), (4.2).

Corollary 4.1. *Assume that the function f is positive on $[0, \infty) \times (0, \infty)^m$, i.e. (2.2) holds. Moreover, assume that, for each $t \geqslant 0$, the function $f(t, \cdot, \dots, \cdot)$ is increasing on $[0, \infty)^m$.*

Let there exist a real number $c > 0$ such that

$$\int_0^\infty f(t, \sigma_1(t), \dots, \sigma_m(t))\,\mathrm{d}t \leqslant c,$$

where, for each $j \in \{1, \dots, m\}$, the function σ_j depends on ϕ, c, r and is defined by

$$\sigma_j(t) = \begin{cases} \phi(t - T_j(t)), & \text{if } 0 \leqslant t \leqslant T_j(t), \\ cR(t - T_j(t)), & \text{if } t \geqslant T_j(t). \end{cases}$$

(*Clearly,* σ_j, $j = 1, \dots, m$, *are non-negative continuous real-valued functions on the interval* $[0, \infty)$.) *Then the BVP (4.1), (1.2), (4.2) has at least one solution* x *such that*

$$0 < x(t) \leqslant cR(t) \quad \textit{for every } t > 0$$

and

$$0 < r(t)x'(t) \leqslant c \quad \textit{for every } t \geqslant 0.$$

By applying Corollary 4.1 to the particular case where $r(t) = 1$ for $t \geqslant 0$, we immediately arrive at the main result in [**31**]. (Note that, in this particular case, we have $R(t) = t$ for $t \geqslant 0$.)

For the sake of completeness, we also give the application of Lemma 2.1 to the BVP (4.1), (1.2), (4.2). By specifying Lemma 2.1 to the BVP (4.3), (1.2), (1.3), we get the next result.

Lemma. *Assume that (2.2) holds. Let* x *be a solution of the BVP (4.1), (1.2), (4.2) that is non-negative on the interval* $(0, \infty)$. *Then* x *is always positive on* $(0, \infty)$; *moreover,* x' *is positive on* $[0, \infty)$ (*and so* x *is strictly increasing on* $[0, \infty)$).

In the particular case where $r(t) = 1$ for $t \geqslant 0$, the above result has been established in [**31**].

Before closing this section, let us consider the particular case where the first equation of (1.1) is linear, i.e. the case of the nonlinear two-dimensional delay differential system

$$x'(t) = q(t)y(t), \qquad y'(t) = -f(t, x(t - T_1(t)), \dots, x(t - T_m(t))), \tag{4.4}$$

where q is a positive continuous real-valued function on the interval $[0, \infty)$. Theorem 3.1 can be applied to the BVP (4.4), (1.2), (1.3). On the other hand, we immediately see that (4.4) can be transformed into the equivalent second-order nonlinear delay differential equation

$$\left[\frac{1}{q(t)}x'(t)\right]' + f(t, x(t - T_1(t)), \dots, x(t - T_m(t))) = 0. \tag{4.5}$$

We associate with (4.5) the initial condition (1.2) and the condition

$$\lim_{t \to \infty} \frac{1}{q(t)}x'(t) = 0. \tag{4.6}$$

It is remarkable that, instead of applying Theorem 3.1 to the BVP (4.4), (1.2), (1.3), one can apply Corollary 4.1 to the BVP (4.5), (1.2), (4.6).

5. Application to nonlinear two-dimensional delay differential systems of Emden–Fowler type and examples

Consider the nonlinear two-dimensional delay differential system of Emden–Fowler type

$$x'(t) = q(t)|y(t)|^{\beta} \operatorname{sgn} y(t), \qquad y'(t) = -\sum_{j=1}^{m} p_j(t)|x(t - \tau_j)|^{\gamma_j} \operatorname{sgn} x(t - \tau_j), \tag{5.1}$$

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where m is a positive integer, q is a positive continuous real-valued function on the interval $[0,\infty)$, p_j, $j=1,\dots,m$, are non-negative continuous real-valued functions on $[0,\infty)$, τ_j, $j=1,\dots,m$, are positive real constants, and β and γ_j, $j=1,\dots,m$, are positive real numbers. Suppose that

$$\sum_{j=1}^{m} p_j(t) > 0 \quad \text{for all } t \geqslant 0.$$

We notice that, as p_j, $j=1,\dots,m$, are non-negative on $[0,\infty)$, the last hypothesis means exactly that, for each $t \geqslant 0$, there exists at least one index $j \in \{1,\dots,m\}$ such that $p_j(t) > 0$.

Set

$$\tau = \max_{j=1,\dots,m} \tau_j.$$

(τ is a positive real number.) Our interest is concentrated on solutions of (5.1) on the *whole* interval $[0,\infty)$. A *solution on* $[0,\infty)$ of (5.1) is a pair of two continuous real-valued functions x and y defined on the intervals $[-\tau,\infty)$ and $[0,\infty)$, respectively, which are continuously differentiable on $[0,\infty)$ and satisfy (5.1) for all $t \geqslant 0$. The initial condition (1.2) and the condition (1.3) are associated with the delay differential system (5.1). Hence, we have the BVP (5.1), (1.2), (1.3).

For convenience, we denote by Q the continuous real-valued function on the interval $[0,\infty)$ defined by the formula

$$Q(t) = \int_0^t q(s)\,\mathrm{d}s \quad \text{for } t \geqslant 0.$$

Note that $Q(0) = 0$ and that Q is positive on $(0,\infty)$.

By applying Theorem 3.1 to the particular case of the BVP (5.1), (1.2), (1.3), we are led to the following corollary.

Corollary 5.1. *Let there exist a real number $c > 0$ such that*

$$\sum_{j=1}^{m} \int_0^{\tau_j} [\phi(t-\tau_j)]^{\gamma_j} p_j(t)\,\mathrm{d}t + \sum_{j=1}^{m} c^{\beta\gamma_j} \int_{\tau_j}^{\infty} [Q(t-\tau_j)]^{\gamma_j} p_j(t)\,\mathrm{d}t \leqslant c.$$

Then the BVP (5.1), (1.2), (1.3) has at least one solution (x, y) such that

$$0 < x(t) \leqslant c^{\beta} Q(t) \quad \textit{for every } t > 0 \tag{5.2}$$

and (3.4) holds.

Now, in order to present some examples demonstrating the applicability of our theorem, we shall concentrate on nonlinear two-dimensional Emden–Fowler-type delay differential systems with one constant delay.

Let us consider the delay differential system of Emden–Fowler type

$$x'(t) = q(t)|y(t)|^{\beta} \operatorname{sgn} y(t), \qquad y'(t) = -p(t)|x(t-\tau)|^{\gamma} \operatorname{sgn} x(t-\tau), \tag{5.3}$$

where p and q are positive continuous real-valued functions on the interval $[0,\infty)$, τ is a positive real constant, and β and γ are positive real numbers.

In the particular case of the BVP (5.3), (1.2), (1.3), Corollary 5.1 is formulated as follows.

Corollary. *Let there exist a real number $c > 0$ such that*

$$\int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t + c^{\beta\gamma} \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t \leqslant c. \tag{5.4}$$

Then the BVP (5.3), (1.2), (1.3) has at least one solution (x, y) such that (5.2) and (3.4) hold.

Example 5.2. Consider the BVP (5.3), (1.2), (1.3) with $\beta\gamma = 1$. In this case, condition (5.4) is written as

$$\int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t + c \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t \leqslant c$$

or

$$\int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t \leqslant c\biggl\{1 - \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t\biggr\}. \tag{5.5}$$

We see that, if

$$\int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t < 1, \tag{5.6}$$

then inequality (5.5) holds true (as an equality) for

$$c = \frac{\int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t}{1 - \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t}. \tag{5.7}$$

Clearly, c is a positive real number. Thus, we arrive at the next result.

Corollary. *Assume that $\beta\gamma = 1$. Let condition (5.6) be satisfied, and let $c > 0$ be the real number given by (5.7). Then the BVP (5.3), (1.2), (1.3) has at least one solution (x, y) such that (5.2) and (3.4) hold.*

Example 5.3. Let us consider the BVP (5.3), (1.2), (1.3) with $\beta\gamma = \frac{1}{2}$. Here, condition (5.4) becomes

$$\int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t + c^{1/2} \int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t \leqslant c,$$

namely

$$c - \biggl\{\int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t\biggr\} c^{1/2} - \int_0^{\tau} [\phi(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t \geqslant 0. \tag{5.8}$$

Assume that

$$\int_{\tau}^{\infty} [Q(t-\tau)]^{\gamma} p(t)\,\mathrm{d}t < \infty. \tag{5.9}$$

Following the lines of [**31**, Example 1], we can show that (5.8) holds with $c > 0$ if and only if

$$c \geqslant \left(\frac{1}{2}\int_{\tau}^{\infty}[Q(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t + \sqrt{\left\{\frac{1}{2}\int_{\tau}^{\infty}[Q(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t\right\}^{2} + \int_{0}^{\tau}[\phi(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t}\right)^{2}.$$

Thus, we conclude that (5.8) is valid (as an equality) for

$$c = \left(\frac{1}{2}\int_{\tau}^{\infty}[Q(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t + \sqrt{\left\{\frac{1}{2}\int_{\tau}^{\infty}[Q(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t\right\}^{2} + \int_{0}^{\tau}[\phi(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t}\right)^{2}. \tag{5.10}$$

Hence, we obtain the following result.

Corollary. *Assume that $\beta\gamma = \frac{1}{2}$. Let condition (5.9) be satisfied, and let $c > 0$ be the real number given by (5.10). Then the BVP (5.3), (1.2), (1.3) has at least one solution (x, y) such that (5.2) and (3.4) hold.*

Example 5.4. Consider the case of the BVP (5.3), (1.2), (1.3) with $\beta\gamma = 2$. In this case, condition (5.4) takes the form

$$\int_{0}^{\tau}[\phi(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t + c^{2}\int_{\tau}^{\infty}[Q(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t \leqslant c,$$

i.e.

$$\left\{\int_{\tau}^{\infty}[Q(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t\right\}c^{2} - c + \int_{0}^{\tau}[\phi(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t \leqslant 0. \tag{5.11}$$

After a long analysis similar to that used in [**31**, Example 2], we can conclude that, if

$$\left\{\int_{\tau}^{\infty}[Q(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t\right\}\int_{0}^{\tau}[\phi(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t \leqslant \tfrac{1}{4}, \tag{5.12}$$

then (5.11) holds (as an equality) for

$$c = \frac{1 - \sqrt{1 - 4\{\int_{\tau}^{\infty}[Q(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t\}\int_{0}^{\tau}[\phi(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t}}{2\int_{\tau}^{\infty}[Q(t-\tau)]^{\gamma}p(t)\,\mathrm{d}t}. \tag{5.13}$$

Thus, we are led to the next result.

Corollary. *Assume that $\beta\gamma = 2$. Let condition (5.12) be satisfied, and let $c > 0$ be the real number given by (5.13). Then the BVP (5.3), (1.2), (1.3) has at least one solution (x, y) such that (5.2) and (3.4) hold.*

Before closing this section and ending the paper, we note that, by the use of the particular results obtained in the above general examples, one can construct specific examples in which our theorem is applicable. For such specific examples for the special case of second-order nonlinear delay differential equations, we refer the reader to [**31**].

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