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EXISTENCE OF SOLUTIONS FOR A SYSTEM INVOLVING SCHRÖDINGER OPERATORS WITH WEIGHTS

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Abstract In this paper, we obtain some results on the existence of solutions for the system

$$(-\Delta + q_i)u_i = \mu_i m_i u_i + f_i(x, u_1, \dots, u_n) \text{ in } \mathbb{R}^N, \quad i = 1, \dots, n,$$

where each of the q_i are positive potentials satisfying $\lim_{|x|\to+\infty} q_i(x) = +\infty$, each of the m_i are bounded positive weights and each of the μ_i are real parameters. Depending upon the hypotheses on f_i, we use either the method of sub- and supersolutions or a bifurcation method.

Keywords: Schrödinger operators; sub- and supersolutions method; bifurcation method

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1. Introduction

1.1. Definition of the problem

In this paper, we study the existence of solutions for the system

$$(-\Delta + q_i)u_i = \mu_i m_i u_i + f_i(x, u_1, \dots, u_n) \text{ in } \mathbb{R}^N, \quad i = 1, \dots, n, \tag{1.1}$$

where, for each $i = 1, \dots, n$, $\mu_i \in \mathbb{R}$, and the following hypotheses hold:

(h1) $q_i \in L^2_{\text{loc}}(\mathbb{R}^N)$, and $\lim_{|x|\to+\infty} q_i(x) = +\infty$, $q_i \geqslant \text{const.} > 0$;

(h2) $m_i \in L^\infty(\mathbb{R}^N)$, and there exists $\beta_i > 0$ such that $0 < m_i(x) \leqslant \beta_i$, for all $x \in \mathbb{R}^N$.

Hypotheses on the functions f_i are specified below.

The variational space is denoted by $V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$, where, for each $i = 1, \dots, n$, $V_{q_i}(\mathbb{R}^N)$ is the completion of $D(\mathbb{R}^N)$, the set of $\mathcal{C}^\infty$ functions with compact supports, with respect to the norm

$$\|u\|^2_{q_i} = \int_{\mathbb{R}^N} |\nabla u|^2 + q_i u^2.$$

We recall that the embedding of each $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact.

We also define the norm

$$\|u\|_{m_i}^2 = \int_{\mathbb{R}^N} m_i u^2 \quad \text{for } u \in L^2(\mathbb{R}^N).$$

Hypothesis (h2) ensures that $\|\cdot\|_{m_i}$ is a norm in $L^2(\mathbb{R}^N)$.

We denote by M_i the operator of multiplication by m_i in $L^2(\mathbb{R}^N)$. The operator

$$(-\Delta + q_i)^{-1} M_i : (L^2(\mathbb{R}^N), \|\cdot\|_{m_i}) \to (L^2(\mathbb{R}^N), \|\cdot\|_{m_i})$$

is positive self-adjoint and compact. Therefore, its spectrum is discrete and consists of a positive sequence tending to 0. We denote by λ_i the inverse of the first eigenvalue and by ϕ_i the corresponding eigenfunction that satisfies

$$(-\Delta + q_i)\phi_i = \lambda_i m_i \phi_i \text{ in } \mathbb{R}^N, \quad \|\phi_i\|_{m_i} = 1. \tag{1.2}$$

We recall that λ_i is simple and that $\phi_i > 0$ [**1**, Theorem 2.2]. By the Courant–Fischer formulae, λ_i is given by

$$\lambda_i = \inf\left\{\frac{\int_{\mathbb{R}^N} |\nabla\phi|^2 + q_i\phi^2}{\int_{\mathbb{R}^N} m_i\phi^2},\ \phi \in D(\mathbb{R}^N)\right\}. \tag{1.3}$$

1.2. Some previous results

The author has already studied the existence of solutions for the system (1.1) in different cases: linear or semilinear systems, cooperative or non-cooperative systems. We recall here some of these earlier results.

For the linear case, we rewrite the system (1.1) in the following form:

$$(-\Delta + q_i)u_i = \sum_{j=1}^{n} a_{ij}u_j + f_i \text{ in } \mathbb{R}^N, \quad i = 1, \dots, n. \tag{1.4}$$

We denote by $\lambda(\rho)$ the first eigenvalue (which is positive and simple) of the operator $-\Delta+\rho$ considered in $L^2(\mathbb{R}^N)$ for any potential ρ that satisfies (h1). We define by $\Lambda = (l_{ij})$ the diagonal matrix such that $l_{ii} = \lambda(q_i - a_{ii})$ for $i = 1, \dots, n$ and by $A = (a_{ij}^*)$ the $n \times n$ matrix of the coefficients of the system (1.4) that is defined by

$$a_{ii}^* = 0 \quad \text{and} \quad a_{ij}^* = \|a_{ij}\|_{L^\infty(\mathbb{R}^N)} \text{ for } i \neq j.$$

For a cooperative system, by using the maximum principle and the Lax–Milgram theorem, Alziary *et al*. [**2**] obtained the following result.

Theorem 1.1 (a cooperative system with constant coefficients [2]). *Assume that (h1) is satisfied. Assume also that, for each $i \neq j$, $a_{ij} \in \mathbb{R}$ and $a_{ij} > 0$, and, for each i, $f_i \in L^2(\mathbb{R}^N)$.*

If $\Lambda - A$ is a non-singular M-matrix, then the system (1.4) has a unique solution. Moreover, if $f_i \geqslant 0$ for each i, then this solution is non-negative.

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This result has been extended for systems with bounded coefficients $a_{ij} \in L^\infty(\mathbb{R}^N)$ either in the case of a cooperative system (i.e. $a_{ij} \geqslant 0$ if $i \neq j$) [**4**] or in the case of a not necessarily cooperative system [**5**] using an approximation method and the Schauder fixed point theorem.

Moreover, for the semilinear case (with weights)

$$(-\Delta + q_i)u_i = \mu_i m_i u_i + \sum_{j=1;\, j\neq i}^{n} a_{ij}u_j + f_i(x, u_1, \dots, u_n) \text{ in } \mathbb{R}^N, \quad i = 1, \dots, n, \quad (1.5)$$

when each of the potentials q_i satisfy (h1), under the assumptions of

(i) non-negativity and regularity for the weights m_i,

$$0 \leqslant m_i \in L^{N/2}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad m_i \neq 0,$$

(ii) relations between the coefficients and the weights,

$$i \neq j \Rightarrow 0 \leqslant a_{ij} \leqslant k_{ij}\sqrt{m_i}\sqrt{m_j} \quad \text{with } k_{ij} \in \mathbb{R}^+,$$

(iii) regularity for each of the f_i: there exists a function $\theta_i \in L^2(\mathbb{R}^N)$ such that $|f_i(x, u_1, \dots, u_n)| \leqslant \theta_i$ for all $u_1, \dots, u_n \in L^2(\mathbb{R}^N)$ and f_i is Lipschitz with respect to u_i, uniformly in x,

we recall from [**6**] the following result.

Let $D = (d_{ij})$ be defined as the $n \times n$ matrix given by $d_{ii} = \lambda_i - \mu_i$ and $d_{ij} = -k_{ij}$ otherwise. If D is a non-singular M-matrix, then the system (1.5) has at least one solution.

Note that in all of these precedent results, we have assumed that either $a_{ii} < \lambda(q_i)$ or $\mu_i < \lambda_i$ for each i.

In this paper, we study the existence of solutions for the system (1.1) in the case of $\mu_i > \lambda_i$, μ_i near λ_i for each i. (We recall that λ_i is defined by (1.3).)

1.3. Notation and main results

In § 2, we will follow a method developed in [**9**] for the p-Laplacian in a bounded domain of $\mathbb{R}^N$. This method was adapted in [**7**] for an equation defined in $\mathbb{R}^N$, involving a Schrödinger operator with a potential and a weight that satisfy hypotheses (h1) and (h2). We write (1.1) in the form

$$(-\Delta + q_i)u_i = \mu_i m_i u_i + \sum_{j=1;\, j\neq i}^{n} a_{ij}u_i^p u_j^q + \sum_{j=1;\, j\neq i}^{n} f_{ij}u_j^{p+q} \text{ in } \mathbb{R}^N, \quad i = 1, \dots, n, \quad (1.6)$$

where

$$N = 3, 4, \quad \gamma = 2^* = \frac{2N}{N-2} = 6, 4, \quad (p, q) \in \mathbb{N}^2 \text{ such that } p + q < \gamma,\ p > 0,\ q > 0.$$

We define, for $C \in \mathbb{R}$, $C > 0$ and C sufficiently large, the set

$$X_{q_i,C} = \{\phi \in V_{q_i}(\mathbb{R}^N),\ \phi_i \leqslant \phi \leqslant C \text{ a.e.}\} \quad (1.7)$$

(which is possible by the properties of ϕ_i).

We use the following hypotheses.

(h3) For each $i, j = 1, \dots, n$, $a_{ij} \in L^\infty(\mathbb{R}^N)$ and $f_{ij} \in L^\infty(\mathbb{R}^N)$.

(h4) For each $i, j = 1, \dots, n$, $f_{ij} \geqslant 0$ a.e.

(h5) For each $i = 1, \dots, n$, there exists $j_i \in \{1, \dots, n\} - \{i\}$ such that the following items hold:

(a) if we define $\Omega_{i,+} := \{x \in \mathbb{R}^N,\ a_{ij_i} > 0\}$ and $\Omega_{i,-} := \{x \in \mathbb{R}^N,\ a_{ij_i} < 0\}$, then $\operatorname{meas}(\Omega_{i,+}) \neq 0$ and $\operatorname{meas}(\Omega_{i,-}) \neq 0$;

(b) for each $k \in \{1, \dots, n\} - \{i, j_i\}$, a_{ik} is a non-negative function, equal to 0 in D_i, where D_i is a measurable subset of $\Omega_{i,-}$ with positive measure;

(c) for each $k \in \{1, \dots, n\}$, $f_{ik} = 0$ in D_i.

(h6) There exist $\varepsilon > 0$ and $l \geqslant 1$ such that for each $i = 1, \dots, n$, $a_{ij_i} \geqslant -\varepsilon m_i$ and $\varepsilon < \mu_i / p(lC)^{p+q-1}$.

(h7) For each $i = 1, \dots, n$, there exists a positive constant k_{ij_i} such that

$$k_{ij_i} \leqslant \frac{(p+q)}{lq(lC)^{p+q-1}} \quad \text{and} \quad a_{ij_j} \geqslant -k_{ij_i} f_{ij_i} \phi_{j_i}^{p+q-1} \text{ a.e.}$$

Note that (h5)–(h7) are technical hypotheses and allow (for each i) a function a_{ij_i} to change sign. We define

$$F_i(u_1, \dots, u_n) := \int_{\mathbb{R}^N} \left[\sum_{j=1,\, j\neq i}^{n} a_{ij} u_i^{p+1} u_j^q + (p+1) \sum_{j=1,\, j\neq i}^{n} f_{ij} u_j^{p+q} u_i \right] \tag{1.8}$$

for all $i = 1, \dots, n$ and for all $(u_1, \dots, u_n) \in V_{q_1}(\mathbb{R}^N) \times \dots \times V_{q_n}(\mathbb{R}^N)$, and we also define

$$H_{\mu_i}(v) := \int_{\mathbb{R}^N} [|\nabla v|^2 + q_i v^2 - \mu_i m_i v^2] \tag{1.9}$$

for all $i = 1, \dots, n$ and for all $v \in V_{q_i}(\mathbb{R}^N)$.

Let

$$\lambda_i^* := \sup_{v_i \in V_{q_i}(\mathbb{R}^N),\, v_i \geqslant 0} \left\{ \inf_{\phi \in \Phi_{v_i}} \left\{ \frac{\int_{\mathbb{R}^N} \nabla v_i \cdot \nabla \phi + q_i v_i \phi}{\int_{\mathbb{R}^N} m_i v_i \phi} \right\} \right\} \tag{1.10}$$

and

$$\lambda_i^{**} := \sup_{v_i \in X_{q_i, C}} \left\{ \inf_{\phi \in \Phi_{v_i}} \left\{ \frac{\int_{\mathbb{R}^N} \nabla v_i \cdot \nabla \phi + q_i v_i \phi}{\int_{\mathbb{R}^N} m_i v_i \phi} \right\} \right\}, \tag{1.11}$$

where

$$\begin{aligned} \Phi_{v_i} := \Bigl\{ &\phi \in D(\mathbb{R}^N),\ \phi \geqslant 0,\ \text{such that, for any } j \neq i, \\ &\text{there exists } v_j \in V_{q_j}(\mathbb{R}^N),\ v_j \geqslant 0 \text{ and } \frac{\partial F_i}{\partial u_i}(v_1, \dots, v_n)(\phi) \geqslant 0 \Bigr\} \end{aligned} \tag{1.12}$$

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and where $\partial F_i/\partial u_i$ denotes the ith partial derivative of F_i. Note that the existence of λ_i^* and λ_i^{**} is due to (h3)–(h5) and also note that $\lambda_i^{**} \leqslant \lambda_i^*$.

We also use the following hypotheses for each $i = 1, \dots, n$:

(h8) $\lambda_i^{**} < \infty$;

(h9) $\lambda_i^* < \infty$.

We obtain the main result of § 2, as follows.

Theorem 1.2. *Assume that (h1)–(h8) are satisfied. If* $\lambda_i + \varepsilon(lC)^{p+q-1} < \mu_i < \lambda_i^{**}$ *for each* $i = 1, \dots, n$, *then the system (1.6) has at least one positive solution in* $X_{q_1,C} \times \dots \times X_{q_n,C}$.

Recall that we have defined λ_i by (1.3), λ_i^{**} by (1.11) and $X_{q_i,C}$ by (1.7).

Finally, in § 3, we obtain a result on the existence of solutions, by considering bifurcation solutions from the zero solution, for the semilinear system (1.1).

We define $V = \prod_{i=1}^n V_{q_i}(\mathbb{R}^N)$ and denote by $\langle\cdot,\cdot\rangle_V$ the inner product in V such that, for all $v = (v_1, \dots, v_n) \in V$ and all $w = (w_1, \dots, w_n) \in V$,

$$\langle v, w\rangle_V = \sum_{i=1}^n \langle v_i, w_i\rangle_{q_i}. \tag{1.13}$$

We define the operator

$$T : \mathbb{R}^n \times V \to V, \quad T = (T^1, \dots, T^n), \tag{1.14}$$

by

$$T^i : \mathbb{R}^n \times V \to V_{q_i}(\mathbb{R}^N)$$

if $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$, $u = (u_1, \dots, u_n) \in V$, $v_i \in V_{q_i}(\mathbb{R}^N)$ and

$$\langle T^i(\mu, u), v_i\rangle_{q_i} = \int_{\mathbb{R}^N} [\nabla u_i \cdot \nabla v_i + q_i u_i v_i - \mu_i m_i u_i v_i - f_i(x, u)v_i].$$

We obtain the main result of § 3 using the following hypothesis.

(h10) (i) For each $i = 1, \dots, n$, $f_i : \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}$, defined by $f_i(x, y_1, \dots, y_n)$ with $x \in \mathbb{R}^N$ and $(y_1, \dots, y_n) \in \mathbb{R}^n$, satisfies $f_i(x, 0, \dots, 0) = 0$ for all $x \in \mathbb{R}^N$.

(ii) For each $i = 1, \dots, n$, f_i is Fréchet differentiable with respect to each variable y_i and each derivative $\partial f_i(x, \cdot)/\partial y_j$ is continuous and bounded, uniformly in x.

(iii) For each $i, j = 1, \dots, n$, $\partial f_i/\partial y_j(x, 0, \dots, 0) = 0$.

Theorem 1.3. *Assume that (h1), (h2) and (h10) are satisfied. There then exist a constant* $\varepsilon_0 > 0$, *a neighbourhood* U *of* $(\lambda, 0)$ *(with* $\lambda = (\lambda_1, \dots, \lambda_n)$ *and* $0 = (0, \dots, 0) \in V$*) and a continuous function* $H : (-\varepsilon_0, \varepsilon_0) \to U$ *such that* $T(H(\varepsilon)) = 0$ *for all* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Remark 1.4. Note that $T(H(\varepsilon)) = 0$, with $H(\varepsilon) = (\mu, u) \in U$ for $\mu = (\mu_1, \dots, \mu_n)$ in a neighbourhood of $\lambda = (\lambda_1, \dots, \lambda_n)$ and $u = (u_1, \dots, u_n)$ in a neighbourhood of $0 = (0, \dots, 0)$, signifies that (μ, u) is a non-trivial solution for the system (1.1).

2. Existence of positive solutions

2.1. Preliminary results

The aim of this section is to study the existence of positive solutions for the system (1.6) when $\mu_i > \lambda_i$ for μ_i near λ_i for each i.

Recall that we have defined the functions F_i by (1.8) and H_{μ_i} by (1.9).

First, note the following lemma.

Lemma 2.1.

(i) *For all* $i = 1, \dots, n$ *and all* $\phi \in D(\mathbb{R}^N)$,

$$\frac{\partial F_i}{\partial u_i}(u_1, \dots, u_n)(\phi) = (p+1) \sum_{j=1,\, j \neq i}^{n} \int_{\mathbb{R}^N} [a_{ij} u_i^p u_j^q \phi + f_{ij} u_j^{p+q} \phi]$$

and

$$H'_{\mu_i}(v)(\phi) = 2 \int_{\mathbb{R}^N} [\nabla v \cdot \nabla \phi + q_i v \phi - \mu_i m_i v \phi].$$

(ii) $(u_1, \dots, u_n) \in V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$ *is a supersolution (respectively, subsolution) of the system (1.6) if and only if, for all* $\phi \in D(\mathbb{R}^N)$, $\phi \geqslant 0$ *and all* $i = 1, \dots, n$,

$$H'_{\mu_i}(u_i)(\phi) \begin{cases} \geqslant \dfrac{2}{p+1} \dfrac{\partial F_i}{\partial u_i}(u_1, \dots, u_n)(\phi), \\ \leqslant \dfrac{2}{p+1} \dfrac{\partial F_i}{\partial u_i}(u_1, \dots, u_n)(\phi), \end{cases}$$

respectively.

(iii) *For all* $i = 1, \dots, n$, $\phi \in D(\mathbb{R}^N)$ *and* $t \in \mathbb{R}^+$, $t > 0$,

$$\frac{\partial F_i}{\partial u_i}(tu_1, \dots, tu_n)(\phi) = t^{p+q} \frac{\partial F_i}{\partial u_i}(u_1, \dots, u_n)(\phi) \quad \text{and} \quad H'_{\mu_i}(tu_i)(\phi) = t H'_{\mu_i}(u_i)(\phi).$$

Proceeding as in [**7**,**9**] (see (1.3) and (1.11) for the definitions of λ_i and λ_i^{**}), we obtain the following lemma.

Lemma 2.2. *We have* $\lambda_i \leqslant \lambda_i^{**}$ *for each* $i = 1, \dots, n$.

Proof. Suppose (for example) that $\lambda_1 > \lambda_1^{**}$. Because of the characterization of λ_1, we have $H_{\lambda_1}(\phi_1) = 0$. By the definition of λ_1^{**} (see (1.11)), we deduce the existence of $\phi \in D(\mathbb{R}^N)$, $\phi \geqslant 0$, such that there exist

$$(v_2, \dots, v_n) \in V_{q_2}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N), \quad v_i \geqslant 0, \quad \frac{\partial F_1}{\partial u_1}(\phi_1, v_2, \dots, v_n)(\phi) \geqslant 0$$

and

$$\frac{\int_{\mathbb{R}^N} [\nabla \phi_1 \cdot \nabla \phi + q_1 \phi_1 \phi]}{\int_{\mathbb{R}^N} m_1 \phi_1 \phi} \leqslant \lambda_1^{**} < \lambda_1.$$

Therefore, $H'_{\lambda_1}(\phi_1)(\phi) < 0$.

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For all $\eta \in \mathbb{R}^+$, $\eta > 0$, we have $H_{\lambda_1}(\phi_1 + \eta\phi) = H_{\lambda_1}(\phi_1) + \eta H'_{\lambda_1}(\phi_1)(\phi) + \|\eta\phi\| h(\eta\phi)$ with $h(\eta\phi) \to 0$ as $\eta \to 0$.

Therefore, for η sufficiently small, we have $H_{\lambda_1}(\phi_1 + \eta\phi) < 0$, which contradicts the definition of λ_1. □

Proposition 2.3. *Assume that (h1)–(h3) and (h9) are satisfied.*

If there exists $i' \in \{1, \dots, n\}$ such that $\mu_{i'} > \lambda^*_{i'}$, then the system (1.6) has no positive solution.

Proof. We can write that, for all $v_{i'} \in V_{q_{i'}}(\mathbb{R}^N)$, $v_{i'} \geqslant 0$, there exists $\phi \in D(\mathbb{R}^N)$, $\phi \geqslant 0$, such that for $j \neq i'$ there exists $v_j \in V_{q_j}(\mathbb{R}^N)$, $v_j \geqslant 0$, which satisfies

$$\frac{\partial F_{i'}}{\partial u_{i'}}(v_1, \dots, v_{i'}, \dots, v_n)(\phi) \geqslant 0 \quad \text{and} \quad \frac{\int_{\mathbb{R}^N} [\nabla v_{i'} \cdot \nabla \phi + q_{i'} v_{i'} \phi]}{\int_{\mathbb{R}^N} m_{i'} v_{i'} \phi} \leqslant \lambda^*_{i'} < \mu_{i'}.$$

Then $H'_{\mu_{i'}}(v_{i'})(\phi) < 0$ and we can deduce that the system (1.6) has no positive solution. □

Now we can prove the main result of this section.

2.2. Proof of Theorem 1.2

The plan of this proof is as follows:

(i) we prove the existence of a supersolution for the system (1.6) (see Proposition 2.4);

(ii) we get a subsolution for the system (1.6) (see Proposition 2.5);

(iii) we use the Schauder fixed point theorem (see Proposition 2.6).

Proposition 2.4. *Assume that (h1)–(h8) are satisfied and that $\mu_i < \lambda^{**}_i$ for each $i = 1, \dots, n$. Then the system (1.6) has a supersolution.*

Proof. Since, for each i, $\mu_i < \lambda^{**}_i$, from the definition of λ^{**}_i (see (1.11)) we deduce the existence of

$$v^*_i \in X_{q_i, C} \quad \text{which satisfies } H'_{\mu_i}(v^*_i)(\phi) > 0 \text{ for any } \phi \in \Phi_{v^*_i}. \tag{2.1}$$

We want to show here that there exists $t \in (0, l)$ such that $(tv^*_1, \dots, v^*_n)$ is a supersolution of the system (1.6).

Suppose this were not the case. Then, for all $t \in (0, l)$, $(tv^*_1, \dots, tv^*_n)$ is not a supersolution of system (1.6).

So, for all $t \in (0, l)$, there exist $i_t \in \{1, \dots, n\}$ and $\psi_{i_t} \geqslant 0$ such that

$$H'_{\mu_{i_t}}(tv^*_{i_t})(\psi_{i_t}) < \frac{2}{p+1} \frac{\partial F_{i_t}}{\partial u_{i_t}}(tv^*_1, \dots, tv^*_n)(\psi_{i_t}).$$

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Consider the sets

$$N_t := \Big\{ i \in \{1,\dots,n\},\ \text{there exists } \psi \in D(\mathbb{R}^N),\ \psi \geqslant 0 \text{ such that } H'_{\mu_i}(tv_i^*)(\psi) < \frac{2}{p+1}\frac{\partial F_i}{\partial u_i}(tv_1^*,\dots,tv_n^*)(\psi) \Big\} \tag{2.2}$$

and, for $i_t \in N_t$,

$$K_{i_t} = \Big\{ \psi \in D(\mathbb{R}^N),\ \psi \geqslant 0,\ H'_{\mu_{i_t}}(tv_{i_t}^*)(\psi) < \frac{2}{p+1}\frac{\partial F_{i_t}}{\partial u_{i_t}}(tv_1^*,\dots,tv_n^*)(\psi) \Big\}. \tag{2.3}$$

We can prove that the inequality

$$\frac{\partial F_i}{\partial u_i}(tv_1^*,\dots,tv_n^*)(\psi) \geqslant 0 \quad \Big(\text{respectively, } \frac{\partial F_i}{\partial u_i}(tv_1^*,\dots,tv_n^*)(\psi) \leqslant 0\Big)$$

is not satisfied for all $t > 0$, $i \in N_t$ and $\psi \in K_{i_t}$ (see Appendices A and B).

Therefore, there exist $t > 0$, $i_t \in N_t$, $\phi \in K_{i_t}$ and $\psi \in K_{i_t}$, which satisfy

$$\frac{\partial F_{i_t}}{\partial u_{i_t}}(tv_1^*,\dots,tv_n^*)(\phi) < 0 \quad \text{and} \quad \frac{\partial F_{i_t}}{\partial u_{i_t}}(tv_1^*,\dots,tv_n^*)(\psi) > 0.$$

So we have

$$H'_{\mu_{i_t}}(v_{i_t}^*)(\phi) < \frac{2}{p+1}t^{p+q-1}\frac{\partial F_{i_t}}{\partial u_{i_t}}(v_1^*,\dots,v_n^*)(\phi) < 0 \tag{2.4}$$

and

$$0 < H'_{\mu_{i_t}}(v_{i_t}^*)(\psi) < \frac{2}{p+1}t^{p+q-1}\frac{\partial F_{i_t}}{\partial u_{i_t}}(v_1^*,\dots,v_n^*)(\psi). \tag{2.5}$$

(Note that $\psi \in \Phi_{v_{i_t}^*}$.)

Since $\partial F_{i_t}(v_1^*,\dots,v_n^*)/\partial u_{i_t}$ is a continuous function, there exists a constant $\alpha \in (0,1)$ such that

$$\frac{\partial F_{i_t}}{\partial u_{i_t}}(v_1^*,\dots,v_n^*)(\alpha\phi + (1-\alpha)\psi) = 0.$$

Thus, we deduce that $\alpha\phi + (1-\alpha)\psi \in \Phi_{v_{i_t}^*}$ and so $H'_{\mu_{i_t}}(v_{i_t}^*)(\alpha\phi + (1-\alpha)\psi) > 0$. But, using (2.4) and (2.5), we have

$$\begin{aligned} 0 &< \alpha H'_{\mu_{i_t}}(v_{i_t}^*)(\phi) + (1-\alpha)H'_{\mu_{i_t}}(v_{i_t}^*)(\psi) \\ &< \frac{2}{p+1}t^{p+q-1}\Big[\alpha\frac{\partial F_{i_t}}{\partial u_{i_t}}(v_1^*,\dots,v_n^*)(\phi) + (1-\alpha)\frac{\partial F_{i_t}}{\partial u_{i_t}}(v_1^*,\dots,v_n^*)(\psi)\Big] = 0 \end{aligned}$$

and we obtain a contradiction.

Therefore, there exists $t \in (0,l)$ such that $(tv_1^*,\dots,tv_n^*)$ is a supersolution of the system (1.6).

So

$$H'_{\mu_i}(tv_i^*)(\phi) \geqslant \frac{2}{p+1}\frac{\partial F_i}{\partial u_i}(tv_1^*,\dots,tv_n^*)(\phi) \quad \text{for all } i = 1,\dots,n \text{ and } \phi \geqslant 0. \tag{2.6}$$

Note that, for all $i = 1,\dots,n$, $tv_i^* \geqslant s\phi_i$ if $0 < s \leqslant t$. □

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Proposition 2.5. *Assume that (h1)–(h8) are satisfied. If $\lambda_i + \varepsilon(lC)^{p+q-1} < \mu_i < \lambda_i^{**}$ for each $i = 1, \dots, n$, then the system (1.6) has a subsolution.*

Proof. We show here that $(s\phi_1, \dots, s\phi_n)$ is a subsolution of the system (1.6) with $s > 0$ such that $s \leqslant t \leqslant l$ and $1/l \leqslant s^{p+q-1}$ (which is possible when $l \geqslant 1$).

We recall that

$$(-\Delta + q_i)(s\phi_i) = \lambda_i m_i s\phi_i, \quad i = 1, \dots, n. \tag{2.7}$$

Moreover, from hypotheses (h4) and (h5), we have

$$\begin{aligned}\mu_i m_i s\phi_i &+ \sum_{j=1;\, j\neq i}^{n} a_{ij} s^{p+q} \phi_i^p \phi_j^q + \sum_{j=1;\, j\neq i}^{n} f_{ij} s^{p+q} \phi_j^{p+q} \\ &= s[\mu_i m_i \phi_i + s^{p+q-1} a_{ij_i} \phi_i^p \phi_{j_i}^q + R_i] \text{ with } R_i \geqslant 0 \text{ a.e.}\end{aligned} \tag{2.8}$$

So, combining (2.7) and (2.8),

$$\begin{aligned}(-\Delta + q_i)(s\phi_i) \leqslant \mu_i m_i s\phi_i &+ \sum_{j=1;\, j\neq i}^{n} a_{ij} s^{p+q} \phi_i^p \phi_j^q + \sum_{j=1;\, j\neq i}^{n} f_{ij} s^{p+q} \phi_j^{p+q} \\ &\iff (\lambda_i - \mu_i) m_i \phi_i \leqslant s^{p+q-1} a_{ij_i} \phi_i^p \phi_{j_i}^q + R_i.\end{aligned} \tag{2.9}$$

Since $s \leqslant l$, we have

$$\lambda_i + \varepsilon s^{p+q-1} C^{p+q-1} < \mu_i \quad \text{and so} \quad \frac{\lambda_i - \mu_i}{s^{p+q-1} \phi_i^{p-1} \phi_{j_i}^q} \leqslant \frac{\lambda_i - \mu_i}{(sC)^{p+q-1}} < -\varepsilon.$$

Using (h6), we find that $(\lambda_i - \mu_i) m_i \phi_i < s^{p+q-1} a_{ij_i} \phi_i^p \phi_{j_i}^q$ and therefore (2.8) and (2.9) imply that $(s\phi_1, \dots, s\phi_n)$ is a subsolution of the system (1.6). □

Now let

$$\sigma = [s\phi_1, tv_1^*] \times \cdots \times [s\phi_n, tv_n^*]. \tag{2.10}$$

Recall that $(s\phi_1, \dots, s\phi_n)$ is a subsolution of the system (1.6) (defined by Proposition 2.5) and that $(tv_1^*, \dots, tv_n^*)$ is a supersolution of the system (1.6) (defined by Proposition 2.4).

Let the operator T^* be defined by $T^*(u_1, \dots, u_n) = (v_1, \dots, v_n)$ with $(v_1, \dots, v_n)$ the solution of

$$(-\Delta + q_i) v_i = \mu_i m_i u_i + \sum_{j=1;\, j\neq i}^{n} a_{ij} u_i^p u_j^q + \sum_{j=1;\, j\neq i}^{n} f_{ij} u_j^{p+q} \text{ in } \mathbb{R}^N, \quad i = 1, \dots, n. \tag{2.11}$$

Proposition 2.6. *Assume that (h1)–(h8) are satisfied and that $\lambda_i + \varepsilon(lC)^{p+q-1} < \mu_i < \lambda_i^{**}$ for each $i = 1, \dots, n$. Then the operator T^* has a fixed point in σ, which is a positive solution of the system (1.6).*

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Proof. First, we must prove that $T^*(\sigma) \subset \sigma$.

Let $(u_1, \dots, u_n) \in \sigma$ and $T^*(u_1, \dots, u_n) = (v_1, \dots, v_n)$. By (2.7) and (2.11), for each $i = 1, \dots, n$, we can write

$$(-\Delta + q_i)(v_i - s\phi_i) = \mu_i m_i u_i + \sum_{j=1;\, j\neq i}^{n} a_{ij} u_i^p u_j^q + \sum_{j=1;\, j\neq i}^{n} f_{ij} u_j^{p+q} - s\lambda_i m_i \phi_i.$$

Since $u_k \geqslant s\phi_k$ for each k, using (h6), $a_{ij_i} \geqslant -\varepsilon m_i$, we can deduce that

$$(-\Delta + q_i)(v_i - s\phi_i) \geqslant [\mu_i - \lambda_i - \varepsilon s^{p+q-1} \phi_i^{p-1} \phi_{j_i}^q] m_i s\phi_i. \tag{2.12}$$

But $\phi_i^{p-1}\phi_{j_i}^q \leqslant C^{p+q-1}$, $s^{p+q-1} \leqslant l^{p+q-1}$ and $\lambda_i + \varepsilon(lC)^{p+q-1} \leqslant \mu_i$ so we obtain

$$(-\Delta + q_i)(v_i - s\phi_i) \geqslant 0.$$

By the maximum principle (see Theorem 1.1 for one equation), we deduce that $v_i \geqslant s\phi_i$ for all $i = 1, \dots, n$.

Moreover, for each $i = 1, \dots, n$, we have

$$(-\Delta + q_i)(tv_i^* - v_i) \geqslant \mu_i m_i (tv_i^* - u_i) + \sum_{j=1;\, j\neq i}^{n} a_{ij}[(tv_i^*)^p (tv_j^*)^q - u_i^p u_j^q] + \sum_{j=1;\, j\neq i}^{n} f_{ij}[(tv_j^*)^{p+q} - u_j^{p+q}]. \tag{2.13}$$

So we can rewrite (2.13) as

$$\begin{aligned}(-\Delta + q_i)(tv_i^* - v_i) &\geqslant \mu_i m_i (tv_i^* - u_i) + \sum_{j=1;\, j\neq i}^{n} a_{ij}[(tv_i^*)^p((tv_j^*)^q - u_j^q) + ((tv_i^*)^p - u_i^p)u_j^q] \\ &\quad + \sum_{j=1;\, j\neq i}^{n} f_{ij}(tv_j^* - u_j)\Bigg[\sum_{k=0}^{p+q-1} (tv_j^*)^k u_j^{p+q-1-k}\Bigg]\end{aligned}$$

and as

$$\begin{aligned}(-\Delta + q_i)(tv_i^* - v_i) &\geqslant \mu_i m_i (tv_i^* - u_i) + \sum_{j=1;\, j\neq i}^{n} a_{ij}(tv_i^* - u_i)u_j^q \Bigg[\sum_{k=0}^{p-1} (tv_i^*)^k u_i^{p-1-k}\Bigg] \\ &\quad + \sum_{j=1;\, j\neq i}^{n} (tv_j^* - u_j)\Bigg[a_{ij}(tv_i^*)^p \Bigg(\sum_{k=0}^{q-1} (tv_j^*)^k u_j^{q-1-k}\Bigg) + f_{ij}\Bigg(\sum_{k=0}^{p+q-1} (tv_j^*)^k u_j^{p+q-1-k}\Bigg)\Bigg].\end{aligned}$$

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Since $(u_1,\dots,u_n)\in\sigma$, we get

$$(-\Delta+q_i)(tv_i^*-v_i)\geqslant(tv_i^*-u_i)\biggl[\mu_i m_i+a_{ij_i}u_{j_i}^q\biggl(\sum_{k=0}^{p-1}(tv_i^*)^k u_i^{p-1-k}\biggr)\biggr]$$
$$+(tv_{j_i}^*-u_{j_i})\biggl[a_{ij_i}(tv_i^*)^p\biggl(\sum_{k=0}^{q-1}(tv_{j_i}^*)^k u_{j_i}^{q-1-k}\biggr)$$
$$+f_{ij_i}\biggl(\sum_{k=0}^{p+q-1}(tv_{j_i}^*)^k u_{j_i}^{p+q-1-k}\biggr)\biggr].\quad(2.14)$$

Since

$$u_{j_i}^q\biggl(\sum_{k=0}^{p-1}(tv_i^*)^k u_i^{p-1-k}\biggr)\leqslant p(lC)^{p+q-1},$$

using (h6) we can deduce that

$$\mu_i m_i+a_{ij_i}u_{j_i}^q\biggl(\sum_{k=0}^{p-1}(tv_i^*)^k u_i^{p-1-k}\biggr)\geqslant 0.\quad(2.15)$$

By the same method, using (h7) and $s^{p+q-1}\geqslant 1/l$, we get

$$\frac{f_{ij_i}(\sum_{k=0}^{p+q-1}(tv_{j_i}^*)^k u_{j_i}^{p+q-1-k})}{(tv_i^*)^p(\sum_{k=0}^{q-1}(tv_{j_i}^*)^k u_{j_i}^{q-1-k})}\geqslant\frac{(p+q)f_{ij_i}(s\phi_{j_i})^{p+q-1}}{q(lC)^{p+q-1}}$$

and so

$$a_{ij_i}(tv_i^*)^p\biggl(\sum_{k=0}^{q-1}(tv_{j_i}^*)^k u_{j_i}^{q-1-k}\biggr)+f_{ij_i}\biggl(\sum_{k=0}^{p+q-1}(tv_{j_i}^*)^k u_{j_i}^{p+q-1-k}\biggr)\geqslant 0.\quad(2.16)$$

Therefore, by (2.14)–(2.16), we obtain $(-\Delta+q_i)(tv_i^*-v_i)\geqslant 0$ and so, by the maximum principle (see Theorem 1.1 for one equation), we deduce that $v_i\leqslant tv_i^*$ for all $i=1,\dots,n$. We conclude that $(v_1,\dots,v_n)\in\sigma$ and so $T^*(\sigma)\subset\sigma$.

Now we prove that T^* is a continuous operator. Let $(u_{1,k},\dots,u_{n,k})_k$ be a convergent sequence in σ, with limit $(u_1,\dots,u_n)$ in the sense that $(u_{i,k})_k$ converges to u_i for the norm $\|\cdot\|_{q_i}$. Let $T^*(u_{1,k},\dots,u_{n,k})=(v_{1,k},\dots,v_{n,k})$ and $T^*(u_1,\dots,u_n)=(v_1,\dots,v_n)$. (Recall that T^* is defined by (2.11).)

For each i and k, we have

$$\|v_{i,k}-v_i\|_{q_i}^2=\int_{\mathbb{R}^N}\mu_i m_i(v_{i,k}-v_i)(u_{i,k}-u_i)$$
$$+\sum_{j=1;\,j\neq i}\int_{\mathbb{R}^N}a_{ij}(v_{i,k}-v_i)(u_{i,k}^p u_{j,k}^q-u_i^p u_j^q)$$
$$+\sum_{j=1;\,j\neq i}\int_{\mathbb{R}^N}f_{ij}(v_{i,k}-v_i)(u_{j,k}^{p+q}-u_j^{p+q}).$$

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So

$$\|v_{i,k}-v_i\|_{q_i}^2 = \int_{\mathbb{R}^N} \mu_i m_i (v_{i,k}-v_i)(u_{i,k}-u_i) + \sum_{j=1;\, j\neq i} \int_{\mathbb{R}^N} a_{ij}(v_{i,k}-v_i)[u_{i,k}^p(u_{j,k}^q-u_j^q)+u_j^q(u_{i,k}^p-u_i^p)] + \sum_{j=1;\, j\neq i} \int_{\mathbb{R}^N} f_{ij}(v_{i,k}-v_i)(u_{j,k}^{p+q}-u_j^{p+q}). \tag{2.17}$$

Since $u_{i,k}, u_i, u_{j,k}, u_j, m_i, a_{ij}, f_{ij}$ are bounded, noting that

$$|u_{i,k}^p-u_i^p| \leqslant \text{const.}\,|u_{i,k}-u_i| \quad \text{and} \quad |u_{j,k}^{p+q}-u_j^{p+q}| \leqslant \text{const.}\,|u_{j,k}-u_j| \tag{2.18}$$

and using the Cauchy–Schwartz inequality, by (2.17) and (2.18) we obtain

$$\|v_{i,k}-v_i\|_{q_i} \leqslant \text{const.} \sum_{j=1}^{n} \|u_{j,k}-u_j\|_{q_j}.$$

Therefore, T^* is a continuous operator.

We finish this proof by showing that T^* is compact. Let $(u_{1,k},\dots,u_{n,k})_k$ be a bounded sequence in $\sigma \subset V_{q_1}(\mathbb{R}^N)\times\cdots\times V_{q_n}(\mathbb{R}^N)$ and let $T^*(u_{1,k},\dots,u_{n,k}) = (v_{1,k},\dots,v_{n,k})$.

Since the embedding of $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact, there exists a convergent subsequence, also denoted by $(u_{1,k},\dots,u_{n,k})_k$, in $(L^2(\mathbb{R}^N))^n$. For each i, m, k, we have

$$\|v_{i,m}-v_{i,k}\|_{q_i}^2 = \int_{\mathbb{R}^N} \mu_i m_i (u_{i,m}-u_{i,k})(v_{i,m}-v_{i,k}) + \sum_{j=1;\, j\neq i} \int_{\mathbb{R}^N} a_{ij}(v_{i,m}-v_{i,k})[u_{i,m}^p u_{j,m}^q - u_{i,k}^p u_{j,k}^q] + \sum_{j=1;\, j\neq i} \int_{\mathbb{R}^N} f_{ij}(v_{i,m}-v_{i,k})(u_{j,m}^{p+q}-u_{j,k}^{p+q}). \tag{2.19}$$

Since $u_{i,m}^p u_{j,m}^q - u_{i,k}^p u_{j,k}^q = u_{i,m}^p[u_{j,m}^q-u_{j,k}^q] + [u_{i,m}^p-u_{i,k}^p]u_{j,k}^q$,

$$|u_{i,m}^p-u_{i,k}^p| \leqslant \text{const.}\,|u_{i,m}-u_{i,k}| \quad \text{and} \quad |u_{j,m}^{p+q}-u_{j,k}^{p+q}| \leqslant \text{const.}\,|u_{j,m}-u_{j,k}| \tag{2.20}$$

and, using the Cauchy–Schwartz inequality, by (2.19) and (2.20) we obtain

$$\|v_{i,m}-v_{i,k}\|_{q_i} \leqslant \text{const.} \sum_{j=i}^{n} \|u_{j,m}-u_{j,k}\|_{L^2(\mathbb{R}^N)}.$$

We can deduce that $(v_{i,k})_k$ is a Cauchy sequence for each $i = 1,\dots,n$ and so T^* is a compact operator.

By the Schauder fixed point theorem, we deduce the existence of at least one positive solution for the system (1.6). □

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2.3. Other results

To finish this section, we obtain some results ensuring the validity of (h9). First, we recall the following lemma (obtained in [**7**] by using a method developed in [**9**]).

Lemma 2.7. *For all $i = 1, \dots, n$, all $u \in V_{q_i}(\mathbb{R}^N)$, $u > 0$, all $\phi \in V_{q_i}(\mathbb{R}^N)$, $\phi \geqslant 0$ and all $\mu_i \in \mathbb{R}$,*

$$H'_{\mu_i}(u)\left(\left(\frac{\phi}{u}\right)^{\alpha}\phi\right) - H'_{\mu_i}(\phi)\left(\left(\frac{\phi}{u}\right)^{\alpha}u\right) \leqslant 0$$

with $\alpha \in \mathbb{N}$, $\alpha > 0$.

Proof. We set

$$A = H'_{\mu_i}(u)\left(\left(\frac{\phi}{u}\right)^{\alpha}\phi\right) - H'_{\mu_i}(\phi)\left(\left(\frac{\phi}{u}\right)^{\alpha}u\right).$$

We then have

$$\begin{aligned} A &= 2\int_{\mathbb{R}^N}\left[\nabla u \cdot \nabla\left(\left(\frac{\phi}{u}\right)^{\alpha}\phi\right) - \nabla\phi \cdot \nabla\left(\left(\frac{\phi}{u}\right)^{\alpha}u\right)\right] \\ &= 2\int_{\mathbb{R}^N}\left[\phi\nabla u \cdot \nabla\left(\left(\frac{\phi}{u}\right)^{\alpha}\right) - u\nabla\phi \cdot \nabla\left(\left(\frac{\phi}{u}\right)^{\alpha}\right)\right]. \end{aligned}$$

Since

$$\nabla\left(\left(\frac{\phi}{u}\right)^{\alpha}\right) = \alpha\left(\frac{\phi}{u}\right)^{\alpha-1}\left[\frac{1}{u}\nabla\phi - \frac{\phi}{u^2}\nabla u\right],$$

we get

$$A = 2\alpha\int_{\mathbb{R}^N}\left(\frac{\phi}{u}\right)^{\alpha-1}\left[2\frac{\phi}{u}\nabla u \cdot \nabla\phi - \left(\frac{\phi}{u}\right)^2|\nabla u|^2 - |\nabla\phi|^2\right] \leqslant 0.$$

□

Therefore, we get the last results of this section, as follows.

Proposition 2.8. *Assume that (h1)–(h5) are satisfied. For each $i = 1, \dots, n$, if $\Omega_{i,+} = \{x \in \mathbb{R}^N,\ a_{ij_i}(x) > 0\}$ is a non-empty, bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega_{i,+}$, then $\lambda_i^* < +\infty$.*

Proof. For $i = 1, \dots, n$ consider the following equation $(-\Delta + q_i)u = \lambda_i m_i u$ defined in $\Omega_{i,+}$ with Dirichlet condition on $\partial\Omega_{i,+}$. We denote by λ_{i+} the first eigenvalue (which is simple and positive) and by ϕ_{i+} the first eigenfunction associated with it, i.e.

$$(-\Delta + q_i)\phi_{i+} = \lambda_{i+}m_i\phi_{i+} \text{ in } \Omega_{i,+}, \quad \phi_{i+} > 0 \text{ in } \Omega_{i+},\ \phi_{i+} = 0 \text{ on } \partial\Omega_{i+}. \tag{2.21}$$

Since $\operatorname{supp}\phi_{i+} \subset \Omega_{i+}$, by the above lemma 2.7, we get

$$H'_{\lambda_{i+}}(u_i)\left(\left(\frac{\phi_{i+}}{u_i}\right)^{\alpha}\phi_{i+}\right) \leqslant 0 \quad \text{for all } u_i \in D(\mathbb{R}^N),$$

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i.e. for all $u_i \in D(\mathbb{R}^N)$, $u_i \geqslant 0$,

$$\int_{\mathbb{R}^N} \left[\nabla u_i \cdot \nabla\left(\left(\frac{\phi_{i+}}{u_i}\right)^{\alpha}\phi_{i+}\right) + q_i u_i \left(\frac{\phi_{i+}}{u_i}\right)^{\alpha}\phi_{i+}\right] \times \left(\int_{\mathbb{R}^N} m_i u_i \left(\frac{\phi_{i+}}{u_i}\right)^{\alpha}\phi_{i+}\right)^{-1} \leqslant \lambda_{i+} < \infty. \quad (2.22)$$

Moreover, for all $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n \geqslant 0$,

$$\frac{\partial F_i}{\partial u_i}(u_1, \dots, u_n)\left(\left(\frac{\phi_{i+}}{u_i}\right)^{\alpha}\phi_{i+}\right) = (p+1) \sum_{j=1;\, j \neq i} \int_{\mathbb{R}^N} \left[a_{ij} u_i^p u_j^q \left(\frac{\phi_{i+}}{u_i}\right)^{\alpha}\phi_{i+} + f_{ij} u_j^{p+q} \left(\frac{\phi_{i+}}{u_i}\right)^{\alpha}\phi_{i+}\right] \geqslant 0 \quad (2.23)$$

by $\operatorname{supp}\phi_{i+} \subset \Omega_{i+}$ and hypotheses (h4) and (h5).

So, by (2.22) and (2.23), for all $u_i \in V_{q_i}(\mathbb{R}^N)$, $u_i \geqslant 0$,

$$\begin{aligned}
&\inf_{\phi \in D(\mathbb{R}^N)} \left\{ \frac{\int_{\mathbb{R}^N} \nabla u_i \cdot \nabla\phi + q_i u_i \phi}{\int_{\mathbb{R}^N} m_i u_i \phi},\ \phi \geqslant 0 \text{ such that, for } j = 1, \dots, n,\ j \neq i, \right. \\
&\qquad \left. \exists v_j \in V_{q_j}(\mathbb{R}^N),\ v_j \geqslant 0 \text{ and } \frac{\partial F_i}{\partial u_i}(v_1, \dots, u_i, \dots, v_n)(\phi) \geqslant 0 \right\} \\
&\quad \leqslant \int_{\mathbb{R}^N} \left[\nabla u_i \cdot \nabla\left(\left(\frac{\phi_{i+}}{u_i}\right)^{\alpha}\phi_{i+}\right) + q_i u_i \left(\frac{\phi_{i+}}{u_i}\right)^{\alpha}\phi_{i+}\right]\left(\int_{\mathbb{R}^N} m_i u_i \left(\frac{\phi_{i+}}{u_i}\right)^{\alpha}\phi_{i+}\right)^{-1} \\
&\quad \leqslant \lambda_{i+} < \infty.
\end{aligned}$$

Therefore, $\lambda_i^* \leqslant \lambda_{i+} < \infty$. □

Proposition 2.9. *Assume that (h1)–(h3) are satisfied.*

(i) *We assume here that for all i, j, $f_{ij} = 0$. If there exists $i \in \{1, \dots, n\}$ such that for $j \neq i$, there exist $u_j \geqslant 0$ which satisfy $F_i(u_1, \dots, u_{i-1}, \phi_i, u_{i+1}, \dots, u_n) \geqslant 0$, then $\lambda_i^* \leqslant \lambda_i$ and, since $\lambda_i^* \geqslant \lambda_i$ is always satisfied, $\lambda_i^* = \lambda_i < \infty$.*

(ii) *If there exists $u_1 \geqslant 0, \dots, u_n \geqslant 0$, such that $F_i(u_1, \dots, u_{i-1}, \phi_i, u_{i+1}, \dots, u_n) < 0$, then $\lambda_i < \lambda_i^*$.*

Note that the condition in Proposition 2.9 (ii) is verified if we assume also that (h4) and (h5) are satisfied and if we take $u_j \geqslant 0$ such that $\operatorname{supp} u_j \subset D_i$.

Proof. For proposition 2.9 (i), we assume that $f_{ij} = 0$ for each i and j. So we have

$$F_i(u_1, \dots, u_n) = \sum_{j=1;\, j \neq i}^{n} \int_{\mathbb{R}^N} a_{ij} u_i^{p+1} u_j^q \quad \text{for all } u_1, \dots, u_n, \quad (2.24)$$

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and

$$\frac{\partial F_i}{\partial u_i}(u_1,\dots,u_n)(\phi) = (p+1)\sum_{j=1;\, j\neq i}^{n} \int_{\mathbb{R}^N} a_{ij} u_i^p u_j^q \phi \quad \text{for all } \phi. \tag{2.25}$$

We suppose here that

$$\text{for } j \neq i, \quad \text{there exists } u_j \geqslant 0, \quad F_i(u_1,\dots,u_{i-1},\phi_i,u_{i+1},\dots,u_n) \geqslant 0. \tag{2.26}$$

We must prove that $\lambda_i^* \leqslant \lambda_i$. To do this we use lemma 2.7, with $\alpha = p$.

We have

$$H'_{\lambda_i}(\phi_i)\left(\left(\frac{\phi_i}{u_i}\right)^p u_i\right) = 0 \quad \text{for all } u_i > 0.$$

So, for all $u_i \geqslant 0$,

$$H'_{\lambda_i}(u_i)\left(\left(\frac{\phi_i}{u_i}\right)^p \phi_i\right) \leqslant 0.$$

i.e.

$$\int_{\mathbb{R}^N}\left[\nabla u_i \cdot \nabla\left(\left(\frac{\phi_i}{u_i}\right)^p \phi_i\right) + q_i u_i \left(\frac{\phi_i}{u_i}\right)^p \phi_i\right]\left(\int_{\mathbb{R}^N} m_i u_i \left(\frac{\phi_i}{u_i}\right)^p \phi_i\right)^{-1} \leqslant \lambda_i < \infty. \tag{2.27}$$

Moreover, using (2.24)–(2.26), for all $u_i > 0$ and $j \neq i$, there exists $u_j \geqslant 0$,

$$\begin{aligned}
\frac{\partial F_i}{\partial u_i}(u_1,\dots,u_{i-1},u_i,u_{i+1},\dots,u_n)&\left(\left(\frac{\phi_i}{u_i}\right)^p \phi_i\right) \\
&= (p+1)\sum_{j=1;\, j\neq i} \int_{\mathbb{R}^N} a_{ij} u_i^p u_j^q \left(\frac{\phi_i}{u_i}\right)^p \phi_i \\
&= (p+1) F_i(u_1,\dots,u_{i-1},\phi_i,u_{i+1},\dots,u_n) \\
&\geqslant 0.
\end{aligned} \tag{2.28}$$

Since

$$\begin{aligned}
\inf_{\phi \in D(\mathbb{R}^N)} \Bigg\{ &\frac{\int_{\mathbb{R}^N} \nabla u_i \cdot \nabla \phi + q_i u_i \phi}{\int_{\mathbb{R}^N} m_i u_i \phi},\ \phi \geqslant 0 \text{ such that} \\
&\qquad \text{for } j \neq i,\ \exists u_j \in V_{q_j}(\mathbb{R}^N),\ u_j \geqslant 0 \text{ and } \frac{\partial F_i}{\partial u_i}(u_1,\dots,u_n)(\phi) \geqslant 0 \Bigg\} \\
&\leqslant \int_{\mathbb{R}^N}\left[\nabla u_i \cdot \nabla\left(\left(\frac{\phi_i}{u_i}\right)^p \phi_i\right) + q_i u_i \left(\frac{\phi_i}{u_i}\right)^p \phi_i\right]\left(\int_{\mathbb{R}^N} m_i u_i \left(\frac{\phi_i}{u_i}\right)^p \phi_i\right)^{-1} \\
&\leqslant \lambda_i < \infty,
\end{aligned}$$

by (2.27) and (2.28), we get $\lambda_i^* \leqslant \lambda_i$ and therefore $\lambda_i^* = \lambda_i$.

For the second claim, we assume that there exist

$$u_1 \geqslant 0, \dots, u_n \geqslant 0, \quad F_i(u_1,\dots,u_{i-1},\phi_i,u_{i+1},\dots,u_n) < 0. \tag{2.29}$$

We define

$$\lambda_i^- = \inf_{\substack{\phi \in D(\mathbb{R}^N),\\ \phi \geqslant 0}} \left\{ \frac{\int_{\mathbb{R}^N} [|\nabla \phi|^2 + q_i |\phi|^2]}{\int_{\mathbb{R}^N} m_i |\phi|^2},\ \phi \text{ such that } F_i(u_1, \dots, u_{i-1}, \phi, u_{i+1}, \dots, u_n) > 0 \right\}. \tag{2.30}$$

Let

$$W_i = \{\phi \in V_{q_i}(\mathbb{R}^N),\ \phi \geqslant 0,\ F_i(u_1, \dots, u_{i-1}, \phi, u_{i+1}, \dots, u_n) > 0\}. \tag{2.31}$$

Since $W_i \subset V_{q_i}(\mathbb{R}^N)$, we have $\lambda_i \leqslant \lambda_i^-$. Since $\phi_i \notin W_i$, by the continuity of the function F_i, we deduce that $\lambda_i < \lambda_i^-$.

We now have to prove that $\lambda_i^- \leqslant \lambda_i^*$.

First we prove that

$$\text{there exists } u_i^- \in W_i, \text{ such that } \lambda_i^- = \frac{\int_{\mathbb{R}^N} [|\nabla u_i^-|^2 + q_i |u_i^-|^2]}{\int_{\mathbb{R}^N} m_i |u_i^-|^2}. \tag{2.32}$$

Suppose that

$$\lambda_i^- < \frac{\int_{\mathbb{R}^N} [|\nabla u|^2 + q_i |u|^2]}{\int_{\mathbb{R}^N} m_i |u|^2} \quad \text{for all } u \in W_i.$$

Let $v \in W_i$ such that $F_i(u_1, \dots, u_{i-1}, v, u_{i+1}, \dots, u_n) > 0$. Then $H_{\lambda_i^-}(v) > 0$.

Since $\lambda_i < \lambda_i^-$, we have $H_{\lambda_i^-}(\phi_i) < 0$ and so $H_{\lambda_i^-}(\eta \phi_i) < 0$ for all $\eta > 0$. Since the function $H_{\lambda_i^-}$ is continuous, we obtain the existence of a constant $\alpha \in (0,1)$ such that $H_{\lambda_i^-}(\alpha \eta \phi_i + (1-\alpha) v) = 0$.

Then $\alpha \eta \phi_i + (1-\alpha) v \notin W_i$ and so

$$F_i(u_1, \dots, u_{i-1}, \alpha \eta \phi_i + (1-\alpha) v, u_{i+1}, \dots, u_n) \leqslant 0.$$

However, since $F_i(u_1, \dots, u_{i-1}, (1-\alpha) v, u_{i+1}, \dots, u_n) > 0$, there exists $\eta > 0$ small enough such that $F_i(u_1, \dots, u_{i-1}, \alpha \eta \phi_i + (1-\alpha) v, u_{i+1}, \dots, u_n) > 0$. Therefore, we get a contradiction and therefore we can deduce the existence of u_i^-.

Finally, we prove that $\lambda_i^- \leqslant \lambda_i^*$. Suppose that $\lambda_i^- > \lambda_i^*$. Then

$$\text{there exists } \phi \in \Phi_{u_i^-}, \quad \frac{\int_{\mathbb{R}^N} [\nabla u_i^- \cdot \nabla \phi + q_i u_i^- \phi]}{\int_{\mathbb{R}^N} m_i u_i^- \phi} \leqslant \lambda_i^* < \lambda_i^-. \tag{2.33}$$

Therefore, $H'_{\lambda_i^-}(u_i^-)(\phi) < 0$.

Since $F_i(u_1, \dots, u_{i-1}, u_i^-, u_{i+1}, \dots, u_n) > 0$, by continuity we have

$$F_i(u_1, \dots, u_{i-1}, u_i^- + \eta \phi, u_{i+1}, \dots, u_n) > 0$$

for sufficiently small $\eta > 0$.

Moreover, by (2.32) and (2.33) we have $H'_{\lambda_i^-}(u_i^-)(\phi) < 0$ and $H_{\lambda_i^-}(u_i^-) = 0$, so we can choose $\eta > 0$ small enough that $H_{\lambda_i^-}(u_i^- + \eta \phi) < 0$.

Therefore, we obtain that

$$\frac{\int_{\mathbb{R}^N} [|\nabla (u_i^- + \eta \phi)|^2 + q_i (u_i^- + \eta \phi)^2]}{\int_{\mathbb{R}^N} m_i (u_i^- + \eta \phi)^2} < \lambda_i^-$$

and this contradicts the definition of λ_i^- (see (2.30)). Hence, $\lambda_i^- \leqslant \lambda_i^*$. □

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3. A bifurcation result

3.1. Preliminary results

In this section, we obtain a result on existence of solutions for the semilinear system (1.1) by considering bifurcating solutions from the zero solution. We suppose that (h1), (h2) and (h10) are satisfied throughout this section.

Note that, for each i, f_i is Lipschitz in $(y_1, \dots, y_n)$ uniformly in x.

Proposition 3.1. *The operator T (see (1.14)) is well defined.*

Proof. Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ and $u = (u_1, \dots, u_n) \in V$. For all $v_i \in V_{q_i}(\mathbb{R}^N)$, we introduce

$$G_i(v_i) = \int_{\mathbb{R}^N} [\nabla u_i \cdot \nabla v_i + q_i u_i v_i - \mu_i m_i u_i v_i - f_i(x, u) v_i].$$

Since m_i is bounded, f_i is Lipschitz in u uniformly in x and $f_i(x, 0, \dots, 0) = 0$, we deduce that

$$|G_i(v_i)| \leqslant \text{const.}\left[\sum_{j=1}^{n} \|u_j\|_{q_j}\right] \|v_i\|_{q_i} \quad \text{for all } v_i \in V_{q_i}(\mathbb{R}^N).$$

The operator G_i is linear and continuous. By the Riesz theorem, the operator T^i is well defined for all i and so T is well defined. □

Proposition 3.2. *For all i, the operator T^i is continuous, Fréchet differentiable with continuous derivatives given by the following formulae (for all $\phi, \psi \in V_{q_i}(\mathbb{R}^N)$):*

$$\text{if } j \neq i,\ T^i_{\mu_j} = 0,\quad \langle T^i_{u_j}(\mu, u)\phi, \psi\rangle_{q_i} = -\int_{\mathbb{R}^N} \frac{\partial f_i}{\partial y_j}(x, u)\phi\psi,$$

$$\text{if } j = i,\quad \langle T^i_{\mu_i}(\mu, u), \phi\rangle_{q_i} = -\int_{\mathbb{R}^N} m_i u_i \phi,$$

$$\text{if } j = i,\quad \langle T^i_{u_i}(\mu, u)\phi, \psi\rangle_{q_i} = \int_{\mathbb{R}^N} \left[\nabla\phi \cdot \nabla\psi + q_i \phi\psi - \mu_i m_i \phi\psi - \frac{\partial f_i}{\partial y_i}(x, u)\phi\psi\right]$$

$$\text{if } j \neq i,\quad T^i_{\mu_j u_i} = 0 = T^i_{\mu_i u_j},$$

$$\text{if } j = i,\quad \langle T^i_{\mu_i u_i}(\mu, u)\phi, \psi\rangle_{q_i} = -\int_{\mathbb{R}^N} m_i \phi\psi \ \text{ and } T^k_{\mu_i u_i} = 0 \text{ if } k \neq i.$$

Proof. Proceeding as in [**7**], we do not give the details of the proof here, which is technical but simple. Since m_i is bounded and f_i is Lipschitz in u uniformly in x, we obtain the continuity of T^i and $T^i_{\mu_i}$. By using the hypothesis that $\partial f_i(x, \cdot)/\partial y_j$ is bounded uniformly in x and using the Lebesgue dominated convergence theorem, we get the continuity of $T^i_{u_i}$ and $T^i_{u_j}$. □

Recall that, for each $i = 1, \dots, n$, $(-\Delta + q_i)\phi_i = \lambda_i m_i \phi_i$ in $\mathbb{R}^N$, $\lambda_i > 0$ and $\phi_i > 0$ (see (1.2) and (1.3)).

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Proposition 3.3. *The operator* $T_u(\lambda,0)$ *is a continuous self-adjoint operator with* $\lambda=(\lambda_1,\dots,\lambda_n)$*. The kernel of* $T_u(\lambda,0)$*, denoted by* $N(T_u(\lambda,0))$ *is generated by* $\{\Phi_1,\dots,\Phi_n\}$*, where, for each* $i=1,\dots,n$, $\Phi_i=(0,\dots,0,\phi_i,0,\dots,0)$.

Moreover, if we denote by $R(T_u(\lambda,0))$ *the range of the operator* $T_u(\lambda,0)$*, we have*

(i) $\operatorname{codim}(R(T_u(\lambda,0)))=n$,

(ii) $T_{\mu_i u}(\lambda,0)\Phi_i\notin R(T_u(\lambda,0))$ *for each* i,

(iii) $\dim(\operatorname{span}\{T_{\mu_i u}(\lambda,0)\Phi_i,\ 1\leqslant i\leqslant n\})=n$.

Proof. First note that we have $T_u(\lambda,0)=(T_u^1(\lambda,0),\dots,T_u^n(\lambda,0))$.

Then, using hypothesis (h10) (iii), we find that, for each $i=1,\dots,n$, $T_u^i(\lambda,0)$ is a continuous self-adjoint operator. Therefore, $T_u(\lambda,0)$ is also a continuous self-adjoint operator. Indeed, for $v=(v_1,\dots,v_n)\in V$ and $w=(w_1,\dots,w_n)\in V$,

$$\begin{aligned}\langle T_u(\lambda,0)v,w\rangle_V&=\sum_{i=1}^n\langle T_u^i(\lambda,0)v,w_i\rangle_{q_i}\\&=\sum_{i,j=1}^n\langle T_{u_j}^i(\lambda,0)v_j,w_i\rangle_{q_i}\\&=\sum_{i=1}^n\int_{\mathbb{R}^N}[\nabla v_i\cdot\nabla w_i+q_iv_iw_i-\lambda_im_iv_iw_i]\\&=\langle v,T_u(\lambda,0)w\rangle_V.\end{aligned}$$

We study here the kernel of $T_u(\lambda,0)$, denoted by $N(T_u(\lambda,0))$. For $v=(v_1,\dots,v_n)\in V$, we have

$$\begin{aligned}&v\in N(T_u(\lambda,0))\\&\quad\Longleftrightarrow\ \text{for all } w\in V,\quad \langle T_u(\lambda,0)v,w\rangle_V=0\\&\quad\Longleftrightarrow\ \text{for all } w\in V,\quad \sum_{i=1}^n\int_{\mathbb{R}^N}[\nabla v_i\cdot\nabla w_i+q_iv_iw_i-\lambda_im_iv_iw_i]=0\\&\quad\Longleftrightarrow\ \text{for all } i \text{ and all } w_i\in V_{q_i}(\mathbb{R}^N),\quad \int_{\mathbb{R}^N}[\nabla v_i\cdot\nabla w_i+q_iv_iw_i-\lambda_im_iv_iw_i]=0\\&\quad\Longleftrightarrow\ \text{for all } i,\quad v_i\in\operatorname{span}\{\phi_i\}\\&\quad\Longleftrightarrow\ v\in\operatorname{span}\{\Phi_1,\dots,\Phi_n\},\quad \text{where } \Phi_i=(0,\dots,0,\phi_i,0,\dots,0).\end{aligned}\tag{3.1}$$

Therefore, $\operatorname{codim} R(T_u(\lambda,0))=n$.

Now we prove that, for each i, $T_{\mu_i u}(\lambda,0)\Phi_i\notin R(T_u(\lambda,0))$. Note that we have identified $T_{\mu_i u}(\lambda,0)\cdot(1,\Phi_i)$ with $T_{\mu_i u}(\lambda,0)\Phi_i$. For each $i=1,\dots,n$, we have

$$\begin{aligned}\langle T_{\mu_i u}(\lambda,0)\Phi_i,\Phi_i\rangle_V&=\langle T^i_{\mu_i u}(\lambda,0)\Phi_i,\phi_i\rangle_{q_i}\\&=\langle T^i_{\mu_i u_i}(\lambda,0)\phi_i,\phi_i\rangle_{q_i}\\&=-\int_{\mathbb{R}^N}m_i\phi_i^2\neq0.\end{aligned}$$

Therefore, $T_{\mu_i u}(\lambda,0)\Phi_i$ is not orthogonal to Φ_i. Since $N(T_u(\lambda,0)) = \operatorname{span}\{\Phi_1,\dots,\Phi_n\}$ (see (3.1)) and $R(T_u(\lambda,0)) = N(T_u(\lambda,0))^{\perp_V}$, we deduce that

$$T_{\mu_i u}(\lambda,0)\Phi_i \notin R(T_u(\lambda,0)).$$

Finally, let $(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n$ be such that

$$\sum_{j=1}^{n}\alpha_j T_{\mu_j u}(\lambda,0)\Phi_j = 0.$$

For each $i = 1,\dots,n$, we have

$$\left\langle \sum_{j=1}^{n}\alpha_j T_{\mu_j u}(\lambda,0)\Phi_j, \Phi_i \right\rangle_V = 0,$$

so that

$$\sum_{j=1}^{n}\alpha_j \langle T^i_{\mu_j u}(\lambda,0)\Phi_j, \phi_i\rangle_{q_i} = 0.$$

Therefore,

$$-\alpha_i \int_{\mathbb{R}^N} m_i\phi_i^2 = 0 \quad \text{and so } \alpha_i = 0.$$

So $\dim(\operatorname{span}\{T_{\mu_i u}(\lambda,0)\Phi_i,\ 1\leqslant i\leqslant n\}) = n$. □

Remark 3.4. Note that

$$T_{\mu_i u}(\lambda,0)\Phi_j = 0 \quad \text{if } i\neq j. \tag{3.2}$$

3.2. Proof of Theorem 1.3

Although we cannot apply directly the results obtained in [**8**, Theorem 1.7], we follow its proof to obtain the result developed in Theorem 1.3.

As in [**8**], we introduce the function $h:\mathbb{R}\times\mathbb{R}^n\times V\to V$ defined for $\alpha\in\mathbb{R}$, $\mu\in\mathbb{R}^n$ and $w\in V$ by

$$h(\alpha,\mu,w) = \begin{cases} \dfrac{1}{\alpha}T(\mu,\alpha\Phi_1+\cdots+\alpha\Phi_n+\alpha w) & \text{if } \alpha\neq 0,\\[2ex] T_u(\mu,0)(\Phi_1+\cdots+\Phi_n+w) & \text{if } \alpha = 0. \end{cases} \tag{3.3}$$

Recall that

$$\Phi_i = (0,\dots,0,\phi_i,0,\dots,0)\in V.$$

We have $h(0,\lambda,0) = T_u(\mu,0)\Phi_1+\cdots+T_u(\mu,0)\Phi_n$. Since, for each $i = 1,\dots,n$, $\Phi_i\in N(T_u(\lambda,0))$, we deduce that $h(0,\lambda,0) = 0$.

Moreover, let $g:\mathbb{R}^n\times V\to V$ be defined for $\mu\in\mathbb{R}^n$ and $w\in V$ by

$$g(\mu,w) = T_u(\mu,0)(\Phi_1+\cdots+\Phi_n+w). \tag{3.4}$$

For $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n$ and $v = (v_1, \dots, v_n) \in V$, we have

$$\mathrm{D}g(\mu, w)(\rho, v) = \sum_{i=1}^{n} \rho_i T_{\mu_i u}(\mu, 0)(\Phi_1 + \cdots + \Phi_n + w) + T_u(\mu, 0)v,$$

where $\mathrm{D}g$ is the Fréchet derivative of g. Therefore, for $\rho \in \mathbb{R}^n$ and $v \in V$, we get

$$\mathrm{D}g(\lambda, 0)(\rho, v) = \sum_{i=1}^{n} \rho_i T_{\mu_i u}(\lambda, 0)\Phi_i + T_u(\lambda, 0)v, \tag{3.5}$$

since $T_{\mu_i u}(\lambda, 0)\Phi_j = 0$ if $j \neq i$ (see (3.2)).

Using Proposition 3.3, we deduce that $\mathrm{D}g(\lambda, 0)$ is a linear homeomorphism from $\mathbb{R}^n \times V$ onto V. Indeed, note that $V/R(T_u(\lambda, 0))$ is isomorphic to $N(T_u(\lambda, 0))$. So, for each $i = 1, \dots, n$, there exist a $\psi_i \in N(T_u(\lambda, 0))$ and an $\omega_i = T_u(\lambda, 0)\zeta_i \in R(T_u(\lambda, 0))$ such that

$$T_{\mu_i u}(\lambda, 0)\Phi_i = \psi_i + \omega_i. \tag{3.6}$$

Since $\dim \operatorname{span}\{T_{\mu_i u}(\lambda, 0)\Phi_i,\ 1 \leqslant i \leqslant n\} = n$, we deduce that

$$N(T_u(\lambda, 0)) = \operatorname{span}\{\psi_1, \dots, \psi_n\}.$$

Therefore, for all $w \in V$, there exist a $v \in V$ and a $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n$ such that

$$w = T_u(\lambda, 0)v + \sum_{i=1}^{n} \nu_i \psi_i \tag{3.7}$$

and so, by (3.5)–(3.7), we have

$$w = T_u(\lambda, 0)\biggl(v - \sum_{i=1}^{n} \nu_i \zeta_i\biggr) + \sum_{i=1}^{n} \nu_i T_{\mu_i u}(\lambda, 0)\Phi_i = \mathrm{D}g(\lambda, 0)\biggl(\nu, v - \sum_{i=1}^{n} \nu_i \zeta_i\biggr).$$

We recall that $g(\mu, w) = h(0, \mu, w)$ (see (3.3) and (3.4)).

The implicit function theorem implies the existence of a neighbourhood U' of $(\lambda, 0)$, of $\varepsilon_0 > 0$ and of a function $K : (-\varepsilon_0, \varepsilon_0) \to U'$ such that $h(\varepsilon, \mu, w) = 0$ with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $K(\varepsilon) = (\mu, w) = (K_1\varepsilon, K_2\varepsilon) \in U'$. Therefore, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $T(K_1\varepsilon, \varepsilon(\Phi_1 + \cdots + \Phi_n + K_2\varepsilon)) = 0$ and if we define $H(\varepsilon) := (K_1\varepsilon, \varepsilon(\Phi_1 + \cdots + \Phi_n + K_2\varepsilon))$, we have

$$T(H(\varepsilon)) = 0 \quad \text{for all } \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

3.3. Other results

Finally, we study the global nature of the continuum of solutions obtained by bifurcation from the $(\lambda, 0)$ solution in a particular case. As in [**7**], we follow a method developed in [**3**] using [**10**, theorems 1.3 and 1.4]. We obtain the following result.

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Theorem 3.5. *Assume that (h1), (h2) and (h10) are satisfied. Assume also that, for all i and j, $\lambda_i = \lambda_j := \lambda_0$ (which is satisfied if $q_i = q$ and $m_i = m$ for each $i = 1, \dots, n$). Then, there exists a continuum C of non-trivial solutions for the system (1.1) obtained by bifurcation form the $(\lambda_0, 0)$ solution, which is either unbounded or contains a point $(\mu, 0)$, where $\mu \neq \lambda_0$ is the inverse of an eigenvalue of the operator $A = (L_1, \dots, L_n)$ (where L_i is defined by $\langle L_i u, \phi\rangle_{q_i} = \int_{\mathbb{R}^N} m_i u_i \phi$, for all $u = (u_1, \dots, u_n) \in V$ and all $\phi \in V_{q_i}(\mathbb{R}^N)$).*

Since λ_0 is simple, the continuum C has two connected subsets, C^+ and C^-, which also satisfy the above alternatives.

Proof. First we define an operator S by setting

$$S(\mu, u) = u - T(\mu, u), \qquad S = (S^1, \dots, S^n),$$

i.e. for $\mu \in \mathbb{R}$, for $u = (u_1, \dots, u_n) \in V$ and for $v_i \in V_{q_i}(\mathbb{R}^N)$,

$$\langle S^i(\mu, u), v_i\rangle_{q_i} = \int_{\mathbb{R}^N} [\mu m_i u_i v_i + f_i(x, u) v_i]. \tag{3.8}$$

So $u = (u_1, \dots, u_n)$ is a solution of the system (1.1) if and only if $u = S(\mu, u)$. For each $i = 1, \dots, n$, $S^i(\mu, u) = \mu L_i u + H_i u$, where $v_i \in V_{q_i}(\mathbb{R}^N)$, we write

$$\langle L_i u, v_i\rangle_{q_i} = \int_{\mathbb{R}^N} m_i u_i v_i \quad \text{and} \quad \langle H_i u, v_i\rangle_{q_i} = \int_{\mathbb{R}^N} f_i(x, u) v_i. \tag{3.9}$$

So

$$S(\mu, u) = \mu A u + H u \tag{3.10}$$

with $Au = (L_1 u, \dots, L_n u)$ and $Hu = (H_1 u, \dots, H_n u)$.

To apply the results in [**10**], we must prove that $S^i : \mathbb{R} \times V \to V_{q_i}(\mathbb{R}^N)$ is continuous and compact, that $L_i : V \to V_{q_i}(\mathbb{R}^N)$ is linear and compact, that $H_i u = O(\|u\|_V)$ for $u = (u_1, \dots, u_n)$ near $0 = (0, \dots, 0)$ uniformly on bounded intervals of μ and that $1/\lambda_0$ is a simple eigenvalue of A (which is true because it is a simple eigenvalue of $(-\Delta + q_i)^{-1} M_i$).

We first show that S^i is continuous and compact. Note that S^i is continuous since T^i is continuous. Let $((\mu_p, u_p))_p$ be a bounded sequence in $\mathbb{R} \times V$, with $u_p = (u_{1p}, \dots, u_{np})$. Since the embedding of each $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact, there exists a convergent subsequence, denoted also by $((\mu_p, u_p))_p$ in $\mathbb{R} \times (L^2(\mathbb{R}^N))^n$.

By (3.8), for all $v_i \in V_{q_i}(\mathbb{R}^N)$, we have

$$\begin{aligned}\langle S^i(\mu_p, u_p) - S^i(\mu_m, u_m), v_i\rangle_{q_i} \\ = \mu_p \int_{\mathbb{R}^N} m_i u_{ip} v_i - \mu_m \int_{\mathbb{R}^N} m_i u_{im} v_i + \int_{\mathbb{R}^N} [f_i(x, u_p) - f_i(x, u_m)] v_i.\end{aligned}$$

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So

$$\begin{aligned}\|S^i(\mu_p,u_p)-S^i(\mu_m,u_m)\|_{q_i}^2 &= (\mu_p-\mu_m)\int_{\mathbb{R}^N} m_i u_{ip}[S^i(\mu_p,u_p)-S^i(\mu_m,u_m)]\\ &\quad + \mu_m\int_{\mathbb{R}^N} m_i(u_{ip}-u_{im})[S^i(\mu_p,u_p)-S^i(\mu_m,u_m)]\\ &\quad + \int_{\mathbb{R}^N}[f_i(x,u_p)-f_i(x,u_m)][S^i(\mu_p,u_p)-S^i(\mu_m,u_m)].\end{aligned}$$

By (h2) and (h10) we deduce that $(S^i(\mu_p,u_p))_p$ is a Cauchy sequence and therefore a convergent sequence. So S^i is compact for all $i=1,\dots,n$, and $S=(S^1,\dots,S^n)$ is also compact.

We next show that L_i is linear and compact. The operator L_i is obviously linear and continuous. Therefore, the operator A, defined by (3.10), is also linear and continuous.

Let $(u_p)_p$, $u_p=(u_{1p},\dots,u_{np})$, be a bounded sequence in V. Since the embedding of each $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact, there exists a convergent subsequence, also denoted by $(u_p)_p$, in $(L^2(\mathbb{R}^N))^n$. By the Cauchy–Schwartz inequality, using (3.9), for each i we obtain

$$\|L_iu_p-L_iu_m\|_{q_i}^2=\int_{\mathbb{R}^N} m_i(u_{ip}-u_{im})[L_iu_p-L_iu_m]$$

and so

$$\|L_iu_p-L_iu_m\|_q\leqslant \text{const.}\,\|u_{ip}-u_{im}\|_{L^2(\mathbb{R}^N)}.$$

Therefore, $(L_iu_p)_p$ is a Cauchy sequence, so L_i is compact and A is also compact.

Finally, we have

$$\|H_iu\|_{q_i}^2=\int_{\mathbb{R}^N} f_i(x,u)H_iu\leqslant \text{const.}\,\|u\|_V\|H_iu\|_{q_i}.$$

So $H_iu=O(\|u\|_V)$ and therefore $Hu=O(\|u\|_V)$. □

Appendix A.

Assume that, for all $t>0$, all $i\in N_t$ and all $\psi\in K_{i_t}$,

$$\frac{\partial F_i}{\partial u_i}(tv_1^*,\dots,tv_n^*)(\psi)\geqslant 0.$$

We shall prove that, for all $\phi\geqslant 0$ and all $i=1,\dots,n$, $H'_{\mu_i}(v_i^*)(\phi)\geqslant 0$.

Step 1. Let $\phi\geqslant 0$ be such that there exists

$$i\in\{1,\dots,n\},\quad \frac{\partial F_i}{\partial u_i}(v_1^*,\dots,v_n^*)(\phi)<0.$$

(This is possible by (h4) and (h5).) Let $t\in(0,l)$. Recall that N_t is defined by (2.2) and K_{i_t} is defined by (2.3).

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If $i \in N_t$, then $\phi \notin K_{i_t}$, and so

$$H'_{\mu_i}(tv_i^*)(\phi) \geqslant \frac{2}{p+1}\frac{\partial F_i}{\partial u_i}(tv_1^*,\dots,tv_n^*)(\phi).$$

If $i \notin N_t$, then

$$H'_{\mu_i}(tv_i^*)(\phi) \geqslant \frac{2}{p+1}\frac{\partial F_i}{\partial u_i}(tv_1^*,\dots,tv_n^*)(\phi).$$

Therefore, for all $t \in (0,l)$, we have

$$H'_{\mu_i}(tv_i^*)(\phi) \geqslant \frac{2}{p+1}\frac{\partial F_i}{\partial u_i}(tv_1^*,\dots,tv_n^*)(\phi),$$

i.e. for all $t \in (0,l)$,

$$t^{p+q-1}\sum_{j=1;\,j\neq i}^{n}\int_{\mathbb{R}^N}[a_{ij}(v_i^*)^p(v_j^*)^q+f_{ij}(v_j^*)^{p+q}]\phi \leqslant \int_{\mathbb{R}^N}[\nabla v_i^*\cdot\nabla\phi+q_iv_i^*\phi-\mu_im_iv_i^*\phi]. \quad \text{(A 1)}$$

Note that i is independent of t. So, taking the limit in (A 1), as $t \to 0$, we get

$$0 \leqslant H'_{\mu_i}(v_i^*)(\phi). \quad \text{(A 2)}$$

Step 2. Now, let $\phi \geqslant 0$ be such that there exists

$$i \in \{1,\dots,n\}, \quad \frac{\partial F_i}{\partial u_i}(v_1^*,\dots,v_n^*)(\phi) \geqslant 0.$$

(This is also possible by (h4) and (h5).)

Since $\phi \in \Phi_{v_i^*}$ (see (1.12)), and by the characterization of v_i^* (see (2.1)), we can conclude that

$$H'_{\mu_i}(v_i^*)(\phi) > 0. \quad \text{(A 3)}$$

Step 3. In conclusion, by (A 2) and (A 3), we obtain

$$H'_{\mu_i}(v_i^*)(\phi) \geqslant 0 \quad \text{for all } \phi \in D(\mathbb{R}^N),\ \phi \geqslant 0, \text{ and for all } i = 1,\dots,n.$$

In particular, for $\phi = \phi_i$, we get

$$(\lambda_i - \mu_i)\int_{\mathbb{R}^N} m_iv_i^*\phi_i \geqslant 0.$$

Since $\lambda_i < \mu_i$ and

$$\int_{\mathbb{R}^N} m_iv_i^*\phi_i > 0,$$

we get a contradiction.

Appendix B.

Assume that for all $t > 0$, all $i \in N_t$ and all $\psi \in K_{i_t}$,

$$\frac{\partial F_i}{\partial u_i}(tv_1^*, \dots, tv_n^*)(\psi) \leqslant 0.$$

Let $\phi \geqslant 0$ be such that there exists

$$i \in \{1, \dots, n\}, \quad \frac{\partial F_i}{\partial u_i}(v_1^*, \dots, v_n^*)(\phi) > 0.$$

(This is possible by (h4) and (h5).) Note that $\phi \in \Phi_{v_i^*}$ and so, by the characterization of v_i^* (see (2.1)), we have $H'_{\mu_i}(v_i^*)(\phi) > 0$.

Let $t \in (0, l)$. If $i \in N_t$, then $\phi \notin K_{i_t}$, and so

$$H'_{\mu_i}(tv_i^*)(\phi) \geqslant \frac{2}{p+1}\frac{\partial F_i}{\partial u_i}(tv_1^*, \dots, tv_n^*)(\phi).$$

If $i \notin N_t$, then

$$H'_{\mu_i}(tv_i^*)(\phi) \geqslant \frac{2}{p+1}\frac{\partial F_i}{\partial u_i}(tv_1^*, \dots, tv_n^*)(\phi).$$

Therefore, for all $t \in (0, l)$, we have

$$H'_{\mu_i}(tv_i^*)(\phi) \geqslant \frac{2}{p+1}\frac{\partial F_i}{\partial u_i}(tv_1^*, \dots, tv_n^*)(\phi),$$

i.e. for all $t \in (0, l)$,

$$\begin{aligned} 0 < t^{p+q-1} \sum_{j=1;\, j\neq i}^{n} \int_{\mathbb{R}^N} [a_{ij}(v_i^*)^p (v_j^*)^q + f_{ij}(v_j^*)^{p+q}]\phi \\ \leqslant \int_{\mathbb{R}^N} [\nabla v_i^* \cdot \nabla \phi + q_i v_i^* \phi - \mu_i m_i v_i^* \phi]. \end{aligned} \tag{B 1}$$

Note that i is still independent of t. Therefore, taking account of (B 1), we get a contradiction for sufficiently large t (because we can take a bigger l).

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