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UPPER AND LOWER FIELDS FOR PROFINITE GROUPS OF GIVEN CARDINALITY

DON KRAKOWSKI

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Profinite groups (that is, compact totally disconnected topological groups) have been characterized as Galois groups in Krakowski (1971) and Leptin (1955). They can, in fact, be realized as groups of permutations on sets of transcendentals where every transcendental has a finite orbit under the group action and the fixed field is generated by the invariant rational functions on these orbits. If $G = \{G_{\leftarrow\alpha} \mid \alpha \in A\}$ we define the cardinality of G as $|G| = |A|$. In this paper we construct, for every cardinal number c and profinite group G with $|G| \leqq c$ two universal fields k_c and K_c so that $G \simeq \mathrm{Gal}\,(K_c/K_G)$ and $G \simeq (K_G/k_c)$ for fields K_G and k_G which depend on G. See Cassels and Frohlich (1967) for a description of the basic properties of profinite groups.

THEOREM 1. *Let p be zero or a prime number, and c any cardinal number. Then there is a field K_c of characteristic p such that for every profinite group G with $|G| \leqq c$ there exists a subfield K_G over which K_c is Galois, and* $\mathrm{Gal}(K_c/K_G) \cong G$.

PROOF. Let L be a field of characteristic p, and I be an index set such that $|I| = c$. Define

$$K_c = L(\{X_{in} \mid i \in I,\ n \in Z^+\}),$$

K_{ij} are independent indeterminates over L.

If $G = \{G_{\leftarrow\alpha} \mid \alpha \in A\}$, with $|A| \leqq c$, regard A as embedded in I and G_α as emuedded in S_{n_α}, the symmetric group, where $n_\alpha = |G_\alpha|$.

If $\sigma \in G$ we define

$$\sigma(X_{\beta j}) = \begin{cases} X_{\alpha\sigma(j)} & \text{if } \alpha = \beta \text{ and } j \leqq n_\alpha \\ X_{\beta j} & \text{if } \alpha \neq \beta \text{ or } \alpha = \beta \text{ and } j > n_\alpha\ . \end{cases}$$

Let $k = L((\bigcup_{\alpha \in A} P) \cup (\bigcup_{\alpha \in A} X_{\alpha j} \mid j > n_\alpha\}))$, where P_α is the set of polynomials in $\{X_{\alpha 1}, \cdots, X_{\alpha n_\alpha}\}$ invariant under G_α. K_c is a union of Galois extensions over k

and hence is itself Galois over k. If we show that $\mathrm{Gal}(K_c/k) = \Pi_{\alpha \in A} G_\alpha$ then, since $G = G_{\leftarrow\alpha}$ is closed in $\Pi_{\alpha\in A} G_\alpha$ [Cassels and Frohlich (1967)] it follows from the fundamental theorem of the infinite Galois theory Jacobson (1964) that there is an intermediate field $k \subseteq K_G \subseteq K_c$ and $G \cong \mathrm{Gal}\ (K_c/K_G)$. To show Gal $(K_c/k) = \Pi_{\alpha \in A} G_\alpha$ we define

$$k_\alpha = k(\{X_{\alpha 1}, \cdots, X_{\alpha n_\alpha}\}) \cup (\bigcup \{X_{\beta j} \mid \beta \neq \alpha\})).$$

Then, as above, the action of G_α is on subscripts and it is evident that k_α is Galois over k, with $\mathrm{Gal}(k_\alpha/k) \cong G_\alpha$. Let $K_\alpha = \bigcup_{\beta \leqq \alpha} k_\beta$. Since $k_\alpha \cap k_\beta = k$ for $\alpha \neq \beta$, and $\Pi_{\beta \leqq \alpha} k_\beta$ (the composition of the k_β's) $= K_\alpha$, we conclude that $\mathrm{Gal}(K_\alpha/k) \simeq \Pi_{\alpha \leqq \beta} G_\alpha$. Noting that $\bigcup_{\alpha \in A} K_\alpha = K_c$, we obtain $\mathrm{Gal}(K_c/K) \simeq \Pi_{\alpha \in A} G_\alpha$ and our conclusion follows.

THEOREM 2. *Let p be a zero or a prime number, c any cardinal number. Then there is a field k_c of characteristic p such that for every profinite group G with $|G| \leqq c$ there exists a Galois extension k_G of k_c with* $\mathrm{Gal}(k_G/k_c) \simeq G$.

PROOF. Let Π be the set of profinite groups G with $|G| \leqq c$, where we take one per isomorphism class. We first show that, without loss of generality, $|G| = c$ for every $G \in \Pi$. If $G = \{G_{\leftarrow\alpha} \mid \alpha \in A\}$, let I be a set with $|I| = c$ and consider A as embedded in I. Pick $\alpha \in A$ and let $\leqq$ be the quasi order on A (as a directed set). For every $i \in I$, $i \notin A$ let α_i be a symbol and define the set $A' = A \cup \{\alpha_i \mid i \in I, i \notin A\}$. Extend the quasi order on A to A' by defining for each pair i, j in $I \backslash A$, and each $a \in A$:

$\alpha_i \leqq \alpha_j$

$\alpha_i \leqq a$ if and only if $\alpha \leqq a$

$a \leqq \alpha_i$ if and only if $a \leqq \alpha$.

This makes A' into a directed set, $|A'| = c$. Now take the profinite group

$$G' = \{G'_{\leftarrow\beta} \mid \beta \in A'; G'_\beta = G_\beta \text{ if } \beta \in A,\ G'_\beta = G_\alpha \text{ if } \beta = \alpha_i\},$$

with defining homomorphism on $\pi^\beta_{\alpha_i} = \pi^\beta_\alpha$, $\pi^{\alpha_i}_\beta = \pi^\alpha_\beta$, $\pi^{\alpha_i}_{\alpha_j} =$ identity. Since A is cofinal in A', we have $G \simeq G'$.

Now let $G_\mu = \{G_{\leftarrow i}^{(\mu)} \mid i \in I\} \in \Pi$, and embed $G_i^{(\mu)}$ in the symmetric group $S_{\mu i}^{(\mu)}$, where $|G_i^{(\mu)}| = n_i^{(\mu)}$. We introduce the set of independent indeterminates

$$\{X_{\mu i n} \mid \mu \in \Pi,\ i \in I,\ n \in Z^+\}$$

upon which G_μ acts as follows: for $\sigma \in G_\mu$ with $\sigma \in G_i^{(\mu)}$

$$\sigma(X_{v i_\mu}) = \begin{cases} X_{\mu i \sigma_i(n)} & \text{if } v = \mu \\ X_{v i n} & \text{if } v \neq \mu. \end{cases}$$

Let L be an arbitrary field of characteristic p, and define $k_c = L(\bigcup_{\mu \in \infty} P_\mu)$ where P_μ is the set of polynomials in $X_{\mu in}(i \in I,\ n \in Z^+)$ invariant under G_μ. Let $k_\mu = k_c(\{X_{\mu in} \mid i \in I,\ n \in Z^+\})$. Then k_μ is Galois over k_c with Galois group G_μ.

REMARK. The mere existence of the fields K_c and k_c can be demonstrated by a more simple argument: Let c be the given cardinal and form the product $P = \Pi G_\alpha$, where the product is extended over all profinite groups G_α, with $|G_\alpha| \leqq c$, and we choose one group per isomorphism class. Since the product of profinite groups is again profinite, P is profinite and hence a Galois group, Krakowski (1971). Thus, there exist fields K_c and k_c, of any prescribed characteristic, such that $P = \mathrm{Gal}(K_c/k_c)$. Then by the fundamental theorem of the infinite Galois theory, for every profinite group G with $|G| \leqq c$ there exist fields K_1 and K_2 with $k_c \subseteq K_i \subseteq K_c$ such that $\mathrm{Gal}(K_1/k_c) \simeq \mathrm{Gal}(K_c/K_2) \simeq G$. The existence of both an upper and lower field follows from the fact that G is both a closed subgroup as well as a homomorphic image of P.

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The Weizmann Institute of Science
Rehovot Israel.

Present address:
Department of Mathematics
Wayne State University
Detroit, Michigan 48202
U.S.A.

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