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Positroid varieties: juggling and geometry

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Abstract

While the intersection of the Grassmannian Bruhat decompositions for all coordinate flags is an intractable mess, it turns out that the intersection of only the *cyclic shifts* of one Bruhat decomposition has many of the good properties of the Bruhat and Richardson decompositions. This decomposition coincides with the projection of the Richardson stratification of the flag manifold, studied by Lusztig, Rietsch, Brown–Goodearl–Yakimov and the present authors. However, its cyclic-invariance is hidden in this description. Postnikov gave many cyclic-invariant ways to index the strata, and we give a new one, by a subset of the affine Weyl group we call *bounded juggling patterns*. We call the strata *positroid varieties*. Applying results from [A. Knutson, T. Lam and D. Speyer, *Projections of Richardson varieties*, J. Reine Angew. Math., to appear, arXiv:1008.3939 [math.AG]], we show that positroid varieties are normal, Cohen–Macaulay, have rational singularities, and are defined as schemes by the vanishing of Plücker coordinates. We prove that their associated cohomology classes are represented by affine Stanley functions. This latter fact lets us connect Postnikov's and Buch–Kresch–Tamvakis' approaches to quantum Schubert calculus.

Contents

1. Introduction, and statement of results

1.1 Some decompositions of the Grassmannian

This paper is concerned with the geometric properties of a stratification of the Grassmannian studied in [BGY06, Lus98, Pos05b, Rie06, Wil07]. It fits into a family of successively finer decompositions:

{Bruhat cells}, {open Richardson varieties}, {open **positroid varieties**}, {GGMS strata}.

We discuss the three known ones in turn, and then see how the family of positroid varieties fits in between.

The *Bruhat decomposition* of the Grassmannian of k-planes in n-space dates back, despite the name, to Schubert in the 19th century. It has many wonderful properties.

– The strata are easily indexed (by partitions in a $k \times (n - k)$ box).

– It is a stratification: the closure (a *Schubert variety*) of one open stratum is a union of others.

– Each stratum is smooth and irreducible (in fact a cell).

– Although the closures of the strata are (usually) singular, they are not too bad: they are normal and Cohen–Macaulay, and even have rational singularities.

The Bruhat decomposition is defined relative to a choice of coordinate flag, essentially an ordering on the basis elements of n-space. The *Richardson decomposition* is the common

refinement of the Bruhat decomposition and the *opposite* Bruhat decomposition, using the opposite order on the basis. Again, many excellent properties hold for this finer decomposition.

– It is easy to describe the nonempty intersections of Bruhat and opposite Bruhat strata (they correspond to *nested* pairs of partitions).

– It is a stratification, each open stratum is smooth and irreducible, and their closures are normal and Cohen–Macaulay with rational singularities [Bri02].

At this point one might imagine intersecting the Bruhat decompositions relative to *all* the coordinate flags, so as not to prejudice one over another. This gives the *GGMS decomposition* of the Grassmannian [GGMS87], and as it turns out, these good intentions pave the road to hell.

– It is infeasible to index the nonempty strata [Vam78].

– It is not a stratification [GGMS87, § 5.2].

– The strata can have essentially any singularity [Mne88]. In particular, the nonempty ones need not be irreducible, or even equidimensional.

This raises the question: can one intersect more than two permuted Bruhat decompositions, keeping the good properties of the Bruhat and Richardson decompositions, without falling into the GGMS abyss?

The answer is yes: we will intersect the *n cyclic permutations* of the Bruhat decomposition. That is to say, we will define an *open positroid variety* to be an intersection of n Schubert cells, taken with respect to the n cyclic rotations of the standard flag. We will define a *positroid variety* to be the closure of an open positroid variety. See § 5 for details.

It is easy to show, though not immediately obvious, that this refines the Richardson decomposition. It is even less obvious, though also true, that the open positroid varieties are smooth and irreducible (as we discuss in § 5.4).

There is a similar decomposition for any partial flag manifold G/P, the projection of the Richardson stratification from G/B. That decomposition arises in the study of several seemingly independent structures:

– total nonnegativity, in e.g. [Lus98, Pos05b, Rie06], see § 1.2;

– prime ideals in noncommutative deformations of G/P [LLR08, Yak10b], and a semiclassical version thereof in Poisson geometry [BGY06, GY09];

– the characteristic p notion of Frobenius splitting [KLS].

We show that the positroid stratification and the projected Richardson stratification coincide. Specifically, we prove the following theorem.

THEOREM (Theorem 5.9). *If X_u^w is a Richardson variety in the full flag manifold $(u, w \in S_n)$, then its image under projection to $\mathrm{Gr}(k, n)$ is a positroid variety. If w is required to be a Grassmannian permutation, then every positroid variety arises uniquely this way.*

Theorem 5.9 has been suspected, but has not previously been proved in print, and is surprisingly difficult in its details. This result was already known on the positive part of $\mathrm{Gr}(k, n)$, as we explain in Remark 1.2.

Once we know that positroid varieties are projected Richardson varieties, the following geometric properties follow from the results of [KLS]. Part (i) of the following Theorem was also established by Billey and Coskun [BC10] for projected Richardson varieties.

THEOREM ([KLS] and Theorem 5.15). (i) *Positroid varieties are normal and Cohen–Macaulay, with rational singularities.*

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(ii) *Though positroid varieties are defined as the closure of the intersection of n cyclically permuted Bruhat cells, they can also be defined (even as schemes) as the intersection of the n cyclically permuted Schubert varieties. In particular, each positroid variety is defined as a scheme by the vanishing of some Plücker coordinates.*

THEOREM [KLS]. *The standard Frobenius splitting on the Grassmannian compatibly splits all positroid varieties. Furthermore, positroid varieties are exactly the compatibly split subvarieties of the Grassmannian.*

Before going on, we mention a very general construction given two decompositions $\{Y_a\}_{a \in A}, \{Z_b\}_{b \in B}$ of a scheme X, one refining the other. Assume that:

- $X = \coprod_A Y_a = \coprod_B Z_b$;
- for each $a \in A$, there exists a subset $B_a \subseteq B$ such that $Y_a = \coprod_{B_a} Z_b$;
- each Y_a is irreducible (hence nonempty), and each Z_b is nonempty (we do not assume that each Z_b is irreducible).

Then there is a natural surjection $B \twoheadrightarrow A$ taking b to the unique a such that $Z_b \subseteq Y_a$, and a natural inclusion $A \hookrightarrow B$ taking a to the unique $b \in B_a$ such that Z_b is open in Y_a. (Moreover, the composite $A \hookrightarrow B \twoheadrightarrow A$ is the identity.) We will call the map $B \twoheadrightarrow A$ the *A-envelope*, and will generally use the inclusion $A \hookrightarrow B$ to identify A with its image. Post this identification, each $a \in A$ corresponds to two strata Y_a, Z_a, and we emphasize that these are usually *not* equal; rather, one only knows that Y_a contains Z_a densely.

To each GGMS stratum X, one standardly associates the set of coordinate k-planes that are elements of $\overline{X}$, called the **matroid of** X. (While 'matroid' has many simple definitions, this is not one of them; only **realizable** matroids arise this way, and characterizing them is essentially out of reach [Vam78].) It is a standard, and easy, fact that the matroid characterizes the stratum, so via the $A \hookrightarrow B$ yoga above, we can index the strata in the Schubert, Richardson, and positroid decompositions by special classes of matroids. Schubert matroids have been rediscovered many times in the matroid literature (and renamed each time; see [BM06]). Richardson matroids are known as *lattice path matroids* [BM06]. The matroids associated to the open positroid varieties are exactly the positroids [Pos05b] (though Postnikov's original definition was different, and we give it in the next section).

In our context, the observation two paragraphs above says that if a matroid M is a positroid, then the positroid stratum of M is usually *not* the GGMS stratum of M, but only contains it densely.

Remark 1.1. For each positroid M, Postnikov gives many parametrizations by $\mathbb{R}^\ell_+$ of the totally nonnegative part (whose definition we will recall in the next section) of the GGMS stratum of M. Each parametrization extends to a rational map $(\mathbb{C}^\times)^\ell \to \mathrm{Gr}(k,n)$; if we use the parametrization coming (in Postnikov's terminology) from the Le-diagram of M then this map is well defined on all of $(\mathbb{C}^\times)^\ell$. The image of this map is neither the GGMS stratum nor the positroid stratum of M (although the nonnegative parts of all three coincide). For example, if $(k,n) = (2,4)$ and M is the 'uniform' matroid in which any two elements of $[4]$ are independent, this parametrization is

$$(a, b, c, d) \mapsto (p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}) = (1 : d : cd : bd : (a+1)bcd : abcd^2).$$

The image of this map is the open set where p_{12}, p_{13}, p_{14}, p_{23} and p_{34} are non-zero. It is smaller than the positroid stratum, where p_{13} can be zero. The image is larger than the GGMS stratum, where p_{24} is also non-zero.

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One may regard this, perhaps, as evidence that matroids are a philosophically incorrect way to index the strata. We shall see another piece of evidence in Remark 5.17.

1.2 Juggling patterns, affine Bruhat order, and total nonnegativity
We now give a lowbrow description of the decomposition we are studying, from which we will see a natural indexing of the strata.

Start with a $k \times n$ matrix M of rank k ($\leqslant n$), and think of it as a list of column vectors $\vec{v}_1, \ldots, \vec{v}_n$. Extend this to an infinite but repeating list $\ldots, \vec{v}_{-1}, \vec{v}_0, \vec{v}_1, \ldots, \vec{v}_n, \vec{v}_{n+1}, \ldots$ where $\vec{v}_i = \vec{v}_j$ if $i \equiv j \mod n$. Then define a function $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(i) = \min\{j \geqslant i : \vec{v}_i \in \mathrm{span}(\{\vec{v}_{i+1}, \vec{v}_{i+2}, \ldots, \vec{v}_j\})\}.$$

Since $\vec{v}_{n+i} = \vec{v}_i$, each $f(i) \leqslant n + i$, and each $f(i) \geqslant i$ with equality only if $\vec{v}_i = \vec{0}$. It is fun to prove that f must be $1 : 1$, and has enough finiteness to then necessarily be onto as well. Permutations of $\mathbb{Z}$ satisfying $f(i + n) = f(i) + n \; \forall i$ are called **affine permutations**, and the group thereof can be identified with the affine Weyl group of GL_n (see e.g. [ER96]).

This association of an affine permutation to each $k \times n$ matrix of rank k depends only on the k-plane spanned by the rows, and so descends to $\mathrm{Gr}(k, n)$, where it provides a complete combinatorial invariant of the strata in the cyclic Bruhat decomposition.

THEOREM (Theorem 3.16, Corollary 3.17).[1] *This map from the set of positroid strata to the affine Weyl group is order-preserving, with respect to the closure order on positroid strata (Postnikov's **cyclic Bruhat order**) and the affine Bruhat order, and identifies the set of positroids with a downward Bruhat order ideal.*

Consequently, the cyclic Bruhat order is Eulerian and EL-shellable (as shown by hand already in [Wil07]).

We interpret these f physically as follows. Consider a juggler who is juggling k balls, one thrown every second, doing a pattern of period n. At time $i \in \mathbb{Z}$, they throw a ball that lands shortly[2] before, to be thrown again at, time $f(i)$. No two balls land at the same time, and there is always a ball available for the next throw. If we let $t_i = f(i) - i$ be the **throw** at time i, this cyclic list of n numbers $(t_1, \ldots, t_n)$ is a *juggling pattern*[3] or *siteswap* (for which our references are [Knu93, Pol03]; see also [BEGW94, CG07, CG08, ER96, War05]). This mathematical model of juggling was developed by several groups of jugglers independently in 1985, and is of great practical use in the juggling community.

If M is generic, then the pattern is the lowest-energy pattern, where every throw is a k-throw.[4] At the opposite extreme, imagine that M only has entries in some k columns. Then $n - k$ of the throws are 0-throws, and k are n-throws.[5]

If one changes the cyclic action slightly, by moving the first column to the end *and multiplying it by* $(-1)^{k-1}$, then one preserves the set of real matrices for which every $k \times k$ submatrix has nonnegative determinant. This, by definition, lies over the **totally nonnegative part** $\mathrm{Gr}(k, n)_{\geqslant 0}$

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[1] This result is extended to projected Richardson varieties in partial flag varieties G/P of arbitrary type by He and Lam [HL11].
[2] Almost exactly at time $f(i) - \frac{1}{2}$, according to video analysis of competent jugglers.
[3] Not every juggling pattern arises this way; the patterns that arise from matrices can only have throws of height $\leqslant n$. This bound is very unnatural from the juggling point of view, as it excludes the standard 3-ball cascade $(t_1 = 3)$ with period $n = 1$.
[4] These juggling patterns are called 'cascades' for k odd and '(asynchronous) fountains' for k even.
[5] These are not the most excited k-ball patterns of length n; those would each have a single kn-throw, all the others being 0-throws. But juggling patterns associated to matrices must have each $t_i \leqslant n$.

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of the Grassmannian. (This action may have period either n or $2n$ up on matrices, but it always has period n down on the Grassmannian.) Postnikov's motivation was to describe those matroids whose GGMS strata intersect this totally nonnegative part; it turns out that they are exactly the positroids, and the totally nonnegative part of each open positroid stratum is homeomorphic to a ball.

Remark 1.2. Now that we have defined the totally nonnegative part of the Grassmannian, we can explain the antecedents to Theorem 5.9. Postnikov [Pos05b] defined the totally nonnegative part of the Grassmannian as we have done above, by nonnegativity of all minors. Lusztig [Lus98] gave a different definition which applied to any G/P. That the two notions agree is not obvious, and was established by K. Rietsch in a private communication (2009). In particular, the cyclic symmetry seems to be special to Grassmannians.[6]

Lusztig, using his definition, gave a stratification of $(G/P)_{\geqslant 0}$ by the projections of Richardson varieties. Theorem 3.8 of [Pos05b] (which relies on the results of [MR04, RW08]) states that Postnikov's and Lusztig's stratifications of $\mathrm{Gr}(k, n)_{\geqslant 0}$ coincide. This result says nothing about how the stratifications behave away from the totally nonnegative region. Theorem 5.9 can be thought of as a complex analogue of [Pos05b, Theorem 3.8]; it implies but does not follow from [Pos05b, Theorem 3.8].

We thank Konni Rietsch for helping us to understand the connections between these results.

1.3 Affine permutations, and the associated cohomology class of a positroid variety
Given a subvariety X of a Grassmannian, one can canonically associate a symmetric polynomial in k variables, in a couple of equivalent ways.

(i) Sum, over partitions λ with $|\lambda| = \mathrm{codim}\, X$, the Schur polynomial $S_\lambda(x_1, \ldots, x_k)$ weighted by the number of points of intersection of X with a generic translate of X_{λ^c} (the Schubert variety associated to the complementary partition inside the $k \times (n - k)$ rectangle).

(ii) Take the preimage of X in the Stiefel manifold of $k \times n$ matrices of rank k, and the closure $\overline{X}$ inside $k \times n$ matrices. (In the $k = 1$ case this is the affine cone over a projective variety, and it seems worth giving it the name 'Stiefel cone' in general.) This has a well-defined class in the equivariant Chow ring $A^*_{\mathrm{GL}(k)}(\mathbb{C}^{k \times n})$, which is naturally the ring of symmetric polynomials in k variables.

The most basic case of X is a Schubert variety X_λ, in which case these recipes give the Schur polynomial S_λ. More generally, the first construction shows that the symmetric polynomial must be 'Schur-positive', meaning a positive sum of Schur polynomials.

In reverse, one has ring homomorphisms

$$\{\text{symmetric functions}\} \to \mathbb{Z}[x_1, \ldots, x_k]^{S_k} \cong A^*_{\mathrm{GL}(k)}(\mathbb{C}^{k \times n}) \to A^*_{\mathrm{GL}(k)}(\text{Stiefel}) \cong A^*(\mathrm{Gr}(k, n))$$

and one can ask for a symmetric function f whose image is the class $[X]$.

THEOREM (Theorem 7.1).[7] *The cohomology class associated to a positroid variety can be represented by the affine Stanley function of its affine permutation, as defined in [Lam06].*

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[6] Milen Yakimov has proven the stronger result that the standard Poisson structure on $\mathrm{Gr}(k, n)$, from which the positroid stratification can be derived, is itself cyclic-invariant [Yak10a].

[7] Snider [Sni10] has given a direct geometric explanation of this result by identifying affine patches on $\mathrm{Gr}(k, n)$ with opposite Bruhat cells in the affine flag manifold, in a way that takes the positroid stratification to the Bruhat decomposition. Also, an analogue of this result for projected Richardson varieties in an arbitrary G/P is established by He and Lam [HL11]: the connection with symmetric functions is absent, but the cohomology classes of projected Richardson varieties and affine Schubert varieties are compared via the affine Grassmannian.

This is a surprising result in that affine Stanley functions are not Schur-positive in general, even for this restricted class of affine permutations. Once restricted to the variables $x_1, \ldots, x_k$, they are! In Theorem 7.12 we give a much stronger abstract positivity result, for positroid classes in T-equivariant K-theory.

Our proof of Theorem 7.1 is inductive. In future work, by embedding the Grassmannian in a certain subquotient of the affine flag manifold, and realizing the positroid decomposition as the transverse pullback of the affine Bruhat decomposition, we hope to give a direct geometric proof of this and Theorem 3.16.

1.4 Quantum cohomology and toric Schur functions

In [BKT03], Buch, Kresch, and Tamvakis related quantum Schubert calculus on Grassmannians to ordinary Schubert calculus on 2-step partial flag manifolds. In [Pos05a], Postnikov showed that the structure constants of the quantum cohomology of the Grassmannian were encoded in symmetric functions he called toric Schur polynomials. We connect these ideas to positroid varieties as in the following theorem.

THEOREM (Theorem 8.1). *Let $S \subset \mathrm{Gr}(k, n)$ be the union of all genus-zero stable curves of degree d which intersect a fixed Schubert variety X and opposite Schubert variety Y. Suppose there is a non-trivial quantum problem associated to X, Y and d. Then S is a positroid variety: as a projected Richardson variety it is obtained by a pull-push from the 2-step flag variety considered in [BKT03]. Its cohomology class is given by the toric Schur polynomial of [Pos05a].*

The last statement of the theorem is consistent with the connection between affine Stanley symmetric functions and toric Schur functions (see [Lam06]).

2. Some combinatorial background

Unless otherwise specified, we shall assume that nonnegative integers k and n have been fixed, satisfying $0 \leqslant k \leqslant n$.

2.1 Conventions on partitions and permutations

For integers a and b, we write $[a, b]$ to denote the interval $\{a, a+1, \ldots, b\}$, and $[n]$ to denote the initial interval $\{1, 2, \ldots, n\}$. If $i \in \mathbb{Z}$, we let $\bar{i} \in [n]$ be the unique integer satisfying $i \equiv \bar{i} \mod n$. We write $\binom{S}{k}$ for the set of k-element subsets of S. Thus $\binom{[n]}{k}$ denotes the set of k-element subsets of $\{1, 2, \ldots, n\}$.

As is well known, there is a bijection between $\binom{[n]}{k}$ and the partitions of λ contained in a $k \times (n - k)$ box. There are many classical objects, such as Schubert varieties, which can be indexed by either of these $\binom{n}{k}$-element sets. We will favor the indexing set $\binom{[n]}{k}$, and will only discuss the indexing by partitions when it becomes essential, in § 7.

We let S_n denote the permutations of the set $[n]$. A permutation $w \in S_n$ is written in one-line notation as $[w(1)w(2) \cdots w(n)]$. Permutations are multiplied from right to left so that if $u, w \in S_n$, then $(uw)(i) = u(w(i))$. Thus multiplication on the left acts on values, and multiplication on the right acts on positions. Let $w \in S_n$ be a permutation. An **inversion** of w is a pair $(i, j) \in [n] \times [n]$ such that $i < j$ and $w(i) > w(j)$. The length $\ell(w)$ of a permutation $w \in S_n$ is the number of its inversions. A factorization $w = uv$ is called **length-additive** if $\ell(w) = \ell(u) + \ell(v)$.

The longest element $[n(n-1)\cdots 1]$ of S_n is denoted w_0. The permutation $[234\cdots n1]$ is denoted χ (for Coxeter element). As a Coxeter group, S_n is generated by the simple transpositions $\{s_i = [12\cdots(i-1)(i+1)i(i+2)\cdots n]\}$.

For $k \in [0, n]$, we let $S_k \times S_{n-k} \subseteq S_n$ denote the parabolic subgroup of permutations which send $[k]$ to $[k]$ and $[k+1, n]$ to $[k+1, n]$. A permutation $w \in S_n$ is called **Grassmannian** (respectively **anti-Grassmannian**) if it is minimal (respectively maximal) length in its coset $w(S_k \times S_{n-k})$; the set of such permutations is denoted $S_{n,k}^{\min}$ (respectively $S_{n,k}^{\max}$).

If $w \in S_n$ and $k \in [0, n]$, then $\sigma_k(w) \in \binom{[n]}{k}$ denotes the set $w([k])$. Often, we just write σ for σ_k when no confusion will arise. The map $\sigma_k : S_n \to \binom{[n]}{k}$ is a bijection when restricted to $S_{n,k}^{\min}$.

2.2 Bruhat order and weak order

We define a partial order $\leqslant$ on $\binom{[n]}{k}$ as follows. For $I = \{i_1 < i_2 < \cdots < i_k\}$ and $J = \{j_1 < j_2 \cdots < j_k\} \in \binom{[n]}{k}$, we write $I \leqslant J$ if $i_r \leqslant j_r$ for $r \in [k]$.

We shall denote the **Bruhat order**, also called the **strong order**, on S_n by $\leqslant$ and $\geqslant$. One has the following well known criterion for comparison in Bruhat order: if $u, w \in S_n$ then $u \leqslant w$ if and only if $u([k]) \leqslant w([k])$ for each $k \in [n]$. Covers in Bruhat order will be denoted by $\lessdot$ and $\gtrdot$. The map $\sigma_k : (S_{n,k}^{\min}, \leqslant) \to (\binom{[n]}{k}, \leqslant)$ is a poset isomorphism.

The **(left) weak order** $\leqslant_{\mathrm{weak}}$ on S_n is the transitive closure of the relations

$$w \leqslant_{\mathrm{weak}} s_i w \quad \text{if } \ell(s_i w) = \ell(w) + 1.$$

The weak order and Bruhat order agree when restricted to $S_{n,k}^{\min}$.

2.3 k-Bruhat order and the poset $\mathcal{Q}(k, n)$

The **k-Bruhat order** [BS98, LS82] $\leqslant_k$ on S_n is defined as follows. Let u and w be in S_n. Then u **k-covers** w, written $u \gtrdot_k w$, if and only if $u \gtrdot w$ and $\sigma_k(u) \neq \sigma_k(w)$. The k-Bruhat order is the partial order on S_n generated by taking the transitive closure of these cover relations (which remain cover relations). We let $[u, w]_k \subset S_n$ denote the interval of S_n in k-Bruhat order. It is shown in [BS98] that every interval $[u, w]_k$ in $(S_n, \leqslant_k)$ is a graded poset with rank $\ell(w) - \ell(u)$. We have the following criterion for comparison in k-Bruhat order.

THEOREM 2.1 [BS98, Theorem A]. *Let $u, w \in S_n$. Then $u \leqslant_k w$ if and only if:*

(i) *$1 \leqslant a \leqslant k < b \leqslant n$ implies $u(a) \leqslant w(a)$ and $u(b) \geqslant w(b)$;*

(ii) *if $a < b$, $u(a) < u(b)$, and $w(a) > w(b)$, then $a \leqslant k < b$.*

Define an equivalence relation on the set of k-Bruhat intervals, generated by the relations that $[u, w]_k \sim [x, y]_k$ if there is a $z \in S_k \times S_{n-k}$ so that we have length-additive factorizations $uz = x$ and $wz = y$. If $u \leqslant_k w$, we let $\langle u, w \rangle$ denote the equivalence class containing $[u, w]_k$. Let $\mathcal{Q}(k, n)$ denote the equivalence classes of k-Bruhat intervals.

We discuss this construction in greater generality in [KLS, §2]. To obtain the current situation, specialize the results of that paper to $(W, W_P) = (S_n, S_k \times S_{n-k})$. The results we describe here are all true in that greater generality.

PROPOSITION 2.2. *If $[u_1, w_1] \sim [u_2, w_2]$ then $u_1^{-1}u_2 = w_1^{-1}w_2$ and this common ratio is in $S_k \times S_{n-k}$. Also $\ell(w_1) - \ell(w_2) = \ell(u_1) - \ell(u_2)$.*

Proof. This is obvious for the defining equivalences and is easily seen to follow for a chain of equivalences. $\qquad\square$

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We will prove a converse of this statement below as Proposition 2.4. The reader may prefer this definition of $\sim$.

PROPOSITION 2.3. *Every equivalence class in $\mathcal{Q}(k,n)$ has a unique representative of the form $[u', w']$ where w' is Grassmannian. If $[u,w]$ is a k-Bruhat interval, and $[u', w']$ is equivalent to $[u,w]$ with w' Grassmannian, then we have length-additive factorizations $u = u'z$ and $w = w'z$ with $z \in S_k \times S_{n-k}$.*

Proof. See [KLS, Lemma 2.4] for the existence of a representative of this form. If $[u', w']$ and $[u'', w'']$ are two such representatives, then $(w')^{-1}w''$ is in $S_k \times S_{n-k}$ and both w' and w'' are Grassmannian, so $w' = w''$. Then $(u')^{-1}u'' = (w')^{-1}w'' = e$ so $u' = u''$ and we see that the representative is unique.

Finally, let $[u', w']$ be the representative with w' Grassmannian, and let $[u,w] \sim [u', w']$. Set $z = (u')^{-1}u = (w')^{-1}w$ with $z \in S_k \times S_{n-k}$. Since w' is Grassmannian, we have $\ell(w) = \ell(w') + \ell(z)$. Then the equation $\ell(w) - \ell(w') = \ell(u) - \ell(u')$ from Proposition 2.2 shows that $\ell(u) = \ell(u') + \ell(z)$ as well. So the products $u = u'z$ and $w = w'z$ are both length-additive, as desired. $\qquad\square$

We can use this observation to prove a more computationally useful version of the equivalence relation as in the following proposition.

PROPOSITION 2.4. *Given two k-Bruhat intervals $[u_1, w_1]$ and $[u_2, w_2]$, we have $[u_1, w_1] \sim [u_2, w_2]$ if and only if $u_1^{-1}u_2 = w_1^{-1}w_2$ and the common ratio lies in $S_k \times S_{n-k}$.*

Proof. The forward implication is Proposition 2.2. For the reverse implication, let $[u_1, w_1]$ and $[u_2, w_2]$ be as stated. Let $[u_1', w_1']$ and $[u_2', w_2']$ be the representatives with w_i' Grassmannian. Since $(w_1')^{-1}w_2'$ is in $S_k \times S_{n-k}$, and both w_i' are Grassmannian, then $w_1' = w_2'$. Since $(u_1')^{-1}u_2' = (w_1')^{-1}w_2' = e$, we deduce that $u_1' = u_2'$. So $[u_1, w_1] \sim [u_1', w_1'] = [u_2', w_2'] \sim [u_2, w_2]$ and we have the reverse implication. $\qquad\square$

We also cite the following theorem.

THEOREM 2.5 [BS98, Theorem 3.1.3]. *If $u \leqslant_k w$ and $x \leqslant_k y$ with $wu^{-1} = yx^{-1}$, then the map $v \mapsto vu^{-1}x$ induces an isomorphism of graded posets $[u,w]_k \to [x,y]_k$.*

We equip $\mathcal{Q}(k,n)$ with a partial order $\leqslant$ given by $q' \leqslant q$ if and only if there are representatives $[u,w]_k \in q$ and $[u', w']_k \in q'$ so that $[u', w'] \subseteq [u,w]$. This partial order was studied by Rietsch [Rie06], see also [GY09, Wil07].

The poset $\mathcal{Q}(2,4)$ already has 33 elements; its Hasse diagram appears in [Wil07]. See also Figure 1.

3. Affine permutations, juggling patterns and positroids

Fix integers $0 \leqslant k \leqslant n$. In this section, we will define several posets of objects and prove that the posets are all isomorphic. We begin by surveying the posets we will consider. The objects in these posets will index positroid varieties, and all of these indexing sets are useful. All the isomorphisms we define are compatible with each other. Detailed definitions, and the definitions of the isomorphisms, will be postponed until later in the section.

We have already met one of our posets, the poset $\mathcal{Q}(k,n)$ from § 2.3.

The next poset will be the poset Bound(k,n) of bounded affine permutations: these are bijections $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(i+n) = f(i) + n$, $i \leqslant f(i) \leqslant f(i) + n$ and $(1/n)\sum_{i=1}^{n}(f(i) - i) = k$. After that will be the poset Jugg(k,n) of bounded juggling patterns. The elements of this

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poset are n-tuples $(J_1, J_2, \ldots, J_n) \in \binom{[n]}{k}^n$ such that $J_{i+1} \supseteq (J_i \setminus \{1\}) - 1$, where the subtraction of 1 means to subtract 1 from each element and our indices are cyclic modulo n. These two posets are closely related to the posets of decorated permutations and of Grassmann necklaces, considered in [Pos05b].

We next consider the poset of cyclic rank matrices. These are infinite periodic matrices which relate to bounded affine permutations in the same way that Fulton's rank matrices relate to ordinary permutations. Finally, we will consider the poset of positroids. Introduced in [Pos05b], these are matroids which obey certain positivity conditions.

The following is a combination of all the results of this section.

THEOREM 3.1. *The posets $\mathcal{Q}(k,n)$, $\mathrm{Bound}(k,n)$, $\mathrm{Jugg}(k,n)$, the poset of cyclic rank matrices of type (k,n) and the poset of positroids of rank k on $[n]$ are all isomorphic.*

The isomorphism between $\mathrm{Bound}(k,n)$ and cyclic rank matrices is Corollary 3.12; the isomorphism between $\mathrm{Jugg}(k,n)$ and $\mathrm{Bound}(k,n)$ is Corollary 3.13; the isomorphism between $Q(k,n)$ and $\mathrm{Bound}(k,n)$ is Theorem 3.16; the isomorphism between $\mathrm{Jugg}(k,n)$ and positroids is Proposition 3.21.

3.1 Juggling states and functions

Define a **(virtual) juggling state** $S \subseteq \mathbb{Z}$ as a subset whose symmetric difference from $-\mathbb{N} := \{i \leqslant 0\}$ is finite. (We will motivate this and other juggling terminology below.) Let its **ball number** be $|S \cap \mathbb{Z}_+| - |-\mathbb{N} \setminus S|$, where $\mathbb{Z}_+ := \{i > 0\}$. Ball number is the unique function on juggling states such that for $S \supseteq S'$, the difference in ball numbers is $|S \setminus S'|$, and $-\mathbb{N}$ has ball number zero.

Call a bijection $f : \mathbb{Z} \to \mathbb{Z}$ a **(virtual) juggling function** if for some (or equivalently, any) $t \in \mathbb{Z}$, the set $f(\{i : i \leqslant t\})$ is a juggling state. It is sufficient (but not necessary) that $\{|f(i) - i| : i \in \mathbb{Z}\}$ be bounded. Let G be the set of such functions: it is easy to see that G is a group, and contains the element $s_+ : i \mapsto i + 1$. Define the **ball number of** $f \in G$ as the ball number of the juggling state $f(-\mathbb{N})$, and denote it $\mathrm{av}(f)$ for reasons to be explained later.

LEMMA 3.2. *The ball number $\mathrm{av} : G \to \mathbb{Z}$ is a group homomorphism.*

Proof. We prove what will be a more general statement, that if S is a juggling state with ball number b, and f a juggling function with ball number b', then $f(S)$ is a juggling state with ball number $b + b'$. This is proved by the following argument: if we add one element to S, this adds one element to $f(S)$, and changes the ball numbers of S, $f(S)$ by 1. We can use this operation and its inverse to reduce to the case that $S = -\mathbb{N}$, at which point the statement is tautological.

Now let $f, g \in G$, and apply the just-proven statement to $S = g(-\mathbb{N})$. $\square$

For any bijection $f : \mathbb{Z} \to \mathbb{Z}$, let

$$\mathrm{st}(f, t) := \{f(i) - t : i \leqslant t\}$$
$$= \mathrm{st}(s_+^t f s_+^{-t}, 0)$$

and if $f \in G$, call it the **juggling state of f at time** t. By the homomorphism property just proven, $\mathrm{av}(f) = \mathrm{av}(s_+^k f s_+^{-k})$, which says that every state of $f \in G$ has the same ball number ('ball number is conserved'). The following lemma lets one work with juggling states rather than juggling functions.

LEMMA 3.3. *Say that a juggling state T **can follow** a state S if $T = \{t\} \cup (s_+^{-1} \cdot S)$, and $t \notin s_+^{-1} \cdot S$. In this case say that a t-**throw takes state S to state T**.*

Then a list $(S_i)_{i \in \mathbb{Z}}$ is the list of states of a juggling function if and only if S_{i+1} can follow S_i for each i. In this case the juggling function is unique.

Proof. If the (S_i) arise from a juggling function f, then the condition is satisfied where the element t_i added to $s_+^{-1} \cdot S_{i-1}$ is $f(i) - i$. Conversely, one can construct f as $f(i) = i + t_i$. $\qquad\square$

In fact the finiteness conditions on juggling states and permutations were not necessary for the lemma just proven. We now specify a further finiteness condition, that will bring us closer to the true functions of interest.

LEMMA 3.4. *The following two conditions on a bijection $f : \mathbb{Z} \to \mathbb{Z}$ are equivalent:*

 (i) *there is a uniform bound on $|f(i) - i|$; or*

 (ii) *there are only finitely many different $\mathrm{st}(f, i)$ visited by f.*

If they hold, f is a juggling function.

Proof. Assume first that f has only finitely many different states. By Lemma 3.3, we can reconstruct the value of $f(i) - i$ from the states S_i, S_{i+1}. So $f(i) - i$ takes on only finitely many values, and hence $|f(i) - i|$ is uniformly bounded.

For the reverse, assume that $|f(i) - i| < N$ for all $i \in \mathbb{Z}$. Then $f(-\mathbb{N}) \subseteq \{i < N\}$, and $f(\mathbb{Z}_+) \subseteq \{i > -N\}$. Since f is bijective, we can complement the latter to learn that $f(-\mathbb{N}) \supseteq \{i \leqslant -N\}$. So $\mathrm{st}(f, 0)$, and similarly each $\mathrm{st}(f, t)$, is trapped between $\{i \leqslant -N\}$ and $\{i < N\}$. There are then only 2^{2N} possibilities, all of which are juggling states. $\qquad\square$

In the next section we will consider juggling functions which cycle periodically through a finite set of states.

Define the **height of the juggling state** $S \subseteq \mathbb{Z}$ as

$$\mathrm{ht}(S) := \sum_{i \in S \cap \mathbb{Z}_+} i - \sum_{i \in -\mathbb{N} \setminus S} i,$$

a sort of weighted ball number. We can now motivate the notation $\mathrm{av}(f)$, computing ball number as an average.

LEMMA 3.5. *Let $a, b \in \mathbb{Z}$, $a \leqslant b$ and let $f \in G$. Then*

$$\sum_{i=a+1}^{b} (f(i) - i) = (b - a) \, \mathrm{av}(f) + \mathrm{ht}(\mathrm{st}(f, b)) - \mathrm{ht}(\mathrm{st}(f, a)).$$

In particular, if f satisfies the conditions of Lemma 3.4, then for any $a \in \mathbb{Z}$,

$$\lim_{b \to \infty} \frac{1}{b - a} \sum_{i=a+1}^{b} (f(i) - i) = \mathrm{av}(f).$$

This equality also holds without taking the limit, if $\mathrm{st}(f, a) = \mathrm{st}(f, b)$.

Proof. It is enough to prove the first statement for $b = a + 1$, and add the $b - a$ many equations together. They are of the form

$$f(a + 1) - (a + 1) = \mathrm{av}(f) + \mathrm{ht}(\mathrm{st}(f, a + 1)) - \mathrm{ht}(\mathrm{st}(f, a)).$$

To see this, start with $S = \text{st}(f, a)$, and use $f(a + 1)$ to calculate $\text{st}(f, a + 1)$. The three sets to consider are

$$S = f(\{i \leqslant a\}) \text{ shifted left by } a$$
$$S' = f(\{i \leqslant a\}) \text{ shifted left by } a + 1$$
$$\text{st}(f, a + 1) = f(\{i \leqslant a + 1\}) \text{ shifted left by } a + 1.$$

By its definition, $\text{ht}(S') = \text{ht}(S) - \text{av}(f)$. And $\text{ht}(\text{st}(f, a + 1)) = \text{ht}(S') + f(a + 1) - (a + 1)$. The equation follows.

For the second, if f only visits finitely many states then the difference in heights is bounded, and dividing by $b - a$ kills this term in the limit. $\qquad\square$

We now motivate these definitions from a juggler's point of view. The canonical reference is [Pol03], though our setting above is more general than considered there. All of these concepts originated in the juggling community in the years 1985–1990, though precise dates are difficult to determine.

Consider a idealized juggler who is juggling with one hand,[8] making one throw every second, of exactly one ball at a time, has been doing so since the beginning of time and will continue until its end. If our juggler is only human (other than being immortal) then there will be a limit on how high the throws may go.

Assume at first that the hand is never found empty when a throw is to be made. The history of the juggler can then be recorded by a function

$$f(t) = \text{the time that a ball thrown at time } t \text{ is next thrown.}$$

The number $f(t) - t$ is usually called the **throw at time** t. If ever the juggler *does* find the hand empty i.e. all the balls in the air, then of course the juggler must wait one second for the balls to come down. This is easily incorporated by taking $f(t) = t$, a 0-**throw**.

While these assumptions imply that f is a juggling function, they would also seem to force the conclusion that $f(i) \geqslant i$, i.e. that *balls land after they are thrown*. Assuming that for a moment, it is easy to compute the number of balls being juggled in the permutation f: at any time t, count how many balls were thrown at times $\{i \leqslant t\}$ that are still in the air, $f(i) > t$. This is of course our formula for the ball number, in this special case. The formula $\text{av}(f) = \text{av}(s_+^t f s_+^{-t})$ then says that balls are neither created nor destroyed.

The state of $f \in G$ at time t is the set of times in the future (of t) that balls in the air are scheduled to land. (This was introduced to study juggling by the first author and, independently, by Jack Boyce, in 1988.) The 'height' of a state does not seem to have been considered before.

Thus, the sub-semigroup of G where $f(t) \geqslant t$ encodes possible juggling patterns. Since we would like to consider G as a group (an approach pioneered in [ER96]), we must permit $f(t) < t$. While it may seem fanciful to view this as describing juggling with antimatter, the 'Dirac sea' interpretation of antimatter is suggestive of the connection with the affine Grassmannian.

3.2 Affine permutations

Let $\tilde{S}_n$ denote the group of bijections, called **affine permutations**, $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(i + n) = i + n \quad \text{for all } i \in \mathbb{Z}.$$

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[8] Or as is more often assumed, rigidly alternating hands.

Plainly this is a subgroup of G. This group fits into an exact sequence

$$1 \to \mathbb{Z}^n \xrightarrow{t} \tilde{S}_n \twoheadrightarrow S_n \to 1$$

where for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$, we define the **translation element** $t_\mu \in \tilde{S}_n$ by $t_\mu(i) = n\mu_i + i$ for $1 \leqslant i \leqslant n$. The map $\tilde{S}_n \twoheadrightarrow S_n$ is evident. We can give a splitting map $S_n \to \tilde{S}_n$ by extending a permutation $\pi : [n] \to [n]$ periodically. By this splitting, we have $\tilde{S}_n \simeq S_n \ltimes \mathbb{Z}^n$, so every $f \in \tilde{S}_n$ can be uniquely factorized as $f = w \, t_\mu$ with $w \in S_n$ and $\mu \in \mathbb{Z}^n$.

An affine permutation $f \in \tilde{S}_n$ is written in one-line notation as $[\cdots f(1)f(2)\cdots f(n)\cdots]$ (or occasionally just as $[f(1)f(2)\cdots f(n)]$). As explained in §1.2, jugglers instead list one period of the periodic function $f(i) - i$ (without commas, because very few people can make 10-throws[9] and higher), and call this the **siteswap**. We adopt the same conventions when multiplying affine permutations as for usual permutations. The ball number $\mathrm{av}(f) = (1/n)\sum_{i=1}^{n}(f(i) - i)$ is always an integer; indeed $\mathrm{av}(w\,t_\mu) = \mathrm{av}(t_\mu) = \sum_i \mu_i$. Define

$$\tilde{S}_n^k = \{f \in \tilde{S}_n \mid \mathrm{av}(f) = k\} = s_+^k \tilde{S}_n^0$$

so that $\tilde{S}_n^0 = \ker \mathrm{av}$ is the Coxeter group with simple generators $s_0, s_1, \ldots, s_{n-1}$, usually called the **affine symmetric group**.[10] Note that if $f \in \tilde{S}_n^a$ and $g \in \tilde{S}_n^b$ then the product fg is in $\tilde{S}_n^{a+b}$. There is a canonical bijection $f \mapsto f \circ (i \mapsto i + b - a)$ between the cosets $\tilde{S}_n^a$ and $\tilde{S}_n^b$. The group $\tilde{S}_n^0$ has a Bruhat order '$\leqslant$' because it is a Coxeter group $\widetilde{A_{n-1}}$. This induces a partial order on each $\tilde{S}_n^a$, also denoted $\leqslant$.

An **inversion** of f is a pair $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that $i < j$ and $f(i) > f(j)$. Two inversions (i, j) and (i', j') are equivalent if $i' = i + rn$ and $j' = j + rn$ for some integer r. The number of equivalence classes of inversions is the **length** $\ell(f)$ of f. This is sort of an 'excitation number' of the juggling pattern; this concept does not seem to have been studied in the juggling community (though see [ER96]).

An affine permutation $f \in \tilde{S}_n^k$ is **bounded** if $i \leqslant f(i) \leqslant i + n$ for $i \in \mathbb{Z}$. We denote the set of bounded affine permutations by $\mathrm{Bound}(k, n)$. The restriction of the Bruhat order to $\mathrm{Bound}(k, n)$ is again denoted $\leqslant$.

LEMMA 3.6. *The subset* $\mathrm{Bound}(k, n) \subset \tilde{S}_n^k$ *is a lower order ideal in* $(\tilde{S}_n^k, \leqslant)$. *In particular,* $(\mathrm{Bound}(k, n), \leqslant)$ *is graded by the rank function* $\ell(f)$.

Proof. Suppose $f \in \mathrm{Bound}(k, n)$ and $g \lessdot f$. Then g is obtained from f by swapping the values of $i + kn$ and $j + kn$ for each k, where $i < j$ and $f(i) > f(j)$. By the assumption on the boundedness of f, we have $i + n \geqslant f(i) > f(j) = g(i) \geqslant j > i$ and $j + n > i + n \geqslant f(i) = g(j) > f(j) \geqslant j$. Thus $g \in \mathrm{Bound}(k, n)$. $\square$

Postnikov, in [Pos05b, §13], introduces 'decorated permutations'. A decorated permutation is an element of S_n, with each fixed point colored either 1 or -1. There is an obvious bijection between the set of decorated permutations and $\coprod_{k=0}^{n} \mathrm{Bound}(k, n)$: given an element $f \in \mathrm{Bound}(k, n)$, form the corresponding decorated permutation by reducing f modulo n and coloring the fixed points of this reduction -1 or 1 according to whether $f(i) = i$ or $f(i) = i + n$ respectively. In [Pos05b, §17], Postnikov introduces the cyclic Bruhat order, CB_{kn}, on those

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[9] The few that do sometimes use $A, B, \ldots$ to denote throws $10, 11, \ldots$, which prompts the question of what words are jugglable. Michael Kleber informs us that THEOREM and TEAKETTLE give valid juggling patterns.

[10] One reason the subgroup $\tilde{S}_n^0$ is more commonly studied than $\tilde{S}_n$ is that it is the Coxeter group $\widetilde{A_{n-1}}$; its relevance for us, is that it indexes the Bruhat cells on the affine flag manifold for the group SL_n. In §7 we will be concerned with the affine flag manifold for the group GL_n, whose Bruhat cells are indexed by all of $\tilde{S}_n$.
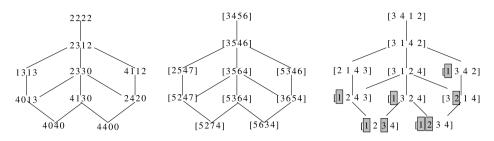
FIGURE 1. The posets of siteswaps, bounded affine permutations, and decorated permutations for $\mathrm{Gr}(2,4)$, each up to cyclic rotation. (The actual posets each have 33 elements.)

decorated permutations corresponding to elements of $\mathrm{Bound}(k,n)$. From the list of cover relations in [Pos05b, Theorem 17.8], it is easy to see that CB_{kn} is anti-isomorphic to $\mathrm{Bound}(k,n)$.

Example 3.7. In the $\mathrm{Gr}(2,4)$ case there are already 33 bounded affine permutations, but only 10 up to cyclic rotation. In Figure 1 we show the posets of siteswaps, affine permuations, and decorated permutations, each modulo rotation. Note that the cyclic symmetry is most visible on the siteswaps, and indeed jugglers draw little distinction between cyclic rotations of the 'same' siteswap.

3.3 Sequences of juggling states

A (k,n)**-sequence of juggling states** is a sequence $\mathcal{J} = (J_1, \ldots, J_n) \in \binom{[n]}{k}^n$ such that for each $i \in [n]$, we have that $J_{i+1} \cup -\mathbb{N}$ follows $J_i \cup -\mathbb{N}$, where the indices are taken modulo n. Let $\mathrm{Jugg}(k,n)$ denote the set of such sequences.

Let $f \in \mathrm{Bound}(k,n)$. Then the sequence of juggling states

$$\ldots, \mathrm{st}(f,-1), \mathrm{st}(f,0), \mathrm{st}(f,1), \ldots$$

is periodic with period n. Furthermore for each $i \in \mathbb{Z}$:

 (a) $-\mathbb{N} \subset \mathrm{st}(f,i)$; and

 (b) $\mathrm{st}(f,i) \cap [n] \in \binom{[n]}{k}$.

Thus

$$\mathcal{J}(f) = (\mathrm{st}(f,0) \cap [n], \mathrm{st}(f,1) \cap [n], \ldots, \mathrm{st}(f,n-1) \cap [n]) \in \mathrm{Jugg}(k,n).$$

LEMMA 3.8. *The map* $f \mapsto \mathcal{J}(f)$ *is a bijection between* $\mathrm{Bound}(k,n)$ *and* $\mathrm{Jugg}(k,n)$.

We now discuss another way of viewing (k,n)-sequences of juggling states which will be useful in §5.1. Let S be a k-ball virtual juggling state. For every integer j, define

$$R_j(S) = k - \#\{x \in S : x > j\}.$$

These $\{R_j\}$ satisfy the following properties:

 – $R_j(S) - R_{j-1}(S)$ is either 0 or 1, according to whether $j \notin J$ or $j \in J$ respectively;

 – $R_j(S) = k$ for j sufficiently positive; and

 – $R_j(S) = j$ for j sufficiently negative.

Conversely, from such a sequence (R_j) one can construct a k-ball juggling state.

Let S_1 and S_2 be two k-ball juggling states. Define a $2 \times \infty$ matrix (r_{ij}) by $r_{ij} = R_{j-i+1}(S_i)$.

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LEMMA 3.9. *The state S_2 can follow S_1 if and only if $r_{1j} - r_{2j} = 0$ or 1 for all $j \in \mathbb{Z}$ and there is no 2×2 submatrix for which $r_{1j} = r_{2j} = r_{2(j+1)} = r_{1(j+1)} - 1$.*

Proof. It is easy to check that $S_2 = s_+^{-1} S_1 \cup \{t\}$ if and only if $r_{1j} = r_{2j} + 1$ for $j \leqslant t$ and $r_{1j} = r_{2j}$ for $j \geqslant t + 1$. If this holds, it immediately follows that $r_{1j} - r_{2j} = 0$ or 1 for all j and that there is no j for which $r_{1j} = r_{2j}$ while $r_{2(j+1)} = r_{1(j+1)} - 1$.

Conversely, suppose that $r_{1j} - r_{2j} = 0$ or 1 for all $j \in \mathbb{Z}$ and there is no j for which $r_{1j} = r_{2j} = r_{2(j+1)} = r_{1(j+1)} - 1$. Then we claim that there is no j for which $r_{1j} = r_{2j}$ and $r_{2(j+1)} = r_{1(j+1)} - 1$. To prove this claim, suppose there were. If $r_{2(j+1)} = r_{2j}$, then we are done by our hypothesis; if $r_{2(j+1)} = r_{2j}$ then $r_{1(j+1)} = r_{1j} + 2$, contradicting that $r_{1(j+1)} - r_{1j} = 0$ or 1. Since $r_{2(j+1)} - r_{2j} = 0$ or 1, we have a contradiction either way. This establishes the claim. Now, we know that $r_{1j} = r_{2j}$ for j sufficiently positive and $r_{1j} = r_{2j} + 1$ for j sufficiently negative, so there must be some t such that $r_{1j} = r_{2j} + 1$ for $j \leqslant t$ and $r_{1j} = r_{2j}$ for $j \geqslant t + 1$. Then S_2 can follow S_1. $\qquad\square$

It is immediate to extend this result to a sequence of juggling states. Let G be the group of juggling functions introduced in §3.1 and let $G^{\mathrm{av}=k}$ be those juggling functions with ball number k. For any f in $G^{\mathrm{av}=k}$, let $\mathcal{J}(f) = (J_1, J_2, \ldots, J_n)$ be the corresponding (k, n)-sequence of juggling states. Define an $\infty \times \infty$ matrix by $r_{ij} = R_{(j-i+1)}(J_i \cup -\mathbb{N})$. Then, applying Lemma 3.9 to each pair of rows of (r_{ij}) gives the following corollary.

COROLLARY 3.10. *The above construction gives a bijection between $G^{\mathrm{av}=k}$ and $\infty \times \infty$ matrices such that:*

(C1) *for each i, there is an m_i such that $r_{ij} = j - i + 1$ for all $j \leqslant m_i$;*

(C2) *for each i, there is an n_i such that $r_{ij} = k$ for all $j \geqslant n_i$;*

(C3) *$r_{ij} - r_{(i+1)j} \in \{0, 1\}$ and $r_{ij} - r_{i(j-1)} \in \{0, 1\}$ for all $i, j \in \mathbb{Z}$; and*

(C4) *if $r_{(i+1)(j-1)} = r_{(i+1)j} = r_{i(j-1)}$ then $r_{ij} = r_{(i+1)(j-1)}$.*

Under this bijection, $r_{ij} = r_{(i+1)j} = r_{i(j-1)} > r_{(i+1)(j-1)}$ if and only if $f(i) = j$.

PROPOSITION 3.11. *Let $f, g \in \tilde{S}_n^k \subset G^{\mathrm{av}=k}$ and let r and s be the corresponding matrices. Then $f \leqslant g$ (in Bruhat order) if and only if $r_{ij} \geqslant s_{ij}$ for all $(i, j) \in \mathbb{Z}^2$.*

Proof. See [BB05, Theorem 8.3.1]. $\qquad\square$

When $r_{ij} = r_{(i+1)j} = r_{i(j-1)} > r_{(i+1)(j-1)}$, we say that (i, j) is a **special entry** of r.
An easy check shows the following.

COROLLARY 3.12. *Under the above bijection, $\mathrm{Bound}(k, n)$ corresponds to $\infty \times \infty$ matrices such that:*

(C1′) *$r_{ij} = j - i + 1$ for all $j < i$;*

(C2′) *$r_{ij} = k$ for all $j \geqslant i + n - 1$;*

(C3) *$r_{ij} - r_{(i+1)j} \in \{0, 1\}$ and $r_{ij} - r_{i(j-1)} \in \{0, 1\}$ for all $i, j \in \mathbb{Z}$;*

(C4) *if $r_{(i+1)(j-1)} = r_{(i+1)j} = r_{i(j-1)}$ then $r_{ij} = r_{(i+1)(j-1)}$; and*

(C5) *$r_{(i+n)(j+n)} = r_{ij}$.*

We call a matrix (r_{ij}) as in Corollary 3.12 a **cyclic rank matrix**. (See [Fu92] for the definition of a rank matrix, which we are mimicking.) We now specialize Proposition 3.11 to the case of $\mathrm{Bound}(k, n)$: define a partial order $\leqslant$ on $\mathrm{Jugg}(k, n)$ by

$$(J_1, \ldots, J_n) \leqslant (J_1', \ldots, J_n') \quad \text{if and only if } J_i \leqslant J_i' \text{ for each } i.$$

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COROLLARY 3.13. *The map $f \mapsto \mathcal{J}(f)$ is an isomorphism of posets from bounded affine permutations* $(\mathrm{Bound}(k, n), \leqslant)$ *to* (k, n)*-sequences of juggling states* $(\mathrm{Jugg}(k, n), \leqslant)$.

Proof. One simply checks that the condition $r_{ij} \geqslant r'_{ij}$ for all j is equivalent to $J_i \leqslant J'_i$. □

Example 3.14. Let $n = 4$ and $k = 2$. Consider the affine permutation $[\cdots 2358 \cdots]$, last seen in Figure 1. Its siteswap is 4112, and the corresponding sequence of juggling states is $(14, 13, 12, 12)$. Below we list a section of the corresponding infinite permutation matrix and cyclic rank matrix. Namely, we display the entries (i, j) for $1 \leqslant i \leqslant 4$ and $i \leqslant j \leqslant i + 4$. The special entries have been underlined

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & & & \\
 & 0 & 1 & 0 & 0 & 0 & & \\
 & & 0 & 0 & 1 & 0 & 0 & \\
 & & & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\qquad
\begin{pmatrix}
1 & \underline{1} & 1 & 2 & 2 & & & \\
 & 1 & \underline{1} & 2 & 2 & 2 & & \\
 & & 1 & 2 & \underline{2} & 2 & 2 & \\
 & & & 1 & 2 & 2 & 2 & \underline{2}
\end{pmatrix}.
$$

In § 1.2 we associated an affine permutation to each $k \times n$ matrix M of rank k; a modification of that rule gives instead a (k, n)-sequence of juggling states. Call a column of M **pivotal** if it is linearly independent of the columns to its left. (If one performs Gaussian elimination on M, a column will be pivotal exactly if it contains a 'pivot' of the resulting reduced row-echelon form.) There will be k pivotal columns, giving a k-element subset of $\{1, \ldots, n\}$; they form the lex-first basis made of columns from M.

Now rotate the first column of M to the end. What happens to the set of pivotal columns? Any column after the first that was pivotal still is pivotal, but (unless the first column was all zeroes) there is a new pivotal column; the new state can follow the previous state. The n cyclic rotations of M thus give a (k, n)-sequence of juggling states.

3.4 From $\mathcal{Q}(k, n)$ to $\mathrm{Bound}(k, n)$

The symmetric group S_n acts on $\mathbb{Z}^n$ (on the left) by

$$
w \cdot (\omega_1, \ldots, \omega_n) = (\omega_{w^{-1}(1)}, \ldots \omega_{w^{-1}(n)}). \tag{1}
$$

If $w \in S_n$ and $t_\lambda, t_{\lambda'} \in \tilde{S}_n$ are translation elements, we have the following relations in $\tilde{S}_n$:

$$
w t_\lambda w^{-1} = t_{w \cdot \lambda} \quad t_\lambda t_{\lambda'} = t_{\lambda + \lambda'}. \tag{2}
$$

Let $\omega_k = (1, \ldots, 1, 0, \ldots, 0)$, with k occurrences of 1, be the kth fundamental weight of $\mathrm{GL}(n)$. Note that $t_{\omega_k} \in \tilde{S}_n^k$. Now fix $\langle u, w \rangle \in \mathcal{Q}(k, n)$, the set of equivalence classes we defined in § 2.3. Define an affine permutation $f_{u,w} \in \tilde{S}_n^k$ by

$$
f_{u,w} = u t_{\omega_k} w^{-1}.
$$

The element $f_{u,w}$ does not depend on the representative $[u, w]_k$ of $\langle u, w \rangle$: if $u' = uz$ and $w' = wz$ for $z \in S_k \times S_{n-k}$ then

$$
u' t_{\omega_k} (w')^{-1} = u z t_{\omega_k} z^{-1} w^{-1} = u t_{z \cdot \omega_k} w^{-1} = u t_{\omega_k} w^{-1}
$$

since z stabilizes ω_k.

PROPOSITION 3.15. *The map* $\langle u, w \rangle \mapsto f_{u,w}$ *is a bijection from* $\mathcal{Q}(k, n)$ *to* $\mathrm{Bound}(k, n)$.

Proof. We first show that $\langle u, w \rangle \mapsto f_{u,w}$ is an injection into $\tilde{S}_n^k$. Suppose that $f_{u,w} = f_{u',w'}$. It is clear from the factorization $\tilde{S}_n \simeq S_n \ltimes \mathbb{Z}^n$ that there is some $z \in S_k \times S_{n-k}$ such that $u = u'z$ and $w = w'z$. Using Proposition 2.4, we have $[u, w] \sim [u', w']$.

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We now show that for $\langle u, w \rangle \in \mathcal{Q}(k, n)$, we have $f_{u,w} \in \mathrm{Bound}(k, n)$. Let $i \in [1, n]$ and $a = w^{-1}(i)$. Then

$$f_{u,w}(i) = \begin{cases} u(a) & \text{if } a > k \\ u(a) + n & \text{if } a \leqslant k. \end{cases}$$

The boundedness of $f_{u,w}$ now follows from Theorem 2.1(i).

Conversely, if $f \in \mathrm{Bound}(k, n)$ then it is clear from (2) that f has a factorization as

$$f = u t_\omega w^{-1}$$

for $u, w \in S_n$ and $\omega \in \{0, 1\}^n$. Since $f \in \tilde{S}_n^k$, the vector ω has k occurrences of 1. By changing u and w, we may further assume that $\omega = \omega_k$ and $w \in S_{n,k}^{\min}$. It remains to check that $u \leqslant_k w$, which we do via Theorem 2.1; its condition (ii) is vacuous when $w \in S_{n,k}^{\min}$ and checking condition (i) is the same calculation as in the previous paragraph. $\qquad \square$

THEOREM 3.16. *The bijection $\langle u, w \rangle \mapsto f_{u,w}$ is a poset isomorphism from the pairs $(\mathcal{Q}(k, n), \leqslant)$ to bounded affine permutations $(\mathrm{Bound}(k, n), \leqslant)$. Furthermore, one has $\ell(f_{u,w}) = \binom{n}{k} - \ell(w) + \ell(u)$.*

Proof. It is shown in [Wil07] that $(\mathcal{Q}(k, n), \leqslant)$ is a graded poset, with rank function given by $\rho(\langle u, w \rangle) = k(n - k) - (\ell(w) - \ell(u))$. It follows that each cover in $\mathcal{Q}(k, n)$ is of the form

(i) $\langle u, w' \rangle \gtrdot \langle u, w \rangle$ where $w' \lessdot w$; or

(ii) $\langle u', w \rangle \gtrdot \langle u, w \rangle$ where $u \lessdot u'$.

We may assume that $w \in S_{n,k}^{\min}$ (Proposition 2.3). Suppose we are in case (i). Then $w' = w(ab)$ where $a \leqslant k < b$ and $w(a) > w(b)$. Here $(ab) \in S_n$ denotes the transposition swapping a and b. Thus $f_{u,w'} = f_{u,w}(w(a)w(b))$. Using the formula in the proof of Proposition 3.15, we see that $f_{u,w}(w(a)) > n$ while $f_{u,w}(w(b)) \leqslant n$. Thus $f_{u,w'} > f_{u,w}$.

Suppose we are in case (ii), and that $u' = u(ab)$ where $a < b$ and $u(a) < u(b)$. It follows that $f_{u,w'} = (u(a)u(b))f_{u,w}$. Suppose first that $a \leqslant k < b$. Then $(t_{\omega_k})^{-1}(a) = a - n$, while $(t_\omega)^{-1}(b) = b$ so we also have $f_{u,w'} = f_{u,w}((w(a) - n)w(b))$ where $w(a) - n$ is clearly less than $w(b)$. Thus $f_{u,w'} > f_{u,w}$. Otherwise suppose that $a, b \geqslant k$ (the case $a, b \leqslant k$ is similar). Then $f_{u,w'} = f_{u,w}(w(a)w(b))$. Since $w \in S_{n,k}^{\min}$, we have $w(a) < w(b)$. Again we have $f_{u,w'} > f_{u,w}$.

We have shown that $\langle u', w' \rangle \geqslant \langle u, w \rangle$ implies $f_{u',w'} \geqslant f_{u,w}$. The converse direction is similar.

The last statement follows easily, using the fact that both of the posets $(\mathcal{Q}(k, n), \leqslant)$ and $(\mathrm{Bound}(k, n), \leqslant)$ are graded. $\qquad \square$

3.5 Shellability of $\mathcal{Q}(k, n)$

A graded poset P is **Eulerian** if for any $x \leqslant y \in P$ such that the interval $[x, y]$ is finite we have $\mu(x, y) = (-1)^{\mathrm{rank}(x) - \mathrm{rank}(y)}$, where μ denotes the Möbius function of P. A labeling of the Hasse diagram of a poset P by some totally ordered set Λ is called an **EL-labeling** if for any $x \leqslant y \in P$:

(i) there is a unique label-(strictly)increasing saturated chain C from x to y;

(ii) the sequence of labels in C is Λ-lexicographically minimal amongst the labels of saturated chains from x to y.

If P has an EL-labeling then we say that P is EL-*shellable*.

Verma [Ver71] has shown that the Bruhat order of a Coxeter group is Eulerian. Dyer [Dye93, Proposition 4.3] showed the stronger result that every Bruhat order (and also its dual) is EL-shellable. (See also [BW82].) Since these properties are preserved under taking convex subsets,

Lemma 3.6, Corollary 3.13 and Theorem 3.16 imply the following result, proved for the dual of $(\mathcal{Q}(k, n), \leqslant)$ by Williams [Wil07].

COROLLARY 3.17. *The posets* $(\mathcal{Q}(k, n), \leqslant)$, $(\mathrm{Bound}(k, n), \leqslant)$, *and* $(\mathrm{Jugg}(k, n), \leqslant)$, *and their duals are Eulerian and EL-shellable.*

Remark 3.18. Williams' result is stronger than Corollary 3.17: in our language, she shows that the poset $\widehat{\mathcal{Q}(k, n)}$, formed by adding a formal maximal element $\hat{1}$ to $\mathcal{Q}(k, n)$, is shellable.

3.6 Positroids

A **matroid** $\mathcal{M}$ **on** $[n]$ **with rank** k is a non-empty collection of k-element subsets of $[n]$, called **bases**, satisfying the unique minimum axiom: for any permutation $w \in S_n$, there is a unique minimal element of $w \cdot \mathcal{M}$, in the partial order $\leqslant$ on $\binom{[n]}{k}$. This is only one of many equivalent definitions of a matroid; see [Bry86] for a compendium of many others, in which this one appears as Axiom $B2^{(6)}$.

Let $\mathcal{M}$ be a matroid of rank k on $[n]$. Define a sequence of k-element subsets $\mathcal{J}(\mathcal{M}) = (J_1, J_2, \ldots, J_n)$ by letting J_r be the minimal base of $\chi^{-r+1}(\mathcal{M})$, which is well defined by assumption. Postnikov proved the following lemmas, in the terminology of Grassmann necklaces.

LEMMA 3.19 [Pos05b, Lemma 16.3]. *For a matroid* $\mathcal{M}$, *the sequence* $\mathcal{J}(\mathcal{M})$ *is a* (k, n)-*sequence of juggling states.*

Let $\mathcal{J} = (J_1, J_2, \ldots, J_n) \in \mathrm{Jugg}(k, n)$. Define

$$\mathcal{M}_{\mathcal{J}} = \left\{ I \in \binom{[n]}{k} \,\middle|\, \chi^{-r+1}(I) \geqslant J_r \right\}.$$

LEMMA 3.20 [Pos05b, Oh11]. *Let* $\mathcal{J} \in \mathrm{Jugg}(k, n)$. *Then* $\mathcal{M}_{\mathcal{J}}$ *is a matroid and* $\mathcal{J}(\mathcal{M}_{\mathcal{J}}) = \mathcal{J}$.

The matroids $\mathcal{M}_{\mathcal{J}}$ are called **positroids**.

PROPOSITION 3.21. *The maps* $\mathcal{J} \mapsto \mathcal{M}_{\mathcal{J}}$ *and* $\mathcal{M} \mapsto \mathcal{J}(\mathcal{M})$ *are inverse isomorphisms between the poset* $\mathrm{Jugg}(k, n)$ *and the poset of positroids;* $\mathcal{J}(\mathcal{M}_1) \leqslant \mathcal{J}(\mathcal{M}_2)$ *if and only if* $\mathcal{M}_1 \supseteq \mathcal{M}_2$.

Proof. The composition $\mathcal{J} \mapsto \mathcal{M}_{\mathcal{J}} \mapsto \mathcal{J}(\mathcal{M}_{\mathcal{J}})$ is the identity by the above lemma and, since the set of positroids is defined as those matroids of the form $\mathcal{M}_{\mathcal{J}}$, the compositions are inverse in the other order as well.

It is easy to see from the definitions, that $M_1 \supseteq M_2$ implies $\mathcal{J}(M_1) \leqslant \mathcal{J}(M_2)$ and that $\mathcal{J}_1 \leqslant \mathcal{J}_2$ implies $\mathcal{M}_{\mathcal{J}_1} \supseteq \mathcal{M}_{\mathcal{J}_2}$. Since these correspondences are inverse, then $\mathcal{J}_1 \leqslant \mathcal{J}_2$ if and only if $\mathcal{M}_{\mathcal{J}_1} \supseteq \mathcal{M}_{\mathcal{J}_2}$. $\square$

If $\mathcal{M}$ is an arbitrary matroid, then we call the positroid $\mathcal{M}_{\mathcal{J}(\mathcal{M})}$ the **positroid envelope** of $\mathcal{M}$ (see the discussion before Remark 1.1). Every positroid is a matroid. The positroid envelope of a positroid is itself.

Example 3.22. Let M_1 and M_2 be the matroids $\{12, 13, 14, 23, 24, 34\}$ and $\{12, 23, 34, 14\}$. In both cases, $\mathcal{J}(M_i)$ is $(12, 12, 12, 12)$ and, thus, M_1 is the positroid envelope of both M_1 and M_2. The corresponding affine permutation is $[\cdots 3456 \cdots]$. On the other hand, if $M_3 = \{12, 13, 14, 23, 24\}$, then $\mathcal{J}(M_3) = \{12, 12, 13, 12\}$, with corresponding affine permutation $[\cdots 3546 \cdots]$.

Remark 3.23. Postnikov [Pos05b] studied the totally nonnegative part $\mathrm{Gr}(k, n)_{\geqslant 0}$ of the Grassmannian. Each point $V \in \mathrm{Gr}(k, n)_{\geqslant 0}$ has an associated matroid $\mathcal{M}_V$. Postnikov showed

that the matroids that can occur this way, called positroids, were in bijection with Grassmann necklaces of type (k, n) (similar to our (k, n)-sequences of juggling states), with decorated permutations of $[n]$ with k anti-exceedances, and with many other combinatorial objects. Oh [Oh11], proving a conjecture of Postnikov, showed that positroids can be defined in the way we have done.

4. Background on Schubert and Richardson varieties

We continue to fix nonnegative integers k and n, satisfying $0 \leqslant k \leqslant n$. For S any subset of $[n]$, let $\mathrm{Project}_S : \mathbb{C}^n \twoheadrightarrow \mathbb{C}^S$ denote the projection onto the coordinates indexed by S. (So the kernel of $\mathrm{Project}_S$ is $\mathrm{Span}_{s \notin S} e_s$.)

4.1 Schubert and Richardson varieties in the flag manifold

Let $F\ell(n)$ denote the variety of flags in $\mathbb{C}^n$. For a permutation $w \in S_n$, we have the **Schubert cell**

$$\mathring{X}_w = \{G_\bullet \in F\ell(n) \mid \dim(\mathrm{Project}_{[j]}(G_i)) = \#\{w([i]) \cap [j]\} \text{ for all } i, j\}$$

and **Schubert variety**

$$X_w = \{G_\bullet \in F\ell(n) \mid \dim(\mathrm{Project}_{[j]}(G_i)) \leqslant \#\{w([i]) \cap [j]\} \text{ for all } i, j\}$$

which both have codimension $\ell(w)$; moreover $X_w = \overline{\mathring{X}_w}$. (For basic background on the combinatorics of Schubert varieties, see [Fu92] or [MS05, ch. 15].) We thus have

$$F\ell(n) = \coprod_{w \in S_n} \mathring{X}_w \quad \text{and} \quad X_w = \coprod_{v \geqslant w} \mathring{X}_v.$$

Similarly, we define the **opposite Schubert cell**

$$\mathring{X}^w = \{G_\bullet \in F\ell(n) \mid \dim(\mathrm{Project}_{[n-j+1,n]}(G_i)) = \#\{w([i]) \cap [n-j+1, n]\} \text{ for all } i, j\}$$

and **opposite Schubert variety**

$$X^w = \{G_\bullet \in F\ell(n) \mid \dim(\mathrm{Project}_{[n-j+1,n]}(G_i)) \leqslant \#\{w([i]) \cap [n-j+1, n]\} \text{ for all } i, j\}.$$

It may be easier to understand these definitions in terms of matrices. Let M be an $n \times n$ invertible matrix and let G_i be the span of the top i rows of M. Then $G_\bullet$ is in $\mathring{X}_w$ (respectively, X_w), if and only if, for all $1 \leqslant i, j \leqslant n$, the rank of the top-left $i \times j$ submatrix of M is the same as (respectively, less than or equal to) the rank of the corresponding submatrix of the permutation matrix w. Similarly, $G_\bullet$ is in $\mathring{X}^w$ (respectively X^w) if the ranks of the top right submatrices of M are equal to (respectively less than or equal to) those of w. (The permutation matrix of w has 1 in positions $(i, w(i))$ and 0 elsewhere.)

Define the **Richardson varieties** as the transverse intersections

$$X_u^w = X_u \cap X^w \quad \text{and} \quad \mathring{X}_u^w = \mathring{X}_u \cap \mathring{X}^w.$$

The varieties X_v^w and $\mathring{X}_v^w$ are nonempty if and only if $v \leqslant w$, in which case each has dimension $\ell(w) - \ell(v)$. Let $E_\bullet$ be the flag $(\mathrm{Span}(e_1), \mathrm{Span}(e_1, e_2), \ldots)$. The coordinate flag $vE_\bullet$ is in X_u^w if and only if $u \leqslant v \leqslant w$.

We will occasionally need to define **Schubert cells and varieties with respect to a flag** $F_\bullet$. We set

$$\mathring{X}_w(F_\bullet) = \{G_\bullet \in F\ell(n) \mid \dim(G_i/(G_i \cap F_{n-j})) = \#\{w([i]) \cap [j]\} \text{ for all } i, j\}$$

and define $X_w(F_\bullet)$ by replacing $=$ with $\leqslant$. Warning: under this definition X_w is $X_w(w_0 E_\bullet)$.

4.2 Schubert varieties in the Grassmannian

Let $\mathrm{Gr}(k,n)$ denote the Grassmannian of k-planes in $\mathbb{C}^n$, and let $\pi : F\ell(n) \to \mathrm{Gr}(k,n)$ denote the natural projection. For $I \in \binom{[n]}{k}$, we let

$$\mathring{X}_I = \{V \in \mathrm{Gr}(k,n) \mid \dim \mathrm{Project}_{[j]}(V)) = \#(I \cap [j])\}$$

denote the Schubert cell labeled by I and

$$X_I = \{V \in \mathrm{Gr}(k,n) \mid \dim \mathrm{Project}_{[j]}(V) \leqslant \#(I \cap [j])\}$$

the Schubert variety labeled by I.

Thus we have $\pi(X_w) = X_{\pi(w)}$ and

$$\mathrm{Gr}(k,n) = \coprod_{I \in \binom{[n]}{k}} \mathring{X}_I \quad \text{and} \quad X_J = \coprod_{I \geqslant J} \mathring{X}_I.$$

We define

$$\mathring{X}^I = \{V \in \mathrm{Gr}(k,n) \mid \dim \mathrm{Project}_{[n-j+1,n]}(V) = \#(I \cap [n-j+1,n])\},$$
$$X^I = \{V \in \mathrm{Gr}(k,n) \mid \dim \mathrm{Project}_{[n-j+1,n]}(V) \leqslant \#(I \cap [n-j+1,n])\}.$$

So, for $J \in \binom{[n]}{k}$, the k-plane $\mathrm{Span}_{j \in J} e_j$ lies in X_I if and only if $I \leqslant J$, and lies in X^K if and only if $J \leqslant K$.

To review: if u and w lie in S_n, then X_u is a Schubert variety, X^w an opposite Schubert and X_u^w a Richardson variety in $F\ell(n)$. If I and J lie in $\binom{[n]}{k}$, then X_I, X^J and X_I^J mean the similarly named objects in $\mathrm{Gr}(k,n)$. (Note that permutations have lower case letters from the end of the alphabet while subsets have upper case letters chosen from the range $\{I, J, K\}$.) The symbol $\mathring{X}$ would indicate that we are dealing with an open subvariety, in any of these cases.

5. Positroid varieties

We now introduce the positroid varieties, our principal objects of study. Like the Schubert and Richardson varieties, they will come in open versions, denoted $\mathring{\Pi}$, and closed versions, denoted Π.[11] The positroid varieties will be subvarieties of $\mathrm{Gr}(k,n)$, indexed by the various posets introduced in §3. For each of the different ways of viewing our posets, there is a corresponding way to view positroid varieties. The main result of this section will be that all of these ways coincide. Again, we sketch these results here and leave the precise definitions until later.

Given $[u,w]_k$, representing an equivalence class in $\mathcal{Q}(k,n)$, we can project the Richardson variety $\mathring{X}_u^w$ (respectively X_u^w) to $\mathrm{Gr}(k,n)$. Given a (k,n)-sequence of juggling states $(J_1, J_2, \ldots, J_n) \in \mathrm{Jugg}(k,n)$, we can take the intersection $\bigcap \chi^{i-1} \mathring{X}_{J_i}$ (respectively $\bigcap \chi^{i-1} X_{J_i}$) in $\mathrm{Gr}(k,n)$. (Recall χ is the cyclic shift $[234 \ldots n1]$.) Given a cyclic rank matrix r, we can consider the image in $\mathrm{Gr}(k,n)$ of the space of $k \times n$ matrices such that the submatrices made of cyclically consecutive columns have ranks equal to (respectively, less than or equal to) the entries of r. Given a positroid M, we can consider those points in $\mathrm{Gr}(k,n)$ whose matroid has positroid envelope equal to (respectively, contained in) M.

THEOREM 5.1. *Choose our $[u,w]_k$, $(J_1, \ldots, J_n)$, r and M to correspond by the bijections in §3. Then the projected open Richardson variety, the intersection of cyclically permuted open*

---

[11] Note that Π stands for 'positroid', 'Postnikov', and 'projected Richardson'.

Schubert varieties, the space of matrices obeying the rank conditions, and the space of matrices whose matroids have the required positroid envelope, will all coincide as subsets of $\mathrm{Gr}(k,n)$.

The equalities of the last three spaces is essentially a matter of unwinding definitions. The equality between the projected open Richardson variety, and the space of matrices obeying the cyclic rank conditions, is non-trivial and is checked in Proposition 5.4.

We call the varieties we obtain in this way **open positroid varieties** or **positroid varieties** respectively, and denote them by $\mathring{\Pi}$ or Π with a subscript corresponding to any of the possible combinatorial indexing sets.

The astute reader will note that we did not describe how to define a positroid variety using a bounded affine permutation (except by translating it into some other combinatorial data). We hope to address this in future work using the geometry of the affine flag manifold. The significance of bounded affine permutations can already be seen in this paper, as it is central in our description in §7 of the cohomology class of Π.

5.1 Cyclic rank matrices

Recall the definition of a cyclic rank matrix from the end of §3.1. As we explained there, cyclic rank matrices of type (k,n) are in bijection with $\mathrm{Bound}(k,n)$ and hence with $\mathcal{Q}(k,n)$ and with bounded juggling patterns of type (k,n).

Let $V \in \mathrm{Gr}(k,n)$. We define an infinite array $r_{\bullet\bullet}(V) = (r_{ij}(V))_{i,j\in\mathbb{Z}}$ of integers as follows: for $i > j$, we set $r_{ij}(V) = j - i + 1$ and for $i \leqslant j$ we have

$$r_{ij}(V) = \dim(\mathrm{Project}_{\{i,i+1,\ldots,j\}}(V))$$

where the indices are cyclic modulo n. (So, if $n = 5$, $i = 4$ and $j = 6$, we are projecting onto $\mathrm{Span}(e_4, e_5, e_1)$.) Note that, when $j \geqslant i + n - 1$, we project onto all of $[n]$. If V is the row span of a $k \times n$ matrix M, then $r_{ij}(V)$ is the rank of the submatrix of M consisting of columns i, $i+1, \ldots, j$.

5.2 Positroid varieties and open positroid varieties

LEMMA 5.2. *Let* $V \in \mathrm{Gr}(k,n)$. *Then* $r_{\bullet\bullet}(V)$ *is a cyclic rank matrix of type* (k,n).

Proof. Conditions (C1′), (C2′), and (C5) are clear from the definitions. Let M be a $k \times n$ matrix whose row span is V; let M_i be the ith column of M. Condition (C3) says that adding a column to a matrix either preserves the rank of that matrix or increases it by one. The hypotheses of condition (C4) state that M_i and M_j are in the span of $M_{i+1}, M_{i+2}, \ldots, M_{j-1}$; the conclusion is that $\dim \mathrm{Span}(M_i, M_{i+1}, \ldots, M_{j-1}, M_j) = \dim \mathrm{Span}(M_{i+1}, \ldots, M_{j-1})$. □

For any cyclic rank matrix r, let $\mathring{\Pi}_r$ be the subset of $\mathrm{Gr}(k,n)$ consisting of those k-planes V with cyclic rank matrix r. We may also write $\mathring{\Pi}_f$, $\mathring{\Pi}_{\mathcal{J}}$ or $\mathring{\Pi}_u^w$ where f is the bounded affine permutation, $\mathcal{J}$ the juggling pattern or $\langle u, w \rangle$ the equivalence class of k-Bruhat interval corresponding to r.

The next result follows directly from the definitions. Recall that $\chi = [23\cdots(n-1)n1] \in S_n$ denotes the long cycle.

LEMMA 5.3. *For any* $\mathcal{J} = (J_1, J_2, \ldots, J_n) \in \mathrm{Jugg}(k,n)$, *we have*

$$\mathring{\Pi}_{\mathcal{J}} = \mathring{X}_{J_1} \cap \chi(\mathring{X}_{J_2}) \cap \cdots \cap \chi^{n-1}(\mathring{X}_{J_n}).$$

By Lemma 5.2 and our combinatorial bijections, we have

$$\mathrm{Gr}(k, n) = \coprod_{\mathcal{J} \in \mathrm{Jugg}(k,n)} \mathring{\Pi}_{\mathcal{J}}.$$

We call the sets $\mathring{\Pi}_{\mathcal{J}}$ **open positroid varieties**. Postnikov [Pos05b] showed that $\mathring{\Pi}_{\mathcal{J}}$ (and even $(\mathring{\Pi}_{\mathcal{J}})_{\geqslant 0}$) is nonempty if $\mathcal{J} \in \mathrm{Jugg}(k, n)$ (this statement also follows from Proposition 5.4 below). We define the **positroid varieties** $\Pi_{\mathcal{J}}$ to be the closures $\Pi_{\mathcal{J}} := \overline{\mathring{\Pi}_{\mathcal{J}}}$.

5.3 From $\mathcal{Q}(k, n)$ to cyclic rank matrices

We now describe a stratification of the Grassmannian due to Lusztig [Lus98], and further studied by Rietsch [Rie06]. (This stratification was also independently discovered by Brown *et al.* [BGY06, GY09], motivated by ideas from Poisson geometry; we will not discuss the Poisson perspective further in this paper.) Lusztig and Rietsch's work applies to any partial flag variety, and we specialize their results to the Grassmannian.

The main result of this section is the following proposition.

PROPOSITION 5.4. *Let $u \leqslant_k w$, and $f_{u,w}$ be the corresponding affine permutation from § 3.4. Recall that $\mathring{X}_u^w$ denotes the open Richardson variety in $F\ell(n)$ and π the map $F\ell(n) \to \mathrm{Gr}(k, n)$. Then $\mathring{\Pi}_f = \pi(\mathring{X}_u^w)$.*

Remark 5.5. If $u \leqslant w$, but $u \not\leqslant_k w$, then $\pi(\mathring{X}_u^w)$ may not be of the form $\mathring{\Pi}_f$. See [KLS, Remark 3.5].

The projection $\pi(\mathring{X}_u^w)$ depends only on the equivalence class of $[u, w]_k$ in $\mathcal{Q}(k, n)$.

PROPOSITION 5.6 [KLS, Lemma 3.1]. *Suppose $[u, w]_k \sim [u', w']_k$ in $\mathcal{Q}(k, n)$. Then $\pi(\mathring{X}_u^w) = \pi(\mathring{X}_{u'}^{w'})$.*

We now introduce a piece of notation which will be crucial in the proof of Proposition 5.4, but will then never appear again. Let $V \in \mathrm{Gr}(k, n)$. Given a flag $F_\bullet$ in $\mathbb{C}^n$, we obtain another flag $F_\bullet(V)$ containing V as the kth subspace, as follows. Take the sequence $F_0 \cap V, F_1 \cap V, \ldots, F_n \cap V$ and remove repetitions to obtain a partial flag $F_\bullet \cap V$ inside $\mathbb{C}^n$, with dimensions $1, 2, \ldots k$. Next take the sequence $V + F_0, V + F_1, V + F_2, \ldots, V + F_n$ and remove repetitions to obtain a partial flag $F_\bullet + V$ inside $\mathbb{C}^n$ of dimensions $k, k + 1, \ldots, n$. Concatenating $F_\bullet \cap V$ and $F_\bullet + V$ gives us a flag $F_\bullet(V)$ in $\mathbb{C}^n$. The flag $F_\bullet(V)$ is the 'closest' flag to $F_\bullet$ which contains V as the kth subspace. This notion of 'closest flag' is related to the notion of 'closest Borel subgroup' in [Rie06, § 5], and many of our arguments are patterned on arguments of [Rie06].

LEMMA 5.7. *Let w be a Grassmannian permutation. Let $F_\bullet$ be a complete flag and let $V = F_k$. Then $F_\bullet \in \mathring{X}^w$ if and only if:*

(i) $V \in \mathring{X}^{\sigma(w)}$*; and*

(ii) $F_\bullet = E_\bullet(V)$.

Proof. The flag $F_\bullet$ is in $\mathring{X}^w$ if and only if, for every i and j,

$$\dim(F_i \cap E_j) = \#(w([i]) \cap [j]) \quad \text{or, equivalently,} \quad \dim(F_i + E_j) = (i + j) - \#(w([i]) \cap [j]).$$

When $i = k$, the equation above is precisely the condition that $V \in \mathring{X}^{\sigma(w)}$. Therefore, when proving either direction of the equivalence, we may assume that $V \in \mathring{X}^{\sigma(w)}$.

Since $V \in \mathring{X}^{\sigma(w)}$,

$$E_\bullet(V) = (E_{w(1)} \cap V, E_{w(2)} \cap V, \ldots, E_{w(k)} \cap V, E_{w(k+1)} + V, E_{w(k+2)} + V, \ldots, E_{w(n)} + V).$$

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Let $i \leqslant k$. If $F_\bullet \in \mathring{X}^w$, then $\dim(F_i \cap E_{w(i)}) = \#(w([i]) \cap [w(i)]) = i$, where we have used that w is Grassmannian. However, $F_i \cap E_{w(i)} \subseteq V \cap E_{w(i)}$, which also has dimension i because $V \in \mathring{X}^{\sigma(w)}$. So $F_\bullet \in \mathring{X}^w$ implies that $F_i = V \cap E_{w(i)}$. Similarly, for $i > k$, the equation $\dim(F_i + E_{w(i)}) = i + w(i) - \#(w([i]) \cap [w(i)])$ implies that $F_i = E_{w(i)} + V$. So, if $F_\bullet \in \mathring{X}^x$ then $F_\bullet = E_\bullet(V)$. The argument is easily reversed. $\qquad\square$

Lemma 5.8. *Let w be a Grassmannian permutation, with $u \leqslant w$ and let $V \in \mathrm{Gr}(k, n)$. Then $V \in \pi(\mathring{X}_u^w)$ if and only if*

$$E_\bullet(V)_i = V \cap E_{w(i)} \quad \text{for } 1 \leqslant i \leqslant k \qquad (3)$$

$$E_\bullet(V)_i = V + E_{w(i)} \quad \text{for } k < i \leqslant n \qquad (4)$$

and

$$E_\bullet(V) \in \mathring{X}_u. \qquad (5)$$

Proof. By definition, $V \in \pi(\mathring{X}_u^w)$ if and only if there is a flag $F_\bullet$ with $V = F_k$ and $F_\bullet \in \mathring{X}_u \cap \mathring{X}^w$. By Lemma 5.7, this flag $F_\bullet$, should it exist, must be $E_\bullet(V)$. By Lemma 5.7, $E_\bullet(V)$ lies in $\mathring{X}^w$ if and only if V lies in $\mathring{X}^{\sigma(w)}$. So $E_\bullet(V) \in \mathring{X}_u \cap \mathring{X}^w$ if and only if $V \in \mathring{X}^{\sigma(w)}$ and $E_\bullet(V) \in \mathring{X}_u$. Now, conditions (3) and (4) determine the dimension of $V \cap E_{w(i)}$ for all i. They are precisely the condition on $\dim(V \cap w(j))$ occurring in the definition of $\mathring{X}^{\sigma(w)}$. So conditions (3)–(5) are equivalent to the condition that $V \in \mathring{X}^{\sigma(w)}$ and $E_\bullet(V) \in \mathring{X}_u$. $\qquad\square$

Proof of Proposition 5.4. First, note that by Proposition 5.6, and the observation that $f_{u,w}$ only depends on the equivalence class $\langle u, w \rangle$ in $\mathcal{Q}(k, n)$, we may replace (u, w) by any equivalent pair in $\mathcal{Q}(k, n)$. We may thus assume that w is Grassmannian (Proposition 2.3). By Lemma 5.8, $V \in \pi(\mathring{X}_u^w)$ if and only if conditions (3)–(5) hold.

Suppose that $V \in \pi(\mathring{X}_u^w)$. Let $r = r_{\bullet\bullet}(V)$. Let $a \in \mathbb{Z}$ and let $b = f_{u,w}(a)$; without loss of generality we may assume that $1 \leqslant a \leqslant n$. Set $i = w^{-1}(a)$. We now check that (a, b) is a special entry of r.

Case 1: $i \notin [k]$. In this case $f_{u,w}(a) = u t_{\omega_k} w^{-1}(a) \in [n]$. Since $f_{u,w} \in \mathrm{Bound}(k, n)$, we deduce that $a \leqslant b \leqslant n$. (Occasionally, our notation will implicitly assume $a < b$, we leave it to the reader to check the boundary case.) By conditions (4) and (5),

$$\dim \mathrm{Project}_{[b]}(V + E_a) = \#(u([i]) \cap [b]).$$

We can rewrite this as

$$\dim(V + E_a + w_0 E_{n-b}) = (n - b) + \#(u([i]) \cap [b])$$

or, again,

$$\dim \mathrm{Project}_{[a+1,b]}(V) = \#(u([i]) \cap [b]) - a.$$

(We have used $a \leqslant b \leqslant n$ to make sure that $\dim E_a + w_0 E_{n-b} = n - b + a$.) In conclusion,

$$r_{(a+1)b} = \#(u([i]) \cap [b]) - a.$$

A similar computation gives us

$$r_{(a+1)(b-1)} = \#(u([i]) \cap [b - 1]) - a.$$

We now wish to compute r_{ab} and $r_{a(b-1)}$. This time, we have $V + E_{a-1} = E_\bullet(V)_{i-1}$. So we deduce from condition (4) that

$$r_{ab} = \#(u([i - 1]) \cap [b]) - (a - 1)$$

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and
$$r_{a(b-1)} = \#(u([i-1]) \cap [b-1]) - (a-1).$$

Now, $u(i) = u(w^{-1}(a)) = b$. So, $r_{(a+1)b} - r_{(a+1)(b-1)} = 1$ and, since $b \notin u([i-1])$, we also have $r_{ab} - r_{a(b-1)} = 0$. So (a,b) is special as claimed.

Case 2: $i \in [k]$. In this case, $b = u(i) + n$ and $n+1 \leqslant b \leqslant a+n$. We mimic the previous argument, using $V \cap E_a \cap w_0 E_{2n-b}$ in place of $V + E_a + w_0 E_{n-b}$; the conclusion again is that (a,b) is a special entry of r.

We have now checked, in both cases, that (a,b) is a special entry of r. Therefore, the affine permutation g associated to r has $g(a) = b$. Since $f_{u,w}(a) = b$, we have checked that $f_{u,w} = g$. We have thus shown that, if $V \in \pi(\mathring{X}^w_u)$, then $V \in \mathring{\Pi}_{f_{u,w}}$.

We must now prove the converse. Let $r_{\bullet\bullet}(V) = r(f_{u,w})$. Let (u',w') be such that $E_\bullet(V) \in \mathring{X}^{w'}_{u'}$, so we know that $V \in \pi(\mathring{X}^{w'}_{u'})$. By Lemma 5.7, w' is Grassmannian. So $r_{\bullet\bullet}(V) = r(f_{u',w'})$ and $f_{u',w'} = f_{u,w}$. However, by Proposition 3.15, this shows that $[u,w]_k$ and $[u',w']_k$ represent the same element of $\mathcal{Q}(k,n)$. Since w and w' are both Grassmannian, this means that $u = u'$ and $w = w'$, and $V \in \pi(\mathring{X}^w_u)$ as desired. $\square$

5.4 Positroid varieties are projected Richardson varieties

Lusztig [Lus98] exhibited a stratification $\coprod P_{(u,v,w)}$ of $\mathrm{Gr}(k,n)$ indexed by triples $(u,v,w) \in S_{n,k}^{\max} \times (S_k \times S_{n-k}) \times S_{n,k}^{\min}$ satisfying $u \leqslant wv$, and showed that his strata satisfy

$$P_{(u,v,w)} = \pi(\mathring{X}^{wv}_u) = \pi(\mathring{X}^w_{uv^{-1}}).$$

Furthermore, the projection $\pi : F\ell(n) \to \mathrm{Gr}(k,n)$ restricts to an isomorphism on $\mathring{X}^{wv}_u$. Using the bijection between the triples (u,v,w) and $\mathcal{Q}(k,n)$ (see § 2.3), the following theorem thus follows from Proposition 5.4.

THEOREM 5.9. *The stratification of $\mathrm{Gr}(k,n)$ by open positroid varieties is identical to Lusztig's stratification. If $f = f_{u,w}$ corresponds to $\langle u,w \rangle$ under the bijection $\mathcal{Q}(k,n) \to \mathrm{Bound}(k,n)$ of § 3.4, then $\pi(\mathring{X}^w_u) = \mathring{\Pi}_f$. The varieties Π_f and $\mathring{\Pi}_f$ are irreducible of codimension $\ell(f)$, and $\mathring{\Pi}_f$ is smooth. For any Richardson variety X^w_u, whether or not $u \leqslant_k w$, the projection $\pi(X^w_u)$ is a closed positroid variety.*

Proof. Open Richardson varieties in the flag manifold are smooth and irreducible (by Kleiman transversality). Lusztig's strata are, by definition, the projected open Richardson varieties, which we have just showed are the same as the open positroid varieties. Lusztig shows that π restricted to $\mathring{X}^w_u$ is an isomorphism on its image, so $\dim \mathring{\Pi}_f = \dim X^w_u = \ell(w) - \ell(u)$ and $\mathring{\Pi}_f$ is irreducible. By Theorem 3.16, $\ell(w) - \ell(u) = k(n-k) - \ell(f)$, so $\mathring{\Pi}_f$ has codimension $\ell(f)$, as does its closure Π_f.

See [KLS, Proposition 3.3] for the fact that the projection of any Richardson X^w_u is equal to the projection of some $X^{w'}_{u'}$ with u' anti-Grassmannian. $\square$

For $u \leqslant_k w$, we shall call X^w_u a **Richardson model** for $\Pi_{f_{u,w}}$. We refer the reader to [KLS] for a discussion of projections of closed Richardson varieties. In particular, for any $u \leqslant w$ (not necessarily a k-Bruhat relation) there exists a bounded affine permutation f such that $\pi(X^w_u) = \Pi_f$.

Postnikov [Pos05b] parametrized the 'totally nonnegative part' of any open positroid variety, showing that it is homeomorphic to an open ball. Before one knows that positroid varieties are actually irreducible, one can use this parametrizability to show that only one component

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intersects the totally nonnegative part of the Grassmannian. *A priori* there might be other components, so it is nice to know that in fact there are not.

We now describe the containments between positroid varieties in the following theorem.

THEOREM 5.10. *Open positroid varieties form a stratification of the Grassmannian. Thus for $f \in \mathrm{Bound}(k, n)$ we have*

$$\Pi_f = \coprod_{f' \geqslant f} \mathring{\Pi}_{f'} = X_{J_1} \cap \chi(X_{J_2}) \cap \cdots \cap \chi^{n-1}(X_{J_n})$$

where $(J_1, J_2, \ldots, J_n) \in \mathrm{Jugg}(k, n)$ corresponds to f.

Proof. Rietsch [Rie06] described the closure relations of Lusztig's stratification of partial flag varieties; see also [BGY06]. The first equality is Rietsch's result, translated from the language of $\mathcal{Q}(k, n)$ to $\mathrm{Bound}(k, n)$.

We know that $X_J = \coprod_{I \geqslant J} \mathring{X}_I$. Using this to expand the intersection $X_{J_1} \cap \chi(X_{J_2}) \cap \cdots \cap \chi^{n-1}(X_{J_n})$ and applying Lemma 5.3 gives the second equality. $\square$

We note that Postnikov [Pos05b] also described the same closure relations for the totally nonnegative Grassmannian, using Grassmann necklaces and decorated permutations.

For a matroid $\mathcal{M}$ let

$$\mathrm{GGMS}(\mathcal{M}) = \{V \in \mathrm{Gr}(k, n) \mid \Delta_I(V) \neq 0 \iff I \in M\}$$

denote the GGMS stratum of the Grassmannian [GGMS87]. Here for $I \in \binom{[n]}{k}$, Δ_I denotes the Plücker coordinate labeled by the columns in I. Recall that in § 3.6, we have defined the positroid envelope of a matroid. It is easy to see that

$$\mathring{\Pi}_f = \coprod_{\mathcal{M}:\mathcal{J}(\mathcal{M})=\mathcal{J}(f)} \mathrm{GGMS}(\mathcal{M}).$$

PROPOSITION 5.11. *Let $\mathcal{M}$ be a positroid. Then $\mathrm{GGMS}(\mathcal{M})$ is dense in $\mathring{\Pi}_{\mathcal{J}(\mathcal{M})}$.*

Proof. Suppose $f \in \mathrm{Bound}(k, n)$ is such that $\mathcal{J}(f) = \mathcal{J}(\mathcal{M})$. Postnikov [Pos05b] showed that the totally nonnegative part $\mathrm{GGMS}(\mathcal{M})_{\geqslant 0}$ of $\mathrm{GGMS}(\mathcal{M})$ is a real cell of dimension $k(n-k) - \ell(f)$. Thus $\mathrm{GGMS}(\mathcal{M})$ has at least dimension $k(n-k) - \ell(f)$. By Theorem 5.9 $\mathring{\Pi}_f$ is irreducible with the same dimension. It follows that $\mathrm{GGMS}(\mathcal{M})$ is dense in $\mathring{\Pi}_{\mathcal{J}(\mathcal{M})}$. $\square$

COROLLARY 5.12. *Let $\mathcal{M}$ be a positroid. Then, as sets,*

$$\Pi_{\mathcal{J}(\mathcal{M})} = \overline{\mathrm{GGMS}(\mathcal{M})} = \{V \in \mathrm{Gr}(k, n) \mid I \notin \mathcal{M} \Rightarrow \Delta_I(V) = 0\}.$$

Proof. The first equality follows from Proposition 5.11. The second follows from Theorem 5.10 and the description of Schubert varieties by vanishing of Plücker coordinates:

$$X_J = \{V \in \mathrm{Gr}(k, n) \mid I < \sigma(J) \Rightarrow \Delta_I(V) = 0\}. \qquad \square$$

Consider the set on the right-hand side of the displayed equation in Corollary 5.12. Lauren Williams conjectured that this set was irreducible; this now follows from Corollary 5.12 and Theorem 5.9.

5.5 Geometric properties of positroid varieties

The following results follow from Theorem 5.9 and the geometric results of [KLS].

THEOREM 5.13. *Positroid varieties are normal, Cohen–Macaulay, and have rational singularities.*

THEOREM 5.14. *There is a Frobenius splitting on the Grassmannian that compatibly splits all the positroid varieties therein. Furthermore, the set of positroid varieties is exactly the set of compatibly split subvarieties of the Grassmannian.*

THEOREM 5.15. *Let $\mathcal{M}$ be a positroid. Then the ideal defining the variety $\Pi_{\mathcal{J}(\mathcal{M})}$ inside $\mathrm{Gr}(k,n)$ is generated by the Plücker coordinates $\{\Delta_I : I \notin \mathcal{M}\}$.*

Proof. By Theorem 5.10, $\Pi_{\mathcal{J}(\mathcal{M})}$ is the set-theoretic intersection of some permuted Schubert varieties. By the Frobenius splitting results of [KLS], it is also the scheme-theoretic intersection.

Hodge proved that Schubert varieties (and hence permuted Schubert varieties) are defined by the vanishing of Plücker coordinates. (See e.g. [Ram87], where a great generalization of this is proven using Frobenius splitting.) The intersection of a family of them is defined by the vanishing of all their individual coordinates.

As explained in [FZ00, Proposition 3.4], it is easy to determine which Plücker coordinates vanish on a T-invariant subscheme X of the Grassmannian; they correspond to the fixed points not lying in X. $\qquad\square$

COROLLARY 5.16. *Let M be a positroid. Embed $\mathrm{Gr}(k,n)$ into $\mathbb{P}^{\binom{n}{k}-1}$ by the Plücker embedding. Then the ideal of $\Pi_{\mathcal{J}(M)}$ in $\mathbb{P}^{\binom{n}{k}-1}$ is generated in degrees 1 and 2.*

Proof. By Theorem 5.15, the ideal of $\Pi_{\mathcal{J}(M)}$ is the sum of a linearly generated ideal and the ideal of $\mathrm{Gr}(k,n)$. It is classical that the ideal of $\mathrm{Gr}(k,n)$ is generated in degree 2. $\qquad\square$

Remark 5.17. For a subvariety $X \subseteq G/P \subseteq \mathbb{P}V$ of a general flag manifold embedded in the projectivization $\mathbb{P}V$ of an irreducible representation, one can ask whether X is defined as a set by the vanishing of extremal weight vectors in V. This is easy to show for Schubert varieties (see [FZ00]) and more generally for Richardson varieties.

Since the above corollary proves this property for positroid varieties, and [FZ00] prove it for Richardson varieties in G/B, one might conjecture that it would be true for projected Richardson varieties in other G/Ps. This is *not* the case: consider the Richardson variety X_{1324}^{4231} projecting to a divisor in the partial flag manifold $\{(V_1 \subset V_3 \subset \mathbb{C}^4)\}$. One can check that the image contains every T-fixed point, so no extremal weight vector vanishes on it.

For any irreducible T-invariant subvariety $X \subseteq G/P$, the set of T-fixed points $X^T \subseteq (G/P)^T \cong W/W_P$ forms a *Coxeter matroid* [BGW03], and X is contained in the set where the extremal weight vectors corresponding to the complement of X^T vanish. If the containment is proper, as in the above example, one may take this as evidence that the Coxeter matroid is not a good description of X. We saw a different knock against matroids in Remark 1.1.

6. Examples of positroid varieties

In this section, we will see that a number of classical objects studied in algebraic geometry are positroid varieties, or closely related to positroid varieties.

First, for any $I \in \binom{[n]}{k}$, the Schubert variety X_I in the Grassmannian is the positroid variety associated to the positroid $\{J : J \geqslant I\}$. Similarly, the cyclically permuted Schubert varieties $\chi^i \cdot X_I$ are also positroid varieties. Similarly, the Richardson varieties X_I^K are also positroid varieties, corresponding to the positroid $\{J : I \leqslant J \leqslant K\}$.

Another collection of objects, closely related to Schubert varieties, are the *graph Schubert varieties*. Let X_w be a Schubert variety in $F\ell(n)$. Considering $F\ell(n)$ as $B_- \backslash \mathrm{GL}_n$ (where B_- is the group of invertible lower triangular matrices), let X_w' be the preimage of X_w in GL_n.

The **matrix Schubert variety** MX_w, introduced in [Fu92], is the closure of X'_w in $\mathrm{Mat}_{n \times n}$. We cut MX_w out of $\mathrm{Mat}_{n \times n}$ by imposing certain rank conditions on the top-left justified submatrices (as was explained in § 4.1). Embed $\mathrm{Mat}_{n \times n}$ into $\mathrm{Gr}(n, 2n)$ by the map Γ which sends a matrix M to the graph of the linear map $\vec{v} \mapsto M\vec{v}$; its image is the **big cell** $\{\Delta_{[n]} \neq 0\}$. In coordinates, $\Gamma(M)$ is the row span of the $n \times 2n$ matrix $\begin{bmatrix} \mathrm{Id} & M \end{bmatrix}$. We will abuse notation by also calling this matrix $\Gamma(M)$. We introduce here the **graph Schubert variety**, GX_w, as the closure of $\Gamma(MX_w)$ in $\mathrm{Gr}(n, 2n)$. Graph Schubert varieties will be studied further in a separate paper by the first author (in preparation).

Let us write $M_{[1,i],[1,j]}$ for the top-left $i \times j$ submatrix of M. Then the rank of $M_{[1,i],[1,j]}$ is $n - i$ less than the rank of the submatrix of $\Gamma(M)$ using rows $\{i+1, i+2, \ldots, n, n+1, \ldots, n+j\}$. So every point of GX_w obeys certain rank bounds on the submatrices of these types. These rank bounds are precisely the rank bounds imposed by $r(f)$, where f is the affine permutation $f(i) = w(i) + n$ for $1 \leqslant i \leqslant n$, $f(i) = i + n$ for $n + 1 \leqslant i \leqslant 2n$. So GX_w is contained in Π_f, with equality on the open set $\Gamma(\mathrm{Mat}_{n \times n})$. But Π_f and GX_w are both irreducible, so this shows that $GX_w = \Pi_f$. In § 7, we will see that cohomology classes of general positroid varieties will correspond to affine Stanley symmetric functions; under this correspondence, graph Schubert varieties give the classical Stanley symmetric functions.

The example of graph Schubert varieties can be further generalized [BGY06, § 0.7]. Let u and v be two elements of S_n and consider the affine permutation $f(i) = u(i) + n$ for $1 \leqslant i \leqslant n$, $f(i) = v^{-1}(i - n) + 2n$ for $n + 1 \leqslant i \leqslant 2n$. (So our previous example was when v is the identity.) Let us look at $\Pi_f \cap \Gamma(\mathrm{Mat}_{n \times n})$. This time, we impose conditions both on the ranks of the upper left submatrices and the lower right submatrices. In fact, $\mathring{\Pi}_f$ lies entirely within $\Gamma(\mathrm{GL}_n)$ and is $\Gamma(B_- u B_+ \cap B_+ v B_-)$. This is essentially Fomin and Zelevinsky's [FZ99] **double Bruhat cell**. Precisely, the double Bruhat cell $\mathrm{GL}_n^{u,v}$ is $B_+ u B_+ \cap B_- v B_-$. So the positroid variety Π_f is the closure in $\mathrm{Gr}(n, 2n)$ of $\Gamma(w_0 \mathrm{GL}_n^{w_0 u, w_0 v})$.

Finally, we describe a connection of positroid varieties to quantum cohomology, which we discuss further in § 8. For C any algebraic curve in $\mathrm{Gr}(k, n)$, one defines the degree of C to be its degree as a curve embedded in $\mathbb{P}^{\binom{n}{k} - 1}$ by the Plücker embedding; this can also be described as $\int_{\mathrm{Gr}(k,n)} [C] \cdot [X_\square]$ where $X_\square$ is the Schubert divisor. Let I, J and K be three elements of $\binom{n}{k}$ and d a nonnegative integer, $d \leqslant k$, such that $\mathrm{codim}\, X_I + \mathrm{codim}\, X_J + \mathrm{codim}\, X_K = k(n-k) + dn$.

Intuitively, the (genus zero) quantum product $\langle X_I X_J X_K \rangle_d$ is the number of curves in $\mathrm{Gr}(k, n)$, of genus zero and degree d, which meet $X_I(F_\bullet)$, $X_J(G_\bullet)$ and $X_K(H_\bullet)$ for a generic choice of flags $F_\bullet$, $G_\bullet$ and $H_\bullet$. This is made precise via the construction of spaces of stable maps, see [FP97].

Define $E(I, J, d)$ to be the space of degree d stable maps of a genus zero curve with three marked points to $\mathrm{Gr}(k, n)$, such that the first marked point lands in X^I and the second marked point lands in X_J. Let $S(I, J, d)$ be the subset of $\mathrm{Gr}(k, n)$ swept out by the third marked point. It is intuitively plausible that $\langle X_I X_J X_K \rangle_d$ is $\int_{\mathrm{Gr}(k,n)} [S(I, J, d)] \cdot [X_K]$ and we will show that, under certain hypotheses, this holds. We will show that (under the same hypotheses) $S(I, J, d)$ is a positroid variety.

7. The cohomology class of a positroid variety

Let $H^*(\mathrm{Gr}(k, n))$, $H_T^*(\mathrm{Gr}(k, n))$ denote the ordinary and equivariant (with respect to the natural action of $T = (\mathbb{C}^*)^n$) cohomologies of the Grassmannian, with integer coefficients. If $X \subset \mathrm{Gr}(k, n)$ is a T-invariant subvariety of the Grassmannian, we let $[X]_0 \in H^*(\mathrm{Gr}(k, n))$ denote its ordinary

cohomology class, and $[X] \in H^*_T(\mathrm{Gr}(k,n))$ denote its equivariant cohomology class. We also write $[X]|_p$ for the restriction of $[X]$ to a T-fixed point p. We index the fixed points of $\mathrm{Gr}(k,n)$ by $\binom{[n]}{k}$. We use similar notation for the flag manifold $F\ell(n)$, whose fixed points are indexed by S_n. Recall that $\pi : F\ell(n) \to \mathrm{Gr}(k,n)$ denotes the (T-equivariant) projection.

In [Lam06], a symmetric function $\tilde{F}_f \in \mathrm{Sym}$ is introduced for each affine permutation f. Let $\psi : \mathrm{Sym} \to H^*(\mathrm{Gr}(k,n))$ denote the natural quotient map. In this section, we show the following theorem.

THEOREM 7.1. *Let* $f \in \mathrm{Bound}(k,n)$. *Then* $\psi(\tilde{F}_f) = [\Pi_f]_0 \in H^*(\mathrm{Gr}(k,n))$.

7.1 Monk's rule for positroid varieties

The equivariant cohomology ring $H^*_T(\mathrm{Gr}(k,n))$ is a module over $H^*_T(\mathrm{pt}) = \mathbb{Z}[y_1, y_2, \ldots, y_n]$. The ring $H^*_T(\mathrm{Gr}(k,n))$ is graded with the real codimension, so that $\deg(y_i) = 2$ and $\deg([X]_T) = 2 \operatorname{codim}(X)$ for an irreducible T-equivariant subvariety $X \subset \mathrm{Gr}(k,n)$.

Let $X_\square \in H^*_T(\mathrm{Gr}(k,n))$ denote the class of the Schubert divisor. Note that $\pi^*(X_\square) \in H^*_T(F\ell(n))$ is the class $[X_{s_k}]$ of the kth Schubert divisor. We recall the equivariant Monk's formula (see for example [KK86]):

$$[X_{s_k}] \cdot [X_w] = ([X_{s_k}]|_w) \cdot [X_w] + \sum_{w \lessdot_k v} [X_v]. \tag{6}$$

PROPOSITION 7.2. *Let* Π_f *be a positroid variety with Richardson model* X^w_u. *Then*

$$X_\square \cdot [\Pi_f] = (X_\square|_{\sigma(u)}) \cdot [\Pi_f] + \sum_{u \lessdot_k u' \leqslant_k w} [\Pi_{f_{u',w}}]. \tag{7}$$

Here σ is the map $\sigma_k : S_n \to \binom{[n]}{k}$.

Proof. Let X^w_u be a Richardson model for Π_f. Then, using the projection formula and (6), we have in $H^*_T(\mathrm{Gr}(k,n))$,

$$
\begin{aligned}
X_\square \cdot [\Pi_f] &= \pi_*(\pi^*(X_\square) \cdot [X^w] \cdot [X_u]) \\
&= ([X_{s_k}]|_u) \cdot [\Pi_f] + \pi_* \left(\sum_{u \lessdot_k u'} [X_{u'}] \cdot [X^w] \right) \\
&= (X_\square|_{\sigma(u)}) \cdot [\Pi_f] + \sum_{u \lessdot_k u'} \pi_*([X^w_{u'}]).
\end{aligned}
$$

But

$$\pi_*([X^w_{u'}]) = \begin{cases} [\Pi_{f_{u',w}}] & \text{if } u' \leqslant_k w, \\ 0 & \text{otherwise.} \end{cases} \qquad \qquad \square$$

COROLLARY 7.3. *Let* Π_f *be a positroid variety with Richardson model* X^w_u, *and let* $X_\square \subseteq \mathrm{Gr}(k,n)$ *denote the Schubert divisor. Then, as a scheme,*

$$(u \cdot X_\square) \cap \Pi_f = \bigcup_{u \lessdot_k u' \leqslant_k w} \Pi_{f_{u',w}}.$$

Proof. The containment $\supseteq$ follows from Theorem 5.15. The above proposition tells us that the two sides have the same cohomology class, hence any difference in scheme structure must occur in lower dimension; this says that $(u \cdot X_\square) \cap \Pi_f$ is generically reduced (and has no other top-dimensional components). But since Π_f is irreducible and normal (Theorems 5.9 and 5.13), a generically reduced hyperplane section of it must be equidimensional and reduced. $\square$

Lemma 7.4. *The collection of positroid classes $[\Pi_f] \in H_T^*(\mathrm{Gr}(k,n))$ are completely determined by:*

(i) $[\Pi_f]$ *is homogeneous with degree* $\deg([\Pi_f]) = 2\ell(f)$;

(ii) *Proposition 7.2; and*

(iii) *the positroid point classes* $\{[\Pi_{t_{w \cdot \omega_k}}] = [\sigma(w)] \mid w \in S_{n,k}^{\min}\}$.

Proof. Let $f \in \mathrm{Bound}(k,n)$. We may assume by induction that the classes $[\Pi_{f'}]$ for $\ell(f') > \ell(f)$ have all been determined. The case $\ell(f) = k(n-k)$ is covered by assumption (3), so we assume $\ell(f) < k(n-k)$. Using Proposition 7.2, we may write

$$(X_\square - X_\square|_{\sigma(v)}) \cdot [\Pi_f] = \sum_{v \lessdot_k v' \leqslant_k w} [\Pi_{f_{(v',w)}}].$$

Now, the class $X_\square - X_\square|_{\sigma(v)}$ does not vanish when restricted to any fixed point $J \neq \pi(v)$ (see [KT03]), so the above equation determines $[\Pi_f]|_J$ for every $J \neq \pi(v)$. Thus if a and b are two classes in $H_T^*(\mathrm{Gr}(k,n))$ satisfying (7), then $a - b$ must be supported on $\pi(v)$. This means that $a - b$ is a multiple of the point class $[\pi(v)]$. But $\deg([\pi(v)]) = 2k(n-k)$ and $\deg(a) = \deg(b) = \ell(f) < 2k(n-k)$ so $a = b$. Thus $[\Pi_f]$ is determined by the three assumptions. $\square$

7.2 Chevalley formula for the affine flag variety

Let $\widetilde{\mathcal{F}\ell}(n)$ denote the affine flag variety of $\mathrm{GL}(n,\mathbb{C})$. We let $\{\xi^f \in H_T^*(\widetilde{\mathcal{F}\ell}(n)) \mid f \in \tilde{S}_n\}$ denote the equivariant Schubert classes, as defined by Kostant and Kumar in [KK86].

Now suppose that $f \in \tilde{S}_n$. We say that f is **affine Grassmannian** if $f(1) < f(2) < \cdots < f(n)$. For any $f \in \tilde{S}_n$, we write $f^0 \in \tilde{S}_n$ for the affine permutation given by $f^0 = [\cdots g(1)g(2) \cdots g(n) \cdots]$ where $g(1), g(2), \ldots, g(n)$ is the increasing rearrangement of $f(1), f(2), \ldots, f(n)$. Then f^0 is affine Grassmannian. Suppose that $f \lessdot g$ and $f^0 \neq g^0$. Then we say that g **0-covers** f and write $f \lessdot_0 g$. These affine analogues of k-covers were studied in [LLMS10].

For a transposition $(ab) \in \tilde{S}$ with $a < b$, we let $\alpha_{(ab)}$ (respectively $\alpha_{(ab)}^\vee$) denote the corresponding positive root (respectively coroot), which we shall think of as an element of the **affine root lattice** $Q = \bigoplus_{i=0}^{n-1} \mathbb{Z} \cdot \alpha_i$ (respectively **affine coroot lattice** $Q^\vee = \bigoplus_{i=0}^{n-1} \mathbb{Z} \cdot \alpha_i^\vee$). We have $\alpha_{(ab)} = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{b-1}$, where the α_i are the simple roots, and the indices on the right-hand side are taken modulo n. A similar formula holds for coroots. Note that $\alpha_{(ab)} = \alpha_{(a+n,b+n)}$.

In the following s_0 denotes $[\cdots k, k+2, k+3, \ldots, k+n-1, k+n+1 \cdots] \in \tilde{S}_n$.

Lemma 7.5. *Suppose that $f \in \mathrm{Bound}(k,n)$. Then*

$$\xi^{s_0} \cdot \xi^f = \xi^{s_0}|_f \cdot \xi^f + \sum_{f \lessdot_0 g \in \mathrm{Bound}(k,n)} \xi^g + other\ terms,$$

where the other terms are a linear combination of Schubert classes not labeled by $\mathrm{Bound}(k,n)$.

Proof. We deduce this formula by specializing the Chevalley formula for Kac–Moody flag varieties in [KK86],[12] which in our situation states that for any $f \in \tilde{S}$,

$$\xi^{s_0} \cdot \xi^f = \xi^{s_0}|_f \cdot \xi^f + \sum_{f \lessdot g = f \cdot (ab)} \langle \alpha_{(ab)}^\vee, \chi_0 \rangle \, \xi^g$$

---

[12] The formula in Kostant and Kumar [KK86], strictly speaking, applies to the affine flag variety $\widetilde{\mathcal{F}\ell}(n)_0$ of $\mathrm{SL}(n)$. But each component of $\widetilde{\mathcal{F}\ell}(n)$ is isomorphic to $\widetilde{\mathcal{F}\ell}(n)_0$.

where χ_0 is a weight of the affine root system satisfying $\langle \alpha_i^\vee, \chi_0 \rangle = \delta_{i0}$. We see that

$$\langle \alpha_{(ab)}^\vee, \chi_0 \rangle = \#(\{\ldots, -2n, -n, 0, n, 2n, \ldots\} \cap [a, b)).$$

Now suppose that $g \in \text{Bound}(k, n)$. Since $i \leqslant g(i) \leqslant i + n$, if $g \cdot (ab) \lessdot g$ then we must have $0 < b - a < n$. In this case, the condition that $[a, b)$ intersects $\{\ldots, -2n, -n, 0, n, 2n, \ldots\}$ is the same as $f \lessdot_0 g$, and furthermore one has $\langle \alpha_{(ab)}^\vee, \chi_0 \rangle = 1$. This proves the lemma. □

7.3 Positroid classes and Schubert classes in affine flags

For the subsequent discussion we work in the topological category. Our ultimate aim is to calculate certain cohomology classes, and changing from the algebraic to the topological category does not alter the answers. We refer the reader to [Mag07, PS86] for background material.

Let U_n denote the group of unitary $n \times n$ matrices and let $T_\mathbb{R} \simeq (S^1)^n$ denote the subgroup of diagonal matrices. We write LU_n for the space of polynomial loops into U_n, and ΩU_n for the space of polynomial based loops into U_n. It is known that $\text{LU}_n/T_\mathbb{R}$ is weakly homotopy equivalent to $\widetilde{\mathcal{Fl}}(n)$, and that $\Omega U_n \simeq \text{LU}_n/U_n$ is weakly homotopy equivalent to the affine Grassmannian (see [PS86]).

The connected components of ΩU_n and LU_n are indexed by $\mathbb{Z}$, using the map $L\det : \text{LU}_n \to \text{LU}_1 = Map(S^1, U(1)) \sim \pi_1(U(1)) = \mathbb{Z}$. We take as our basepoint of the k-component of ΩU_n the loop $t \mapsto \text{diag}(t, \ldots, t, 1, \ldots, 1)$, where t appears k times. Abusing notation, we write $t_{\omega_k} \in \Omega U_n$ for this point, identifying the basepoint with a translation element.

The group LU_n acts on ΩU_n by the formula

$$(a \cdot b)(t) = a(t)b(t)a(t)^{-1}$$

where $a(t) \in \text{LU}_n$ and $b(t) \in \Omega U_n$. The group U_n embeds in LU_n as the subgroup of constant loops. The action of LU_n on ΩU_n restricts to the conjugation action of $U(n)$ on $U(n)$. It then follows that the orbit of the basepoint under the action of U_n

$$U_n \cdot t_{\omega_k} \simeq U_n/(U_k \times U_{n-k}) \tag{8}$$

is isomorphic to the Grassmannian $\text{Gr}(k, n)$.

Thus we have a map $q : \text{Gr}(k, n) \hookrightarrow \Omega U_n$. Let $r : \Omega U_n \to \text{LU}_n/T$ be the map obtained by composing the natural inclusion $\Omega U_n \hookrightarrow \text{LU}_n$ with the projection $\text{LU}_n \twoheadrightarrow \text{LU}_n/T$. We let

$$p := r \circ q : \text{Gr}(k, n) \longrightarrow \text{LU}_n/T$$

denote the composition of q and r. All the maps are $T_\mathbb{R}$-equivariant, so we obtain a ring homomorphism $p^* : H_T^*(\widetilde{\mathcal{Fl}}(n)) \to H_T^*(\text{Gr}(k, n))$.

LEMMA 7.6. *Suppose $w \in S_{n,k}^{\min}$ and $I = \sigma(w)$, which we identify with a T-fixed point of $\text{Gr}(k, n)$. Then $p(I) = t_{w \cdot \omega_k} T_\mathbb{R} \in \text{LU}_n/T_\mathbb{R}$.*

Proof. It follows from the action of $S_n \leqslant U_n$ on ΩU_n that $q(I) = t_{w \cdot \omega_k} \in \Omega U_n$. But by definition $r(t_{w \cdot \omega_k}) = t_{w \cdot \omega_k} \in \text{LU}_n$. □

LEMMA 7.7. (i) *Suppose $w \in S_{n,k}^{\min}$. Then $p^*(\xi^{t_{w \cdot \omega_k}}) = [\sigma(w)]$.*

(ii) *Furthermore, $p^*(\xi^{s_0}) = X_\square$.*

Proof. We prove (i). Let $f = t_{w \cdot \omega_k}$. It is enough to check that $\xi^f|_{t_{u \cdot \omega_k}} = [\pi(w)]|_{\sigma(u)}$ for each $u \in S_{n,k}^{\min}$. We have $[\pi(w)]|_{\sigma(u)} = 0$ unless $u = w$. By [KK86, Proposition 4.24(a)], $\xi^f|_g = 0$ unless $f \leqslant g$.

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Since f is maximal in $\mathrm{Bound}(k, n)$, it is enough to calculate

$$\xi^f|_f = \prod_{\alpha \in f^{-1}(\Delta_-) \cap \Delta_+} \theta(\alpha).$$

Here Δ_+ (respectively Δ_-) are the positive (respectively negative) roots of the root system of $\tilde{S}_n$, and $\theta(\alpha) \in H_T^*(\mathrm{pt}) = \mathbb{Z}[y_1, y_2, \ldots, y_n]$ denotes the image of α under the linear map defined by

$$\theta(\alpha_i) = \begin{cases} y_i - y_{i+1} & \text{for } i = 1, 2, \ldots, n-1, \\ y_n - y_1 & \text{for } i = 0. \end{cases}$$

Applying θ corresponds to specializing from $H_{T \times S^1}^*(\widetilde{\mathcal{F}\ell}(n))$ to $H_T^*(\widetilde{\mathcal{F}\ell}(n))$.

With this terminology, $\alpha_{(ab)} \in f^{-1}(\Delta_-)$ if and only if $f(a) > f(b)$. We have $\theta(\alpha_{(ab)}) = y_a - y_b$, where the indices are taken modulo n. Thus

$$\xi^f|_f = \prod_{i \in \sigma(w) \text{ and } j \in [n] \setminus \sigma(w)} (y_i - y_j)$$

which is easily seen to agree with $[\pi(w)]|_{\sigma(w)}$.

Now we prove (ii). The class $p^*(\xi^{s_0}) \in H_T^*(G/P)$ is of degree 2. So by [KT03, Lemma 1], it is enough to show that it vanishes when restricted to the identity basepoint, and equals $X_\square|_{s_k} = y_k - y_{k+1}$ when restricted to s_k. We know that

$$\xi^{s_0}|_{\mathrm{id}} = 0 \quad \text{and} \quad \xi^{s_0}|_{s_0} = y_k - y_{k+1}$$

since $\mathrm{id} < s_0$, and the inversions of s_0^{-1} are exactly $\{\alpha_k\}$. (Here id denotes $[\cdots k+1, k+2, \ldots, k+n \cdots]$.) But we have that t_{ω_k} is in the same (right) S_n-coset as id and $t_{s_k \cdot \omega_k}$ is in the same S_n-coset as s_0. Since ξ^{s_0} is a Grassmannian class, it follows that [KK86] $\xi^{s_0}|_{\mathrm{id}} = \xi^{s_0}|_{t_{\omega_k}}$ and $\xi^{s_0}|_{s_0} = \xi^{s_0}|_{t_{s_k \cdot \omega_k}}$. Applying Lemma 7.6, we see that $p^*(\xi^{s_0})$ has the desired properties. $\qquad\square$

THEOREM 7.8. *For each $f \in \tilde{S}_n^k$, we have in $H_T^*(\mathrm{Gr}(k, n))$,*

$$p^*(\xi^f) = \begin{cases} [\Pi_f] & \text{if } f \in \mathrm{Bound}(k, n), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $f \notin \mathrm{Bound}(k, n)$. Then by [KK86, Proposition 4.24(a)], $\xi^f|_g = 0$ unless $f \leqslant g$, so that $\xi^f|_g = 0$ for $g \in \mathrm{Bound}(k, n)$ (using Lemma 3.6). It follows that $p^*(\xi^f)$ vanishes at each T-fixed point of $\mathrm{Gr}(k, n)$, and so it is the zero class.

We shall show that the collection of classes $\{p^*(\xi^f) \mid f \in \mathrm{Bound}(k, n)\}$ satisfies the conditions of Lemma 7.4. Part (i) is clear, and part (iii) follows from Lemma 7.7(i). We check part (ii). The map p^* is a ring homomorphism, so the formula in Lemma 7.5 holds for the classes $p^*(\xi^f)$ as well. Suppose $f \lessdot g \in \mathrm{Bound}(k, n)$ and $f = f_{u,w}$ and $g = f_{u',w'}$. As in the proof of Theorem 3.16, we may assume that either (i) $u' = u$ and $w' \lessdot w$, or (ii) $u' \gtrdot u$ and $w' = w$. If $f \lessdot_0 g$, then writing $f_{u,w} = t_{u \cdot \omega_k} u w^{-1}$ and recalling that right multiplication by uw^{-1} acts on the positions, we see that we must have $u \cdot \omega_k \neq u' \cdot \omega_k$. This implies that we are in case (ii), and that $u' \gtrdot_k u$. Conversely, if $w' = w$ and $u' \gtrdot_k u$ then we must have $f \lessdot_0 g$. Comparing Lemma 7.5 and Proposition 7.2, and using Lemma 7.7(ii), we see that we may apply Lemma 7.4 to the classes $\{p^*(\xi^f) \mid f \in \mathrm{Bound}(k, n)\}$.

Thus $p^*(\xi^f) = [\Pi_f]$ for every $f \in \mathrm{Bound}(k, n)$. $\qquad\square$

7.4 Affine Stanley symmetric functions

Let Sym denote the ring of symmetric functions over $\mathbb{Z}$. For each $f \in \tilde{S}_n^0$, a symmetric function $\tilde{F}_f \in \text{Sym}$, called the *affine Stanley symmetric function*, is defined in [Lam06]. This definition extends to all $f \in \tilde{S}_n$ via the isomorphisms $\tilde{S}_n^k \simeq \tilde{S}_n^0$.

We will denote the simple reflections of the Coxeter group $\tilde{S}_n^0$ by $s_0, s_1, \ldots, s_{n-1}$, where the indices are taken modulo n. Let $w \in \tilde{S}_n^0$. We say that w is **cyclically decreasing** if there exists a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for w such that (a) no simple reflection is repeated, and (b) if s_i and s_{i+1} both occur, then s_{i+1} precedes s_i. Then the **affine Stanley symmetric function** $\tilde{F}_f$ is defined by letting the coefficient of $x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}$ in $\tilde{F}_f(x_1, x_2, \ldots)$ be equal to the number of factorizations $w = w^{(1)} w^{(2)} \cdots w^{(r)}$, where each $w^{(i)}$ is cyclically decreasing, $\ell(w^{(i)}) = a_i$, and $\ell(w) = \ell(w^{(1)}) + \ell(w^{(2)}) + \cdots + \ell(w^{(r)})$.

For example, consider $k = 2$, $n = 4$ and $f = [5, 2, 7, 4]$. The corresponding element of $\tilde{S}_n^0$ is $f_0 = [3, 0, 5, 2]$; the reduced words for f_0 are $s_1 s_3 s_0 s_2$, $s_1 s_3 s_2 s_0$, $s_3 s_1 s_0 s_2$ and $s_3 s_1 s_2 s_0$. So the coefficient of $x_1 x_2 x_3 x_4$ in $\tilde{F}_f$ is 4, corresponding to these four factorizations. Similar computations yield that $\tilde{F}_f = 4m_{1111} + 2m_{211} + m_{22} = s_{22} + s_{211} - s_{1111}$ where the monomial symmetric functions are denoted by m and the Schur functions are denoted by s. Note that affine Stanley symmetric functions are not necessarily Schur-positive!

The ordinary cohomology $H^*(\Omega \text{SU}_n)$ can be identified with a quotient of the ring of symmetric functions:

$$H^*(\Omega \text{SU}_n) \simeq \text{Sym}/\langle m_\lambda \mid \lambda_1 > n \rangle,$$

where m_λ denotes the monomial symmetric function labeled by λ. We refer to [Sta99] for general facts concerning symmetric functions, and to [Lam08] for more about $H^*(\Omega \text{SU}_n)$.

Let $s_\lambda \in \text{Sym}$ denote the Schur functions, labeled by partitions. As each component of ΩU_n is homeomorphic to ΩSU_n, the inclusion $q : \text{Gr}(k, n) \to \Omega U_n$ (defined after (8)) induces a map $\tilde{\psi} : H^*(\Omega \text{SU}_n) \to H^*(\text{Gr}(k, n))$. Let $\psi : \text{Sym} \to H^*(\text{Gr}(k, n))$ denote the composition of the quotient map $\text{Sym} \twoheadrightarrow H^*(\Omega \text{SU}_n)$ with $\tilde{\psi} : H^*(\Omega \text{SU}_n) \to H^*(\text{Gr}(k, n))$.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_1 \leqslant n - k$, let $\sigma(\lambda)$ be $\{\lambda_k + 1, \lambda_{k-1} + 2, \ldots, \lambda_1 + k\}$. This is a bijection from partitions with at most k parts and largest part at most $n - k$ to $\binom{[n]}{k}$. We denote the set of such partitions by $\text{Par}(k, n)$.

LEMMA 7.9. *The map $\psi : \text{Sym} \to H^*(\text{Gr}(k, n))$ is the natural quotient map defined by*

$$\psi(s_\lambda) = \begin{cases} [X_{\sigma(\lambda)}]_0 & \lambda \in \text{Par}(k, n), \\ 0 & otherwise. \end{cases}$$

Proof. The copy of $\text{Gr}(k, n)$ inside ΩU_n is the union of the $\binom{n}{k}$ Schubert varieties labeled by the translation elements $\{t_{w \cdot \omega_k} \mid w \in S_{n,k}^{\min}\}$. It follows that the map $\tilde{\psi} : H^*(\Omega \text{SU}_n) \to H^*(\text{Gr}(k, n))$ sends Schubert classes to Schubert classes.

It is well known that $H^*(\text{Gr}(k, n))$ is isomorphic to the quotient ring of Sym as stated in the lemma. To check that the quotient map agrees with ψ, it suffices to check that they agree on the homogeneous symmetric functions $h_i \in \text{Sym}$, which generate Sym. In [Lam08, Theorem 7.1] it is shown that the Schubert classes of $H^*(\Omega \text{SU}_n)$ are the 'affine Schur functions', denoted $\tilde{F}_\lambda$. When λ is a single row, we have $\tilde{F}_{(r)} = h_r \in \text{Sym}/\langle m_\lambda \mid \lambda_1 > n \rangle$. Furthermore, the finite-dimensional Schubert variety in ΩU_n with dual Schubert class $\tilde{F}_{(r)}$ lies in $\text{Gr}(k, n) \subset \Omega U_n$ exactly when $r \leqslant n - k$. It follows that $\psi(h_r) = [X_{(r)}]_0 \in H^*(\text{Gr}(k, n))$ for $r \leqslant n - k$, and $\psi(h_r) = 0$ for $r \geqslant n - k$. Thus ψ is the stated map. $\square$

Proof of Theorem 7.1. Let $\xi_0^f \in H^*(\widetilde{\mathcal{F}\ell}(n))$ denote the non-equivariant Schubert classes. It is shown[13] in [Lam08, Remark 8.6] that we have $r^*(\xi_0^f) = \tilde{F}_f \in H^*(\Omega SU_n)$, where we identify $\tilde{F}_f \in \mathrm{Sym}$ with its image in $H^*(\Omega SU_n) = \mathrm{Sym}/\langle m_\lambda \mid \lambda_1 > n \rangle$. Thus we calculate using the non-equivariant version of Theorem 7.8

$$[\Pi_f]_0 = p^*(\xi_0^f) = q^* r^*(\xi_0^f) = \psi(\tilde{F}_f). \qquad \square$$

Recall our previous example where $k = 2$, $n = 4$ and $f = [5, 2, 7, 4]$, with siteswap 4040. This positroid variety is a point. The affine Stanley function $\tilde{F}_f$ was $s_{22} + s_{211} - s_{1111}$, so $\psi(\tilde{F}_f) = \psi(s_{22})$, the class of a point.

Example 7.10. Stanley invented Stanley symmetric functions in order to prove that the number of reduced words for the long word w_0 in S_m was equal to the number of standard Young tableaux of shape $(m - 1, m - 2, \ldots, 21)$. He showed that $F_{w_0} = s_{(m-1)(m-2)\cdots21}$, so the number of reduced words for w_0 is the coefficient of the monomial $m_{11\cdots1}$ in the Schur polynomial $s_{(m-1)(m-2)\cdots21}$, as required. See [Sta84] for more background. We show how to interpret this result using positroid varieties.

Let $(k, n) = (m, 2m)$. The Stanley symmetric function F_{w_0} is the affine Stanley associated to the affine permutation

$$v : i \mapsto \begin{cases} i + n & 1 \leqslant i \leqslant n \\ w_0(i - n) + 2n & n + 1 \leqslant i \leqslant 2n. \end{cases}$$

As discussed in §6, the positroid variety Π_v is a graph matrix Schubert variety, and can be described as the Zariski closure, within $G(m, 2m)$, of m-planes that can be represented in the form

$$\mathrm{RowSpan} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \end{pmatrix}.$$

(The example shown is for $m = 5$.)

Reordering columns turns Π_v into the Zariski closure of those m-planes that can be represented in the form

$$\mathrm{RowSpan} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * & 0 & * \\ 0 & 0 & 1 & * & 0 & * & 0 & * & 0 & * \\ 1 & * & 0 & * & 0 & * & 0 & * & 0 & * \end{pmatrix}.$$

This is the Schubert variety $X_{1357\cdots(2m-1)}$, which is associated to the partition $(m-1)(m-2)\cdots321$. Reordering columns acts trivially in $H^\bullet(G(k, n))$, so the cohomology classes of Π_v and $X_{135\cdots(2m-1)}$ are the same, and they thus correspond to the same symmetric function. This shows that $F_{w_0} = s_{(m-1)(m-2)\cdots21}$.

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[13] The setup in [Lam08] involves ΩSU_n, but each component of ΩU_n is isomorphic to ΩSU_n so the results easily generalize.

7.5 The K- and K_T-classes of a positroid variety

We conjecture that the K-class of a positroid variety is given by the affine stable Grothendieck polynomials defined in [Lam06]. These symmetric functions were shown in [LSS10] to have the same relationship with the affine flag manifold as affine Stanley symmetric functions, with K-theory replacing cohomology.

CONJECTURE 7.11.[14] The K-theory class of the structure sheaf of a positroid variety Π_f is given by the image of the affine stable Grothendieck polynomial $\tilde{G}_f$, when $K^*(\mathrm{Gr}(k,n))$ is identified with a ring of symmetric functions as in [Buc02].

This conjecture would follow from suitable strengthenings of Proposition 7.2, and Lemmas 7.4, and 7.5. We have the necessary characterization of the K_T positroid classes: Corollary 7.3 and the main result of [Knu09] give the K_T-analogue of Proposition 7.2. The degree-based argument used in Lemma 7.4 must be modified, in the absence of a grading on K-theory, to comparing pushforwards to a point, and it is easy to show using [KLS, Theorem 4.5] that the pushforward of a positroid class is 1. What is currently missing are the two corresponding results on affine stable Grothendieck polynomials.

While the class associated to an algebraic subvariety of a Grassmannian is always a positive combination of Schubert classes, this is not visible from Theorem 7.1, as affine Stanley functions are not in general positive combinations of Schur functions s_λ.

The following theorem gives a much stronger positivity result on positroid classes.

THEOREM 7.12. *Let X be a positroid variety, and $[\mathcal{O}_X] \in K_T(\mathrm{Gr}(k,n))$ the class of its structure sheaf in equivariant K-theory. Then in the expansion $[\mathcal{O}_X] = \sum_\lambda a_\lambda [\mathcal{O}_\lambda]$ into classes of Schubert varieties, the coefficient $a_\lambda \in K_T(pt)$ lies in $(-1)^{|\lambda| - \dim X} \mathbb{N}[\{e^{-\alpha_i} - 1\}]$, where the $\{\alpha_i\}$ are the simple roots of $\mathrm{GL}(n)$.*

Proof. This is just the statement of [AGM11, Corollary 5.1], which applies to any T-invariant subvariety X of a flag manifold such that X has rational singularities (as positroid varieties do, [KLS, Corollary 4.8]). $\square$

After our first version of this preprint was circulated, a very direct geometric proof of Theorem 7.8 was given in [Sni10], which also proves the corresponding statement in equivariant K-theory. Snider identifies each affine patch on $\mathrm{Gr}(k,n)$ with an opposite Bruhat cell in the affine flag manifold, T-equivariantly, in a way that takes the positroid stratification to the Bruhat decomposition, thereby corresponding the K_T-classes.

8. Quantum cohomology, toric Schur functions, and positroids

8.1 Moduli spaces of stable rational maps to the Grassmannian

For background material on stable maps we refer the reader to [FW04]. Let $I, J \in \binom{[n]}{k}$, which we assume to be fixed throughout this section. We now investigate the variety $S(I, J, d)$ consisting of points lying on a stable rational map of degree d, intersecting $X_J \subset \mathrm{Gr}(k,n)$ and $X^I \subset \mathrm{Gr}(k,n)$. Let $M_{0,3}(d)$ denote the moduli space of stable rational maps to $\mathrm{Gr}(k,n)$ with three marked points and degree d. Write $p_1, p_2, p_3 : M_{0,3}(d) \to \mathrm{Gr}(k,n)$ for the evaluations at the three marked points.

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[14] This conjecture has been established in [HL11].

Denote by $E(I, J, d)$ the subset

$$E(I, J, d) = p_1^{-1}(X_J) \cap p_2^{-1}(X^I) \subset M_{0,3}(d).$$

It is known [FW04] that $E(I, J, d)$ is reduced and locally irreducible, with all components of dimension $\dim(X_J) + \dim(X^I) + dn - k(n-k)$. Furthermore, the pushforward $(p_3)_*([E(I, J, d)]) \in H^*(\mathrm{Gr}(k, n))$ is a generating function for three-point, genus zero, Gromov–Witten invariants, in the sense that

$$(p_3)_*([E]) \cdot \sigma = \langle [X_J], [X^I], \sigma \rangle_d \tag{9}$$

for any class $\sigma \in H^*(\mathrm{Gr}(k, n))$. We now define $S(I, J, d) := p_3(E(I, J, d))$. Let us say that **there is a non-zero quantum problem for** (I, J, d) if $\langle [X_J], [X^I], \sigma \rangle_d$ is non-zero for some $\sigma \in H^*(\mathrm{Gr}(k, n))$. It follows from (9) that $S(I, J, d)$ and $E(I, J, d)$ have the same dimension whenever there is a non-zero quantum problem for (I, J, d), namely,

$$\dim(S(I, J, d)) = \dim(E(I, J, d)) = \dim(X_J) + \dim(X^I) + dn - k(n-k). \tag{10}$$

The torus T acts on $M_{0,3}(d)$ and, since X_J and X^I are T-invariant, the space $E(I, J, d)$ also has a T-action. The torus-fixed points of $E(I, J, d)$ consist of maps $f : C \to \mathrm{Gr}(k, n)$, where C is a tree of projective lines, such that $f_*(C)$ is a union of T-invariant curves in $\mathrm{Gr}(k, n)$ whose marked points are T-fixed points, satisfying certain stability conditions. Since p_3 is T-equivariant, we have $S(I, J, d)^T = p_3(E(I, J, d)^T)$. The T-invariant curves in $\mathrm{Gr}(k, n)$ connect pairs of T-fixed points labeled by $I, J \in \binom{[n]}{k}$ satisfying $|I \cap J| = k - 1$. We'll write $T(I, J, d)$ for $S(I, J, d)^T$, considered as a subset of $\binom{[n]}{k}$.

We now survey the rest of this section. In § 8.2, we use the ideas of the previous paragraph to give an explicit combinatorial description of $T(I, J, d)$. We then define an explicit affine permutation f associated to (I, J, d) in (11) below. We say that (I, J, d) is **valid** if for $i \in I$, we have $i + k \leqslant f(I, J, d)(i) \leqslant i + n$ and, for $m \in [n] \backslash I$, we have $m \leqslant f(I, J, d)(m) \leqslant m + k$. In particular, (I, J, d) is valid implies $f(I, J, d)$ is bounded. The main result of this section is given in the following theorem.

THEOREM 8.1. *When (I, J, d) is valid, the image $p_3(E(I, J, d))$ is Π_f. Moreover, there is one component F_0 of $E(I, J, d)$ for which $p_3 : F_0 \to \Pi_f$ is birational; on any other component F of $E(I, J, d)$, we have $\dim p_3(F) < \dim F$.*

When (I, J, d) is not valid, then $\dim p_3(F) < \dim F$ for every component F of $E(I, J, d)$. Thus, (I, J, d) is valid if and only if there is a non-zero quantum problem for (I, J, d).

Our key combinatorial result is given in the following proposition.

PROPOSITION 8.2. *Let (I, J, d) be valid. Then $T(I, J, d)$ is the positroid corresponding to the bounded affine permutation $f(I, J, d)$.*

We should point out that we use previously known formulas for Gromov–Witten invariants to establish part of Theorem 8.1. Namely, when f is valid, we can establish directly that $p_3(E(I, J, d)) \subseteq \Pi_f$. To prove that $(p_3)_*([E]) = [\Pi_f]$, we combine previous work of Postnikov with Theorem 7.1.

It was shown in [Lam06] that $\psi(\tilde{F}_f)$ is Postnikov's 'toric Schur function'. Postnikov showed that this toric Schur function computed Gromov–Witten invariants but did not provide a subvariety of $\mathrm{Gr}(k, n)$ representing his class; Theorem 8.1 can thus be viewed as a geometric explanation for toric Schur functions.

8.2 Formulas for $T(I,J,d)$ and $f(I,J,d)$

We proceed to describe $S(I,J,d)^T$ explicitly.

If $I \in \binom{[n]}{k}$, then we let $A(I)$ (respectively $B(I)$) denote the upper (respectively lower) order ideals generated by I. Thus $A(J) = \{K \in \binom{[n]}{k} \mid K \geqslant J\}$ is the set of T-fixed points lying in X_J and $B(I)$ is the set of T-fixed points lying in X^I. Define the undirected **Johnson graph** $G_{k,n}$ with vertex set $\binom{[n]}{k}$, and edges $I \leftrightarrow J$ if $|I \cap J| = k - 1$. The distance function $\operatorname{dist}(I, J)$ in $G_{k,n}$ is given by $\operatorname{dist}(I, J) = k - |I \cap J|$. Then one has

$$S(I,J,d)^T = T(I,J,d) := \left\{ K \in \binom{[n]}{k} \mid \operatorname{dist}(K, B(I)) + \operatorname{dist}(K, A(J)) \leqslant d \right\}.$$

For a pair (I, J), it is shown in [FW04] that the minimal d such that $T(I,J,d)$ is nonempty (or equivalently, that there is a path of length d from I to J in $G_{k,n}$), is equal to the minimal d such that a non-zero quantum problem for (I, J, d) exists.

Denote by $M = \{m_1 < m_2 < \cdots < m_{n-k}\}$ the complement of I in $[n]$ and similarly $L = \{l_1 < l_2 < \cdots < l_{n-k}\}$ the complement of J. We define a bi-infinite sequence $\tilde{i}$ such that $\tilde{i}_a = i_a$ for $1 \leqslant a \leqslant k$ and $\tilde{i}_{a+k} = \tilde{i}_a + n$. Similarly, we extend J, M and L to bi-infinite sequences $\tilde{j}$, $\tilde{m}$ and $\tilde{l}$, such that $\tilde{j}_{a+k} = \tilde{j}_a + n$, $\tilde{m}_{a+n-k} = \tilde{m}_a + n$ and $\tilde{l}_{a+n-k} = \tilde{l}_a + n$. Define an affine permutation $f(I, J, D)$ by

$$f(I,J,d)(\tilde{i}_r) = \tilde{j}_{r+k-d} \quad f(I,J,d)(\tilde{m}_r) = \tilde{l}_{r+d}. \tag{11}$$

We say that (I, J, d) is **valid** if, for $i \in I$, we have $i + k \leqslant f(I,J,d)(i) \leqslant i + n$ and, for $m \in M$, we have $m \leqslant f(I,J,d)(m) \leqslant m + k$. In particular, if (I, J, d) is valid, then $f(I, J, d)$ is bounded.

For example, let $k = 2$ and $n = 6$. Pick $I = \{1, 4\}$, $J = \{2, 4\}$, $d = 1$. Then $M = \{2, 3, 5, 6\}$ and $L = \{1, 3, 5, 6\}$. The equation $f(\tilde{i}_r) = \tilde{j}_{r+k-d}$ gives $f(1) = 4$ and $f(4) = 8$. The equation $f(\tilde{m}_r) = \tilde{l}_{r+d}$ gives $f(2) = 3$, $f(3) = 5$, $f(5) = 6$ and $f(6) = 7$. Thus $f(I, J, d) = [\cdots 435867 \cdots]$.

Our next task is to prove Proposition 8.2; we shorten $f(I, J, d)$ to f. Our approach is to first find the cyclic rank matrix for f. Let $\tilde{I}$ and $\tilde{J}$ be the preimages of I and J under the projections $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. For integers $a \leqslant b$, we adopt the shorthand $I[a, b]$ for $\#(\tilde{I} \cap [a, b])$ and similar notations $J[a, b]$ etc. For $1 \leqslant a \leqslant b \leqslant a + n$, define

$$r_{ab} := \min(b - a + 1, d + J[1, b] - I[1, a], k) \tag{12}$$

and define r_{ab} for all r_{ab} such that $r_{a+n,b+n} = r_{ab}$; by $r_{ab} = b - a + 1$ for $a > b$ and by $r_{ab} = k$ for $a + n < b$.

LEMMA 8.3. *If (I, J, d) is valid, then the matrix r_{ab} is the cyclic rank matrix for f.*

It will be convenient to introduce the functions $\alpha_1(a, b) = b - a + 1$, $\alpha_2(a, b) = d + J[1, b] - I[1, a]$ and $\alpha_3(a, b) = k$, so that $r_{ab} = \min(\alpha_1(a, b), \alpha_2(a, b), \alpha_3(a, b))$.

Proof. We first check that r_{ab} is a cyclic rank matrix, meaning that it obeys the conditions in Corollary 3.12. Conditions (C1$'$), (C2$'$) and (C5) hold by definition.

For $r = 1, 2$ or 3, it is easy to see that, $\alpha_r(a, b) - \alpha_r(a + 1, b)$ and $\alpha_r(a, b + 1) - \alpha_r(a, b)$ are clearly either 0 or 1, and are integer valued. So $r_{ab} - r_{(a+1)b}$ and $r_{ab} - r_{a(b-1)}$ are either 0 or 1. This verifies (C3).

Similarly, for any (a, b) and $r = 1, 2$ or 3, the 2×2 matrix $\begin{pmatrix} \alpha_r(a,b-1) & \alpha_r(a,b) \\ \alpha_r(a+1,b-1) & \alpha_r(a+1,b) \end{pmatrix}$ is of one of the forms

$$\begin{pmatrix} s & s \\ s & s \end{pmatrix} \quad \begin{pmatrix} s & s+1 \\ s & s+1 \end{pmatrix} \quad \begin{pmatrix} s+1 & s+1 \\ s & s \end{pmatrix} \quad \begin{pmatrix} s+1 & s+2 \\ s & s+1 \end{pmatrix}$$

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for some integer s. Looking at what happens when we take the minimum of three matrices of this form (where we never take both the two middle matrices), we see that we always get one of the above forms, or $\left(\begin{smallmatrix} s+1 & s+1 \\ s & s+1 \end{smallmatrix}\right)$. In particular, condition (C4) holds. We have now shown that r_{ab} is a cyclic rank matrix. Let g be the associated permutation.

We now show that $g = f$. Let $(a, b) = (i_r, \tilde{j}_{r+k-d})$ for some r between 1 and k. Then $\#(\tilde{I} \cap [1, a)) = r - 1$ and $\#(\tilde{J} \cap [1, b]) = r + k - d$, so $\alpha_2(a, b) = k + 1$. Similar arguments let us compute

$$\begin{pmatrix} \alpha_2(a, b-1) & \alpha_2(a, b) \\ \alpha_2(a+1, b-1) & \alpha_2(a+1, b) \end{pmatrix} = \begin{pmatrix} k & k+1 \\ k-1 & k \end{pmatrix}.$$

The assumption that $i_r + k \leqslant f(I, J, d)(i_r) = \tilde{j}_{r+k-d}$ translates into $b - a \geqslant k$, so the α_1 term in (12) has no effect and we deduce that

$$\begin{pmatrix} r_{a,b-1} & r_{a,b} \\ r_{a+1,b-1} & r_{a+1,b} \end{pmatrix} = \begin{pmatrix} k & k \\ k-1 & k \end{pmatrix}.$$

By the last sentence of Corollary 3.10, this means that $g(a) = b$.

Similarly, if $(a, b) = (m_r, \tilde{l}_{r+d})$ we can show that $g(a) = b$. So, for every $a \in \mathbb{Z}$, we have shown that $g(a) = f(a)$, as desired. □

We now begin proving the lemmas which will let us prove Proposition 8.2.

LEMMA 8.4. *We have* $\mathrm{dist}(K, B(I)) \leqslant s$ *if and only if for all* $r \in [n]$, *one has* $I[1, r] - K[1, r] \leqslant s$.

Proof. Suppose $\mathrm{dist}(K, B(I)) \leqslant s$. Then $\mathrm{dist}(K, L) \leqslant s$ for some $L \leqslant I$. Thus for each r, we have $I[1, r] - K[1, r] \leqslant L[1, r] - K[1, r] \leqslant s$.

Now suppose $I[1, r] - K[1, r] \leqslant s$ for each r. Construct $L \leqslant I$ recursively, starting with $L = \emptyset$. Assume $L \cap [1, r)$ is known. If $r \in K$, place r in L. Otherwise, if $r \notin K$, place r in L only if $r \in I$ and $L[1, r] = I[1, r]$. Repeat until we have constructed a k-element subset L which clearly satisfies $L \leqslant I$. The elements in $L \backslash K$ are all in I. Let ℓ be the largest element in $L \backslash K$. Then $I[1, \ell]$ differs from $K[1, \ell]$ by $|L \backslash K|$, and so $|L \backslash K| \leqslant s$. Thus $\mathrm{dist}(K, L) \leqslant s$. □

LEMMA 8.5. *We have* $K \in T(I, J, d)$ *if and only if*

$$I[1, r] - K[1, r] + K[1, s] - J[1, s] \leqslant d \tag{13}$$

for all $1 \leqslant r, s, \leqslant n + 1$.

Proof. By Lemma 8.4, we have $K \in T(I, J, d)$ if and only if

$$\max(I[1, r] - K[1, r], 0) + \max(K[1, s] - J[1, s], 0) \leqslant d \tag{14}$$

for all $1 \leqslant r, s, \leqslant n + 1$. Equation (14) certainly implies the stated condition. Conversely, if (13) holds, but (14) fails, then we must have $I[1, r] - K[1, r] > d$ or $K[1, s] - J[1, s] > d$ for some r, s. In the first case setting $s = 1$ in (13) gives a contradiction. In the second case, setting $r = 1$ gives a contradiction. □

Proof of Proposition 8.2. First, suppose that $K \in T(I, J, d)$.

By Lemma 8.3, K is in the positroid corresponding to $f(I, J, d)$ if and only if for each cyclic interval $[a, b] \subset [n]$ we have

$$K[a, b] \leqslant \min(b - a + 1, d + J[1, b] - I[1, a], k).$$

Since $\#[a, b] = b - a + 1$ and $\#(K) = k$, we always have $K[a, b] \leqslant b - a + 1$ and $K[a, b] \leqslant k$, so we must check that $K[a, b] \leqslant d + J[1, b] - I[1, a]$.

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First, suppose that $1 \leqslant a \leqslant b \leqslant n$. Then $K[a,b] = K[1,b] - K[1,a)$ so the required equation is

$$K[1,b] - K[1,a) \leqslant d + J[1,b] - I[1,a).$$

This is easily equivalent to (13) for $(r,s) = (a,b)$. Now, suppose that $a \leqslant n < b$. Then $K[a,b] := K[1,b] - K[1,a)$ and we again want to show that $K[1,b] - K[1,a) \leqslant d + J[1,b] - I[1,a)$. Let $b = b' + n$. Then $K[1,b] = K[1,b'] + n$ and $J[1,b] = J[1,b'] + n$. So it is equivalent to show
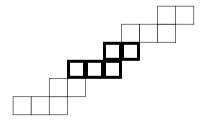
$$K[1,b'] - K[1,a) \leqslant d + J[1,b'] - I[1,a)$$

which is (13) for $(r,s) = (a,b')$.

The reverse implication is similar. $\qquad\square$

8.3 Toric shapes and toric Schur functions

In [Pos05a], Postnikov introduced a family of symmetric polynomials, called *toric Schur polynomials*, and showed that the expansion coefficients of these symmetric functions in terms of Schur polynomials gave the three-point, genus-zero, Gromov–Witten invariants of the Grassmannian. In [Lam06], it was shown that toric Schur functions were special cases of affine Stanley symmetric functions. We now put these results in the context of Theorem 7.1 and equation (9): the subvariety $S(I,J,d) \subset \mathrm{Gr}(k,n)$ is a positroid variety whose cohomology class is a toric Schur polynomial.

We review the notion of a toric shape and refer the reader to [Pos05a] for the notion of a toric Schur function. A **cylindric shape** is a connected, row and column convex subset of $\mathbb{Z}^2$ which is invariant under the translation $(x,y) \mapsto (x + n - k, y - k)$. Also, every row or column of a cylindric shape must be finite, and in addition the 'border' of a cylindric shape is an infinite path which has steps going north and east only (when read from the southwest). A **toric shape** is a cylindric shape such that every row has at most $n - k$ boxes, and every column has at most k boxes. For example, the following is a toric shape for $k = 2$, $n = 5$:

where a fundamental domain for the action of the translation has been highlighted. In [Pos05a], Postnikov associated a toric shape $\theta(I,J,d)$ to each triple (I,J,d) for which a non-trivial quantum problem could be posed involving the Schubert varieties X^I and X_J, and rational curves of degree d. The steps of the upper border of θ are determined by I, the lower border by J. The gap between the two borders is determined by d. We do not give a precise description of Postnikov's construction here as our notation differs somewhat from Postnikov's.

If θ is a cylindric shape, we can obtain an affine permutation as follows. First label the edges of the upper border of θ by integers, increasing the labels from southwest to northeast. Now label the edges of the lower border of θ by integers, so that if e and e' are edges on the upper border and lower border respectively, and they lie on the same northwest-southeast diagonal, then e' has a label which is k bigger than that of e. One then defines $f(\theta)$ as follows: if $a \in \mathbb{Z}$ labels a vertical step of the upper border, then $f(a)$ is the label of the step of the lower border on the same row; if $a \in \mathbb{Z}$ labels a horizontal step, then $f(a)$ is the label of the step of the lower border

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on the same column. This determines $\theta(I, J, d)$ from $f(I, J, d)$ up to a translation: the equations (11) say that the labels inside I or J are vertical steps, while labels in M and L are horizontal steps.

The condition that (I, J, d) is valid translates to $\theta(I, J, d)$ being toric. In our language, Postnikov [Pos05a, Lemma 5.2 and Theorem 5.3] shows that a non-trivial quantum problem exists for (I, J, d) if and only if the toric shape $\theta(I, J, d)$ is well defined. Thus we have the following lemma.

LEMMA 8.6. *A non-trivial quantum problem exists for (I, J, d) if and only if (I, J, d) is valid.*

LEMMA 8.7. *Suppose (I, J, d) is valid. Then*

$$\ell(f(I, J, d)) = |\theta(I, J, d)| = \operatorname{codim}(X_J) + \operatorname{codim}(X^I) - dn,$$

where $|\theta(I, J, d)|$ is the number of boxes in a fundamental domain for $\theta(I, J, d)$.

Proof. The first equality follows from [Lam06], and can be explained simply as follows: each box in a fundamental domain for $\theta(I, J, d)$ corresponds to a simple generator in a reduced expression for $f(I, J, d)$. Indeed, the equations (11) can be obtained by filling $\theta(I, J, d)$ with a wiring diagram, where each wire goes straight down (respectively across) from a horizontal (respectively vertical) step. The second equality follows from [Pos05a]. A simple proof is as follows: if we decrease d by 1, then the lower border of d is shifted one step diagonally southeast, increasing $|\theta(I, J, d)|$ by n. When the upper and lower borders are far apart, then changing $\operatorname{codim}(X^I)$ or $\operatorname{codim}(X_J)$ by one also changes $|\theta(I, J, d)|$ by one. Finally, when $I = J$ and $d = 0$, one checks that $|\theta(I, J, d)|$ is $k(n - k)$. □

Proof of Theorem 8.1. Suppose that (I, J, d) is valid.

Consider any index $K \in \binom{[n]}{k} \setminus T(I, J, d)$. Then the Plücker coordinate p_K is zero on $T(I, J, d)$, and hence on $S(I, J, d)$. By Corollary 5.12, Π_f is cut out by $\{p_K = 0 \mid K \notin T(I, J, d)\}$, so $S(I, J, d) \subseteq \Pi_f$.

By Lemma 8.7, (10) and Theorem 5.9, $S(I, J, d)$ and $\Pi_{f(I,J,d)}$ have the same dimension and Π_f is irreducible. So $S(I, J, d) = \Pi_f$. Now, let $F_1, F_2, \ldots, F_r$ be the components of $E(I, J, d)$; let $c_1, c_2, \ldots, c_r$ be the degrees of the maps $p_3 : F_i \to S(I, J, d)$. Using again that Π_f is irreducible, we know that $(p_3)_*(E(I, J, d)) = (\sum_{i=1}^r c_i)[\Pi_f]$. By the main result of [Pos05a], the left-hand side of this equation is the toric Schur polynomial with shape $\theta(I, J, d)$ and by [Lam06, Proposition 33], this is the affine Stanley function $\psi(\tilde{F}_f)$. But by Theorem 7.1, the right-hand side is $(\sum_{i=1}^r c_i)\psi(\tilde{F}_f)$. So $\sum_{i=1}^r c_i = 1$. We deduce that p_3 is birational on one component of $S(I, J, d)$ and collapses every other component.

Finally, if (I, J, d) is not valid, then there is no non-zero quantum product for (I, J, d) by Lemma 8.6, so p_3 must collapse all components of $E(I, J, d)$ in this case. □

8.4 Connection with two-step flag varieties

Let $F\ell(k - d, k, k + d; n)$ and $F\ell(k - d, k + d; n)$ be the spaces of three-step and two-step flags of dimensions $(k - d, k, k + d)$ and $(k - d, k + d)$ respectively. We have maps $q_1 : F\ell(k - d, k, k + d; n) \to \operatorname{Gr}(k, n)$ and $q_2 : F\ell(k - d, k, k + d; n) \to F\ell(k - d, k + d; n)$. For a subvariety $X \subset \operatorname{Gr}(k, n)$ we define, following [BKT03],

$$X^{(d)} = q_2(q_1^{-1}(X)) \subset F\ell(k - d, k + d; n).$$

Let us now consider the subvariety

$$Y(I, J, d) = (X_J)^{(d)} \cap (X^I)^{(d)} \subset F\ell(k - d, k + d; n).$$

Buch–Kresch–Tamvakis studied varieties similar to $Y(I, J, d)$, which arise from intersections of three Schubert varieties, and they also showed in a bijective manner that these intersections solved quantum problems. Let us now consider the subvariety $q_1(q_2^{-1}(Y(I, J, d))) \subset \mathrm{Gr}(k, n)$. The subvarieties $(X_J)^{(d)}, (X^I)^{(d)} \subset F\ell(k - d, k + d; n)$ are Schubert (and opposite Schubert) subvarieties. Thus $q_1(q_2^{-1}(Y(I, J, d))) \subset \mathrm{Gr}(k, n)$ is a positroid variety by Theorem 5.9.

The following result can also be deduced directly from [BM11].

PROPOSITION 8.8. *Suppose* (I, J, d) *is valid. Then* $q_1(q_2^{-1}(Y(I, J, d))) = S(I, J, d)$.

Proof. Let us first show that $S(I, J, d) \subset q_1(q_2^{-1}(Y(I, J, d)))$. Since (I, J, d) is valid, we know that $\dim S(I, J, d) = \dim(X_J) + \dim(X^I) + dn - k(n - k) =: N$. Choose $K \in \binom{[n]}{d}$ such that codim $X_K = N$ and $\langle [S(I, J, d)], [X_K] \rangle \neq 0$. Then, for a general flag $F_\bullet$, $S(I, J, d)$ intersects $X_K(F_\bullet)$ at a finite set of points. Moreover, the set of all points that occur as such intersections is dense in $S(I, J, d)$. (If this set were contained in a subvariety of smaller dimension, then $X_K(F_\bullet)$ would miss $S(I, J, d)$ for generic $F_\bullet$, contradicting our choice of K.)

So, for V in a dense subset of $S(I, J, d)$, we know that V also lies on some $X_K(F_\bullet)$ and we can impose furthermore that $F_\bullet$ is in general position with both $E_\bullet$ and $w_0 E_\bullet$. It follows from [BKT03, Theorem 1] that there is a corresponding point $W \in Y(I, J, d)$ such that $V \in q_1(q_2^{-1}(W))$. Thus $S(I, J, d) \subset q_1(q_2^{-1}(Y(I, J, d)))$.

Conversely, let $W \in Y(I, J, d)$ be a generic point, and $Z = q_1(q_2^{-1}(W)) \subset \mathrm{Gr}(k, n)$. The space Z is isomorphic to $\mathrm{Gr}(d, 2d)$. Pick a point $U \in Z \cap X_J$ and $V \in Z \cap X^I$, and another generic point $T \in Z$. By [BKT03, Proposition 1], there is a morphism $f : \mathbb{P}^1 \to Z \subset \mathrm{Gr}(k, n)$ of degree d which passes through U, V, and T. It follows that a generic point in Z lies in $S(I, J, d)$. Thus $q_1(q_2^{-1}(Y(I, J, d))) \subset S(I, J, d)$. $\qquad\square$

8.5 An example

Let $k = 2$ and $n = 5$. We take $I = J = \{1, 4\}$ and $d = 1$. The affine permutation $f(I, J, d)$ is $[\cdots 43567 \cdots]$, with siteswap 31222. The positroid $T(I, J, d)$ is $\{12, 13, 14, 15, 24, 25, 34, 35, 45\}$ and the juggling states are $\mathcal{J}(f(I, J, d)) = (12, 13, 12, 12, 12)$. If we pull back $Y(I, J, d)$ to $F\ell(n)$ we get the Richardson variety X_{12435}^{45132}. (Following the description given in [BKT03], we obtained 12435 by sorting the entries of the Grassmannian permutation 14235 in positions $k - d + 1, k - d + 2, \ldots, k + d$ in increasing order. For 45132, we first applied w_0 to $J = \{1, 4\}$ to get $\{2, 5\}$. Then we did the sorting, and left-multiplied by w_0 again.)

By [KLS, Proposition 3.3], we have $\pi(X_{12435}^{45132}) = \pi(X_{21543}^{54312})$. With $(u, w) = (21543, 54312)$, we have $f_{u,w} = [2, 1, 5, 4, 3] \cdot t_{(1,1,0,0,0)} \cdot [4, 5, 3, 2, 1] = [4, 3, 5, 6, 7]$, agreeing with $f(I, J, d)$. Alternatively, one can check that the T-fixed points inside X_{21543}^{54312}, that is, the interval $[21543, 54312]$, project exactly to $T(I, J, d)$.

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