THE EULER-LAGRANGE EXPRESSION AND DEGENERATE LAGRANGE DENSITIES

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1. Introduction and motivation

It is well known that many of the field equations from theoretical physics (e.g. Einstein field equations, Maxwell’s equations, Klein-Gordon equation) can be obtained from a variational principle with a suitably chosen Lagrange density. In the case of the Einstein equations the corresponding Lagrangian is degenerate (i.e., the associated Euler-Lagrange equations are of second order whereas in general these would be of fourth order), while in the cases of the Maxwell and Klein-Gordon equations the Lagrangian usually used is not degenerate. However, it is not generally realized that there exist degenerate Lagrange densities which also give rise to these last two field equations. In this note the general structure of this type of degenerate Lagrange density is examined.

We shall concentrate our attention on \( m \) quantities \( \rho^A (A = 1, \ldots, m) \) which in general are each functions of position i.e.

\[
\rho^A = \rho^A(x^i).
\]

Under transformations of the type

\[
\bar{x}^a = \bar{x}^a(x^i)
\]

we shall assume that the \( \rho^A \) transform according to the law

\[
\bar{\rho}^A = C_B^A \rho^B,
\]

where the \( C_B^A \) are functions of \( \bar{x}^a \) (or \( x^I \)) and are completely determined by the transformation (1.1). To fix ideas we cite four examples all of which fall into the above category.

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1 Unless otherwise noted the summation convention will apply, whereby repeated capital indices \( A, B, \ldots \) will be summed from 1 to \( m \) and repeated latin indices \( a, b, \ldots, i, j, \ldots \) from 1 to \( n \).
(i) **Scalar field** \( \phi \). If \( \phi \) is a scalar field the counterpart of (1.2) reads

\[
\bar{\phi} = \phi,
\]

in which case \( m = 1 \), \( \rho^A = \phi \) and \( C_b^A = 1 \).

(ii) **Vector field** \( \psi_i \). In this case \( m = n \), \( \rho^A = \psi_i \) and \( C_b^A = B_a^i (= \partial \psi_i / \partial \bar{x}^a) \) since a vector field transforms according to the law\(^1\)

\[
\bar{\psi}_a = B_a^i \psi_i.
\]

(iii) **Tensor field** \( g_{ij} \). Here \( m = n^2 \), \( \rho^A = g_{ij} \) and \( C_b^A = B_a^i B_b^j \) corresponding to

\[
\bar{g}_{ab} = B_a^i B_b^j g_{ij}.
\]

(iv) **Non-tensorial field** \( \alpha_{ij} \). If \( \alpha_{ij} \) are \( n^2 \) quantities which transform according to\(^2\)

\[
\bar{\alpha}_{ab} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \sum_{c=1}^{n} \sum_{d=1}^{n} B_{ac}^i B_{bd}^j A_i^c A_j^d + B_a^i B_b^j \right) \alpha_{ij}
\]

where we have put

\[
B_{ac}^i = \partial B_a^i / \partial \bar{x}^c \quad \text{and} \quad A_i^c = \partial \bar{x}^c / \partial x^i,
\]

then under these circumstances

\[
C_b^A = \sum_{c=1}^{n} \sum_{d=1}^{n} B_{ac}^i B_{bd}^j A_i^c A_j^d + B_a^i B_b^j.
\]

From these examples it is evident that \( \rho^A \) may represent, on the one hand, the components of an arbitrary relative tensor field and, on the other hand, certain quantities which are manifestly non-tensorial in character.

We now assume that we are given a quantity \( L \)–the Lagrangian. It is furthermore supposed that \( L \) is a function of \( \rho^A \) and its first \( M \) partial derivatives together with \( q \) arbitrary preassigned functions of position \( \lambda^a (a = 1, \ldots, q) \) and the first \( Q \) partial derivatives of \( \lambda^a \), i.e.

\[
(1.3) \quad L = L(\rho^A; \rho^A, i_1; \ldots; \rho^A, i_1 \ldots i_M; \lambda^a; \lambda^a, i_1; \ldots; \lambda^a, i_1 \ldots i_Q),
\]

where a comma denotes partial differentiation.

With \( L \) we can always associate the Euler-Lagrange expression \( E_A(L) \) defined by

\[
(1.4) \quad E_A(L) = \frac{\partial L}{\partial \rho^A} + \sum_{r=1}^{M} (-1)^r \left( \partial^r \right) \left( \frac{\partial L}{\partial \rho^A, i_1 \ldots i_r} \right),
\]

where, as mentioned above, the \( \lambda^a \) are not varied but are assumed to be preass-

\(^2\) The summation convention does not apply to this example.
signed functions of position. In general the Euler-Lagrange expression (1.4) will be of order $2M$ in $\rho^A$ (and order $M + Q$ in $\lambda^a$). If the order of this expression in $\rho^A$ is less than $2M$ the corresponding Lagrangian is called degenerate. For examples of this see [2], [3] and [4].

In order to ensure that the so-called action integral corresponding to (1.3) viz.

$$\int \ldots \int L \, dx^1 \ldots dx^n$$

is an invariant, we assume that $L$ is a scalar density, i.e. under (1.1)

(1.5) \[ \bar{L} = BL \]

where \[ B = \det |B^i| \]

In theoretical physics the role played by the Euler-Lagrange equations

(1.6) \[ E_A(L) = 0 \]

is well known. It is usually possible to derive the field equations of physics from a variational principle with a suitably chosen Lagrangian $L$. To illustrate this we briefly discuss three important cases.

(a) Symmetric tensor field: Einstein vacuum field equations.

Consider the Lagrangian $L$ given by

(1.7) \[ L(g_{ij}; g_{ij,k}; g_{ij,kl}) = \sqrt{g} g^{ij} R^h_{ih} \]

where $g = \det |g_{ij}|$, $g^{ij}$ are characterised by

$$g^{ij} g_{kj} = \delta_k^i$$

and

$$R^h_{ijk} = \left\{ \begin{array}{c} h \\
  \{ i \} \\
  \{ j \}
\end{array} \right\} - \left\{ \begin{array}{c} h \\
  \{ i \} \\
  \{ k \}
\end{array} \right\} + \left\{ \begin{array}{c} r \\
  \{ i \} \\
  \{ j \}
\end{array} \right\} - \left\{ \begin{array}{c} r \\
  \{ i \} \\
  \{ k \}
\end{array} \right\} - \left\{ \begin{array}{c} h \\
  \{ k \} \\
  \{ j \}
\end{array} \right\}$$

with

$$\left\{ \begin{array}{c} h \\
  \{ i \} \\
  \{ j \}
\end{array} \right\} = \frac{1}{2} g^{hk}(g_{ik,j} + g_{jk,i} - g_{ij,k}).$$

With the correspondence

$$\rho^A = g_{ij}, \quad \lambda^a = 0,$$

in (1.3) the associated Euler-Lagrange expression (1.4) is

(1.8) \[ E^{ij}(L) = -\sqrt{g} \left( g^{ih} g^{jk} R^l_{hkl} - \frac{1}{2} g^{ij} g^{hk} R^l_{hkl} \right), \]

3 See e.g. [5] p. 258.
and the Euler-Lagrange equations (1.6) are just the Einstein field equations in vacuo.

(b) Scalar field: Klein-Gordon Equation.

If we consider the Lagrangian

\[ L(\phi; \phi_1; g_{ij}) = \frac{1}{2} \sqrt{g} (\phi^{ij} \phi_1 + k^2 \phi^2) , \]

(where \( k = \text{constant} \)), and use the correspondence

\[ \rho^A = \phi, \lambda^a = g_{ij} , \]

then (1.4) becomes

\[ E(L) = \sqrt{g} (\phi^{ij} \phi_1 - k^2 \phi) , \]

where the vertical bar denotes covariant differentiation with respect to \( g_{ij} \). The Euler-Lagrange equation in this case is the Klein-Gordon equation.

(c) Vector field: Maxwell’s equation

By choosing

\[ L(\psi_1; \psi_1, j; g_{ij}) = \frac{1}{2} \sqrt{g} F_{ij} F_{kl} g^{ik} g^{jk} , \]

where

\[ F_{ij} = \psi_{i,j} - \psi_{j,i} , \]

and by identifying

\[ \rho^A = \psi_i, \lambda^a = g_{ij} , \]

we find that (1.4) reads

\[ E^i(L) = 2 \sqrt{g} g^{ik} g^{jk} F_{kl} \]

The corresponding Euler-Lagrange equations are Maxwell’s equations in the absence of sources.

Of these three Lagrangians i.e. (1.7), (1.9) and (1.11), only one is degenerate viz. (1.7). However, it is not generally realized that in the other two cases it is possible to choose Lagrange densities which are degenerate but which still yield (1.10) and (1.12) as the corresponding Euler-Lagrange expressions. These Lagrangians are

\[ L(\phi; \phi_1; \phi_1; g_{ij}; g_{ij}) = \frac{1}{2} \sqrt{g} \phi (\phi^{ij} \phi_1 - k^2 \phi) , \]

and

\[ L(\psi_1; \psi_1, j; \psi_1, jk; g_{ij}; g_{ij}) = \sqrt{g} \psi_1 g^{ik} g^{jk} F_{kl} \]

respectively. Aside from their degeneracy, the Lagrangians (1.7), (1.13) and (1.14) have something else in common—they all have the same structure viz.
where $a$ is a constant, in the general case. This raises three obvious questions:

1. If $L$ is a scalar density and $\rho_A$ transform according to (1.2) is $\rho^A E_A(L)$ a scalar density?

2. If $\rho^A E_A(L)$ is a scalar density and we regard it as a new Lagrangian, is it always degenerate?

3. If $\rho^A E_A(L)$ is a degenerate Lagrange density are its Euler-Lagrange equations always $E_A(L) = 0$, up to a constant?

These three questions will be answered in the next sections by means of Theorems 2, 4 and 6 respectively.

2. Certain properties of the Euler-Lagrange expression

The main purpose of this section is to establish the transformation law for $E_A(L)$ under (1.1) where it is assumed that $\rho^A$ and $L$ transform according to (1.2) and (1.5) respectively.

To this end we introduce the following notation. If $F^{i_0...i_r}$ is any function of $x^i$ then

$$F^{i_0...i_r} = F^{i_0...i_r},$$

and

$$F^{i_0...i_r} = \frac{\partial F^{i_0...i_r}}{\partial x^{i_r}}$$

for $r = 1, 2, \ldots$. Similarly if $G^{a_0...a_r}$ is any function of $x^a$ then

$$G^{a_0...a_r} = G^{a_0...a_r},$$

and

$$G^{a_0...a_r} = \frac{\partial G^{a_0...a_r}}{\partial x^{a_r}}$$

for $r = 1, 2, \ldots$. In view of this the summation convention will not apply to the indices $i_0$ or $a_0$.

It is possible to rewrite (1.4) in a slightly more concise form by introducing the following definitions:

$$L_{A_0} = \frac{\partial L}{\partial \rho^A}, i_0(= \frac{\partial L}{\partial \rho^A}),$$

and

$$L_{A_1...i_r} = \frac{\partial L}{\partial \rho^A}, i_0...i_r$$

for $1 \leq r \leq M$. In this case (1.4) becomes

$$E_A(L) = \sum_{r=0}^{M} (-1)^r L_{A_1...i_r}^{i_0...i_r}. i_0...i_r.$$
transformation properties of $\tilde{\rho}^A, a_{0a_1...a_k}$ for all $k$, $0 \leq k \leq M$. We shall obtain these properties in the form of a recurrence relation. In view of (1.2) it is clear that $\tilde{\rho}^A, a_{0a_1...a_k}$ will be linear in $\rho^B, l_{0i_1...i_j}$ for all $j$, $0 \leq j \leq k$, and can thus be expressed in the form

\begin{equation}
(2.1) \quad \tilde{\rho}^A, a_{0a_1...a_k} = \sum_{j=0}^{k} \alpha_{Ba_0a_1...a_k}^{A_{i_0j...i_j}} \rho^B, l_{i_0i_1...i_j},
\end{equation}

where $\alpha_{Ba_0a_1...a_k}^{A_{i_0j...i_j}}$ are functions of $x^a$. Differentiation of (2.1) with respect to $x^{a_{k+1}}$ yields

\begin{equation}
(2.2) \quad \tilde{\rho}^A, a_{0a_1...a_k+1} = \sum_{j=1}^{k+1} \left[ \alpha_{Ba_0a_1...a_k}^{A_{i_0j...i_j-1}B_{i_j}^{i_0j...i_j}} + \alpha_{Ba_0a_1...a_k+1}^{A_{i_0j...i_j}} \right] \rho^B, l_{i_0i_1...i_j} + \\
+ \alpha_{Ba_0a_1...a_k,a_{k+1}}^{A} \rho^B,
\end{equation}

with the understanding that

$$
\alpha_{Ba_0a_1...a_k}^{A_{i_0j...i_k+1}} = 0.
$$

A comparison of (2.2) with (2.1) [with $k$ replaced by $(k + 1)$ in the latter] yields the following. If $k$ is any integer, $0 \leq k + 1 \leq M$,

\begin{equation}
(2.3) \quad \alpha_{Ba_0a_1...a_k+1}^{A_{i_0j...i_j}} =
\begin{cases}
\alpha_{Ba_0a_1...a_k}^{A_{i_0j...i_j+1}} & \text{if } j = 0 \text{ and } k \geq 0, \\
\alpha_{Ba_0a_1...a_k}^{A_{i_0j-1}B_{i_j}^{i_0j...i_j}} + \alpha_{Ba_0a_1...a_k+1}^{A_{i_0j...i_j}} & \text{if } 1 \leq j \leq k + 1, \\
0 & \text{if } j > k + 1.
\end{cases}
\end{equation}

Obviously

\begin{equation}
(2.4) \quad \alpha_{Ba_0}^{A_0} = C_B^A.
\end{equation}

By means of (2.3) we can prove the

**Lemma.** If under (1.1) $\rho^A$ and $L$ transform according to (1.2) and (1.5) respectively then for $1 \leq k \leq M - 1$

\begin{equation}
(2.5) \quad \sum_{p=0}^{k-1} (-1)^p L_B^{l_{0i_1...i_{M-p}}, a_{M-p-1}...a_{M-k}+1} + (-1)^k L_B^{l_{0i_1...i_{M-k}}} = \\
= \sum_{j=0}^{k-1} (-1)^j \sum_{p=0}^{k-j-1} (-1)^p L_A^{a_0a_1...a_{M-p}, a_{M-p-1}...a_{M-k}+j+1} + \\
+ (-1)^{k-j} L_A^{a_0a_1...a_{M-k+j}+1} \alpha_{Ba_0a_1...a_{M-k+j+1}}^{A_{i_0j...i_{M-k}} B_{i_j}^{i_0j...i_j}} + \\
+ (-1)^{k-j} L_A^{a_0a_1...a_{M-k}} \alpha_{Ba_0a_1...a_{M-k-1}}^{A_{i_0j...i_{M-k-1}} B_{i_j}^{i_0j...i_j}} / B.
\end{equation}
PROOF. We shall prove this by induction over \( k \). With \( k = 1 \) (2.5) reads

\[
L_B^{i_0i_1\ldots i_M}_{,i_M} - L_B^{i_0i_1\ldots i_M}_{,i_M-1} = (L_A^{a_0a_1\ldots a_M}_{,a_M} - L_A^{a_0a_1\ldots a_M}_{,a_M-1}) \times \\
\times \tilde{\alpha}_{B_0a_1\ldots a_M-2} B^i_{a_M-1} / B - L_A^{a_0a_1\ldots a_M}_{,a_M} \tilde{\alpha}_{B_0a_1\ldots a_M-1} B^i_{a_M-1} / B.
\]

From (1.5) and (2.1) we find

\[
L_B^{i_0i_1\ldots i_M} = L_A^{a_0a_1\ldots a_M}_{,a_M} \tilde{\alpha}_{B_0a_1\ldots a_M} / B,
\]

which, by virtue of (2.3), can be written as

\[
L_B^{i_0i_1\ldots i_M} = L_A^{a_0a_1\ldots a_M}_{,a_M} \tilde{\alpha}_{B_0a_1\ldots a_M-1} B^i_{a_M} / B.
\]

In view of the fact that

\[
\frac{\partial}{\partial x^i}(B^i_{a} / B) = 0,
\]

we thus have

\[
L_B^{i_0i_1\ldots i_M} = (L_A^{a_0a_1\ldots a_M}_{,a_M} \tilde{\alpha}_{B_0a_1\ldots a_M-1} + L_A^{a_0a_1\ldots a_M}_{,a_M} \tilde{\alpha}_{B_0a_1\ldots a_M-1,a_M}) / B.
\]

From (1.5) and (2.1) we also see that

\[
L_B^{i_0i_1\ldots i_M-1} = (L_A^{a_0a_1\ldots a_M}_{,a_M} \tilde{\alpha}_{B_0a_1\ldots a_M-1} + L_A^{a_0a_1\ldots a_M}_{,a_M} \tilde{\alpha}_{B_0a_1\ldots a_M-1,a_M}) / B.
\]

Subtraction of (2.8) from (2.7) yields

\[
L_B^{i_0i_1\ldots i_M-1} = (L_A^{a_0a_1\ldots a_M}_{,a_M} \tilde{\alpha}_{B_0a_1\ldots a_M-1} + L_A^{a_0a_1\ldots a_M}_{,a_M} \tilde{\alpha}_{B_0a_1\ldots a_M-1,a_M}) / B.
\]

which is (2.6) when account is taken of (2.3). Hence (2.5) is valid for \( k = 1 \).

We now assume (2.5) to be true for fixed \( k \) and establish the validity of (2.5) with \( k \) replaced by \( k + 1 \), i.e. we wish to show that (2.5) implies

\[
\sum_{p=0}^{k} (-1)^p L_B^{i_0i_1\ldots i_M-p}_{,i_M-p} \cdot i_M-p_{i_M-p-1} \ldots i_M-k_+1{i_M-k} + (-1)^{k+1} L_B^{i_0i_1\ldots i_M-k-1} =
\]

\[
= \sum_{j=0}^{k} (-1)^{j} \left\{ \sum_{p=0}^{k-j} (-1)^p L_A^{a_0a_1\ldots a_M}_{,a_M-p} \cdot a_M-p_{a_M-p-1} \ldots a_M-k+j+1a_M-k+j+1 +
\]

\[
+ (-1)^{k-j+1} L_A^{a_0a_1\ldots a_M}_{,a_M-k} \cdot a_M-k_{a_M-k-2} B_{a_M-k+j-1} / B +
\]

\[
+ (-1)^{k+1} L_A^{a_0a_1\ldots a_M}_{,a_M} \tilde{\alpha}_{B_0a_1\ldots a_M-1} B^i_{a_M-1} / B.
\]

We see that the first term on the left hand side of (2.9) is the derivative of the left hand side of (2.5) with respect to \( x^{i_M-k} \). Consequently the left hand side of (2.9) can be expressed as the sum of six expressions, viz.
The Euler-Lagrange expression 489

\[ \sum_{j=0}^{k-1} (-1)^j \sum_{p=0}^{k-j} (-1)^p L_A a_{0} \ldots a_{j-1} \cdot \alpha_{M-p \ldots a_M} = -1 \cdot a_M - k + j + a_M - k + j \cdot \alpha_{B a_0 \ldots a_M - k + j - 1} |B| + \]

\[ + \sum_{j=0}^{k-1} (-1)^j \sum_{p=0}^{k-j} (-1)^p L_A a_{0} \ldots a_{j-1} \cdot \alpha_{M-p \ldots a_M} - k + j + a_M - k + j \cdot \alpha_{B a_0 \ldots a_M - k + j - 1} |B| + \]

\[ + \sum_{j=0}^{k-1} (-1)^k \sum_{A} (-1)^p L_A a_{0} \ldots a_{M-k+j} A_{i_0 \ldots i_M-k-1} |B| + \]

\[ + (-1)^k L_A a_{0} \ldots a_{M-k+1} |B| + \]

The fourth expression can be absorbed in the first by extending the summation in the latter from \( j = k - 1 \) to \( j = k \). In the second and third expressions we replace the summation over \( j \) by one over \( j - 1 \) and include a \( j = 0 \) term (which is zero by (2.3)). From the sixth expression we extract the term \( j = k + 1 \) and combine it with the fifth expression. Finally we group the remaining terms in the sixth expression with those in the third to find that the left hand side of (2.9) reads

\[ \sum_{j=0}^{k} (-1)^j \left\{ \sum_{p=0}^{k-j} (-1)^p L_A a_{0} \ldots a_{j-1} \cdot \alpha_{M-p \ldots a_M} = -1 \cdot a_M - k + j \right\} \times \]

\[ \left( \alpha_{B a_0 \ldots a_M - k + j - 1} - \alpha_{B a_0 \ldots a_M - k + j + 2 \cdot a_M - k + j - 1} \right) |B| + \]

\[ + (-1)^{k+1} L_A a_{0} \ldots a_{M-k+1} (\alpha_{B a_0 \ldots a_M - k-1} - \alpha_{B a_0 \ldots a_M - k+1}) |B| . \]

By virtue of (2.3) this is the right hand side of (2.9), which establishes the lemma.

With the aid of this lemma we are now in a position to prove

**THEOREM 1.** If under (1.1) \( \rho A \) and \( L \) transform according to (1.2) and (1.5) respectively then

\[ BE_A(L) = C_A B E_B(L) . \]

**PROOF.** In (2.5) we set \( k = M - 1 \) to find

\[ \sum_{p=0}^{M-2} (-1)^p L_B a_{0} \ldots a_{M-n} \cdot l_{M-p \ldots l_2} + (-1)^{M-1} L_B a_{0} = \]

\[ = \sum_{j=0}^{M-2} (-1)^j \left\{ \sum_{p=0}^{M-j-2} (-1)^p L_A a_{0} \ldots a_{M-p \ldots a_M} = a_{j+2} + \right\} \times \]

\[ \left( \alpha_{B a_0 \ldots a_M - k + j + 1} a_{j+1} + 1 / B \right) + (-1)^{M-j-1} L_A a_{0} \ldots a_{M-k} a_{j+1} B_{a_{j+1}} / B . \]
By differentiating the latter with respect to $x^k$, multiplying by $(-1)^{kM}$ and recalling (1.4) we see that

\[
E_B(L) = \sum_{j=0}^{M-1} (-1)^j \sum_{p=0}^{M-j-1} (-1)^{pM} F_A^{a_0...a_{M-p}...a_{j+1}} \alpha_{B_{a_0...a_j}}/B + \\
\sum_{j=0}^{M-1} \sum_{p=0}^{M-k-2} (-1)^{pM} F_A^{a_0...a_{M-p}...a_{j+2}} + (-1)^{j+1} F_A^{a_0...a_{j+1}} \times \\
x \alpha_{B_{a_0...a_j...a_{j+1}}}/B - F_A^{a_0...a_{M-1}a_{M}} \alpha_{B_{a_0...a_{M-1}a_{M}}}/B + \\
\sum_{j=0}^{M} L_A^{a_0...a_j} \alpha_{B_{a_0...a_j}}/B.
\]

By virtue of (2.3) only the $j = 0$ terms in the first and final expressions survive on the right hand side—all others cancel. We thus find

\[
E_B(L) = \sum_{p=0}^{M-1} (-1)^{pM} F_A^{a_0...a_{M-p}...a_1} + F_A^{a_0} \alpha_{B_{a_0}}/B,
\]

which, in view of (2.4), is (2.10).

From Theorem 1 we see immediately that if $\rho^A$ are the components of a relative tensor of covariant valency $r$, contravariant valency $s$ and weight $w$, then $E_A(L)$ are the components of a relative tensor of covariant valency $s$, contravariant valency $r$ and weight $(1 - w)$.

We remark that Theorem 1 can be extended to quantities $\rho^A$ which transform according to

\[
\tilde{\rho}^A = C_B^A \rho^B + \theta^A
\]

where the $C_B^A$ and $\theta^A$ are functions of $\tilde{x}^a$ and are completely determined by the transformation (1.1). [A typical example of this would be the symmetric affine connection $\Gamma^h_{ij}$ which transforms according to

\[
\tilde{\Gamma}^a_{bc} = B^m_{bc} \Gamma^a_{rm} + A^a_{bh} B^h_{bc}.
\]

In fact we can prove the

**Theorem.** If under (1.1) $L$ is a scalar density and $\rho^A$ transform according to (2.11) then

\[
BE_A(L) = C_B^E_B(L).
\]

However, we will not consider this case in any further detail, but will return to (1.2). If we multiply (2.10) by $\rho^A$ and note (1.2) we have

**Theorem 2.** If under (1.1) $\rho^A$ and $L$ transform according to (1.2) and (1.5) respectively then $\rho^A E_A$ is a scalar density.
We shall now give a direct proof of the frequently asserted

**Theorem 3.** A divergence satisfies the Euler-Lagrange equations identically\(^4\), i.e. if a Lagrangian \(L\) is of the form

\[
L_{(D)} = S_i^i_{,i}
\]

where

\[
S_i^i = S_i^i(\rho_A^i; \rho_A^{i_1}; \ldots; \rho_A^{i_{m-1}}; \lambda^2; \lambda^{i_1}; \ldots; \lambda^{i_{n-1}}),
\]

and \(p\) is any positive integer

then

\[
E_A(L) = 0.
\]

**Proof.** It is clear that

\[
L_{(D)} = \sum_{r=0}^{M-1} S_i^i_{,j_0j_1 \ldots j_r} \rho_B^i_{,j_0j_1 \ldots j_r,i} + \sum_{r=0}^{p} S_i^i_{,j_0j_1 \ldots j_r} \lambda^2_{,j_0j_1 \ldots j_r,i}
\]

where

\[
S_i^i_{,j_0j_1 \ldots j_r} = \partial S_i^i_B/\partial \rho_B^i_{,j_0j_1 \ldots j_r}
\]

and

\[
S_i^i_{,j_0j_1 \ldots j_r} = \partial S_i^i_A/\partial \lambda^2_{,j_0j_1 \ldots j_r}.
\]

For \(1 \leq k \leq M - 1\) it is easily seen, from (2.12), that

\[
L_{(D)}^{i_0i_1 \ldots i_k} = \sum_{r=0}^{M-1} (\partial S_i^i_{,j_0j_1 \ldots j_r}/\partial \rho_A^i_{,i_0i_1 \ldots i_k}) \rho_B^i_{,j_0j_1 \ldots j_r,i} + \sum_{r=0}^{p} (\partial S_i^i_{,j_0j_1 \ldots j_r}/\partial \rho_A^i_{,i_0i_1 \ldots i_k}) \lambda^2_{,j_0j_1 \ldots j_r,i} + S^{[i_k; \|i_0\|i_1 \ldots i_{k-1}]},
\]

where the square bracket denotes complete symmetrisation over \(i_1 \ldots i_k\) (since \(i_0\) is excluded from this symmetrisation process we place it in braces). However, for \(1 \leq k \leq M - 1\), it is easily seen that

\[
S_A^{i_0i_1 \ldots i_k}_{,i} = \sum_{r=0}^{M-1} (\partial S_A^{i_0i_1 \ldots i_k}/\partial \rho_B^i_{,j_0j_1 \ldots j_r}) \rho_B^i_{,j_0j_1 \ldots j_r,i} + \sum_{r=0}^{p} (\partial S_A^{i_0i_1 \ldots i_k}/\partial \lambda^2_{,j_0j_1 \ldots j_r}) \lambda^2_{,j_0j_1 \ldots j_r,i},
\]

\(^4\) This result is not at variance with [1] p. 121.

\(^5\) Summation over \(\alpha\) from 1 to \(q\).
which, when taken together with (2.13), yields

\[ L(l_1 \ldots l_k) = S_A^{l_1 \ldots l_k} + S_A^{l_1 \ldots l_k \ldots l_{k+1}}. \]

From the latter we see that for \( 1 \leq k \leq M - 1 \)

\[ L(l_1 \ldots l_k) = S_A^{l_1 \ldots l_{k+1} \ldots l_k} + S_A^{l_1 \ldots l_k \ldots l_{k+1}}. \]

from which we conclude that

\[ \sum_{k=1}^{M-1} (-1)^k L(l_1 \ldots l_k) + (-1)^M S_A^{l_1 \ldots l_{M-1}} + S_A^{l_1 \ldots l_M} = 0. \]

In view of (2.12) the last two terms on the left hand side are respectively the \( k = M \) and \( k = 0 \) terms of the first expression, so that

\[ E_A(L) = 0. \]

3. Degenerate Lagrange densities

For the reasons indicated in section 1 we now wish to investigate the consequences of adopting the quantity

(3.1) \[ \mathcal{L} = \rho^A E_A(L) \]

as a Lagrangian. If it is assumed that the \( \rho^A \) transform according to (1.2) then Theorem 2 assures us that \( \mathcal{L} \) is a scalar density — one of the requirements usually made of a Lagrangian. Furthermore, if \( L \) is of the type (1.3) then in general \( E_A(L) \) will involve derivatives of \( \rho^A \) up to order \( 2M \) and derivatives of \( \lambda^\alpha \) up to order \( M + \sigma \), in which case

\[ \mathcal{L} = \mathcal{L}(\rho^A; \rho^A_{i_1}; \ldots; \rho^A_{i_1i_2 \ldots i_{2M}}; \lambda^\alpha; \lambda^\alpha_{i_1}; \ldots; \lambda^\alpha_{i_1i_2 \ldots i_{M+\sigma}}). \]

This in turn suggests that the associated Euler-Lagrange expression \( E_A(\mathcal{L}) \) \([i.e. (1.4)]\) with \( L \) replaced by \( \mathcal{L} \) and \( M \) replaced by \( 2M \) will be of order \( 4M \) in \( \rho^A \). In fact this is not the case as is shown by

**Theorem 4.** If

\[ L = L(\rho^A; \rho^A_{i_1}; \ldots; \rho^A_{i_1i_2 \ldots i_M}; \lambda^\alpha; \lambda^\alpha_{i_1}; \ldots; \lambda^\alpha_{i_1i_2 \ldots i_{M+\sigma}}) \]

and

\[ \mathcal{L} = \rho^A E_A(L) \]

then \( E_A(\mathcal{L}) \) is at most of order \( 2M \) in \( \rho^A \) i.e. \( \mathcal{L} \) is a degenerate Lagrange density.
The Euler-Lagrange expression

Proof. For any positive integer $r$, $2 \leq r \leq M$ consider the quantity

$$S^{i_{r-j}, i_{r-j}}$$

where

$$S^{i_{r-j}, i_{r-j}} = (-1)^{r-j} \rho_{i_0 i_{r-1} \ldots i_r - j+1}^{A} L_{A}^{i_0 \ldots i_r - j+1,i_0 \ldots i_r - j-1}$$

and $1 \leq j \leq r-1$. Clearly we have

$$S^{i_{r-j}, i_{r-j}} = (-1)^{r-j} \rho_{i_0 i_{r-1} \ldots i_r - j}^{A} L_{A}^{i_0 \ldots i_r - j,i_0 \ldots i_r - j-1} + (-1)^{r-j+1} \rho_{i_0 \ldots i_r - j+1}^{A} L_{A}^{i_0 \ldots i_r - j+1,i_0 \ldots i_r - j},$$

from which we conclude that

$$\sum_{j=1}^{r-1} S^{i_{r-j}, i_{r-j}} = - \rho_{i_0 \ldots i_r}^{A} L_{A}^{i_0 \ldots i_r} - (-1)^{r} \rho_{i_r}^{A} L_{A}^{i_0 \ldots i_r - 1}.$$

In view of the fact that the last term on the right hand side can be expressed in the form

$$(-1)^{r+1} \{ \rho_{i_0 \ldots i_r}^{A} \},$$

we find that for $2 \leq r \leq M$

$$(-1)^{r} \rho_{i_0 \ldots i_r}^{A} L_{A}^{i_0 \ldots i_r} = \rho_{i_0 \ldots i_r}^{A} L_{A}^{i_0 \ldots i_r} + \sum_{j=1}^{r-1} S^{i_{r-j}, i_{r-j}} + (-1)^{r} \rho_{i_0 \ldots i_r}^{A} L_{A}^{i_0 \ldots i_r - 1}.$$

Consequently we find

$$\sum_{r=0}^{M} (-1)^{r} \rho_{i_0 \ldots i_r}^{A} L_{A}^{i_0 \ldots i_r} = \sum_{r=0}^{M} \rho_{i_0 \ldots i_r}^{A} L_{A}^{i_0 \ldots i_r} + \sum_{r=2}^{M} \sum_{j=1}^{r-1} S^{i_{r-j}, i_{r-j}} + \sum_{r=1}^{M} (-1)^{r} \rho_{i_0 \ldots i_r}^{A} L_{A}^{i_0 \ldots i_r - 1}\{i_r\},$$

which, in view of (3.1), is of the form

$$\mathcal{L} = \mathcal{L}_1 + T_{i_r}^{i_r}$$

where

$$\mathcal{L}_1 = \sum_{r=0}^{M} \rho_{i_0 \ldots i_r}^{A} L_{A}^{i_0 \ldots i_r}$$

and

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However, by virtue of the fact that \( T^i_{:,i} \) is a divergence, Theorem 3 assures us that it will not contribute to \( E_A(\mathcal{L}) \). Therefore we find from (3.3) that

\[
E_A(\mathcal{L}) = E_A(\mathcal{L}_1).
\]

Furthermore it is clear from (3.4) that

\[
\mathcal{L}_1 = \mathcal{L}_1(\rho^A; \rho^{A,1}; \ldots; \rho^{A,1 \ldots 1_0}; \lambda^a; \lambda_1^a; \ldots; \lambda_{1 \ldots Q}^a),
\]

so that \( E_A(\mathcal{L}) \) is at most of order \( 2M \) in \( \rho^A \). This establishes the theorem.

We have shown that if \( \mathcal{L} \), given by (3.1), is used as a Lagrangian then the corresponding Euler-Lagrange equations are also obtained from \( \mathcal{L}_1 \), given by (3.4). We also know that \( \mathcal{L} \) is a scalar density. It is obviously of interest to establish the tensorial character of \( \mathcal{L}_1 \). This can easily be accomplished as follows. Under (1.1) we have

\[
\sum_{r=0}^{M} \sum_{i_0 \ldots i_r} \rho^B_{,i_0 \ldots i_r} L_A^{i_0 \ldots i_r} = \sum_{r=0}^{M} \sum_{i_0 \ldots i_r} \rho^B_{,i_0 \ldots i_r} L^{a_0 \ldots a_r}_{A_{a_0 \ldots a_r}} = \sum_{j=0}^{M} \left( \sum_{r=0}^{M} \rho^B_{,i_0 \ldots i_r} A_{a_0 \ldots a_r}^{A_{a_0 \ldots a_r}} \right) L^{a_0 \ldots a_r}_{A_{a_0 \ldots a_r}}.
\]

By (2.1) and 2.3) the quantity in brackets on the right hand side is \( \hat{\rho}^A_{,a_0 \ldots a_r} \), which establishes

**Theorem 5.** The Euler-Lagrange expressions obtained from

\[
\mathcal{L} = \rho^A E_A(L)
\]

and

\[
\mathcal{L}_1 = \sum_{r=0}^{M} \rho^A_{,i_0 \ldots i_r} L_A^{i_0 \ldots i_r}
\]

are identical, and, furthermore, both \( \mathcal{L} \) and \( \mathcal{L}_1 \) are scalar densities.

We are now in a position to give a partial answer to the third question posed in section 1. From Theorem 5 we see that if \( \mathcal{L}_1 \) is proportional to \( L \) i.e. if

\[
\mathcal{L}_1 = kL
\]

where \( k \) is a non-zero constant then

\[
E_A(\mathcal{L}) = kE_A(L).
\]

By virtue of the definition of \( \mathcal{L}_1 \) we thus have
Theorem 6. If $L$ is homogeneous of degree $k(\neq 0)$ in the variables $(\rho^A; \rho^A_{,i_1}; \ldots; \rho^A_{,i_1 \ldots i_M})$, i.e. if
\[ \sum_{r=0}^{M} \rho^A_{,i_1 \ldots i_r} L_A^{i_1 \ldots i_r} = k L, \]
then the Lagrange density $\rho^A E_A(L)$ will also have as its Euler-Lagrange equations precisely
\[ E_A(L) = 0. \]

References


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