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Some Rigidity Results Related to Monge–Ampère Functions

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Abstract. The space of Monge–Ampère functions, introduced by J. H. G. Fu, is a space of rather rough functions in which the map $u \mapsto \operatorname{Det} D^2 u$ is well defined and weakly continuous with respect to a natural notion of weak convergence. We prove a rigidity theorem for Lagrangian integral currents that allows us to extend the original definition of Monge–Ampère functions. We also prove that if a Monge–Ampère function u on a bounded set $\Omega \subset \mathbb{R}^2$ satisfies the equation $\operatorname{Det} D^2 u = 0$ in a particular weak sense, then the graph of u is a developable surface, and moreover u enjoys somewhat better regularity properties than an arbitrary Monge–Ampère function of 2 variables.

1 Introduction

The space of Monge–Ampère functions, introduced by J. H. G. Fu [7,8], is the largest known space of functions $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ for which all minors of the Hessian $D^2 u$, including, in particular, the determinant $\det D^2 u$, are well defined as signed Radon measures and weakly continuous in a certain natural sense. This makes it an interesting function space from the point of view of analysis and nonlinear potential theory, and also possibly useful for some problems in the calculus of variations and non-smooth geometry.

Technical restrictions in Fu's work forced him to work with Monge–Ampère functions that are locally Lipschitz. The first goal of this paper is to show that basic properties of Monge–Ampère functions, in particular, an underlying theorem about rigidity of Lagrangian intergal currents, which guarantees that the measures associated with $\det D^2 u$ and with other minors of the Hessian are in some sense canonical, remain valid without this local Lipschitz condition. This is carried out in Theorem 4.1 and its corollaries, and it allows us to expand the space of Monge–Ampère functions to what we believe is its natural generality and to strengthen Fu's weak continuity results. In [13], examples were constructed showing that in n-dimensions there exists a Monge–Ampère function u that is *not* $C^{0,\gamma}_{\mathrm{loc}}$ for any $\gamma > \frac{2}{n+1}$, so the local Lipschitz assumption that we remove is genuinely restrictive.

The other main result of this paper, Theorem 6.1, establishes a rigidity property of a function $u\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ solving the equation $\det D^2 u = 0$, where $\det D^2 u$ is now understood in the sense of Monge–Ampère functions. If $u\colon \Omega \to \mathbb{R}$ is a sufficiently smooth function and $\det D^2 u(x) = 0$ in Ω, then it is a classical fact that the graph of u is a developable surface, in the sense that for every $x \in \Omega$, either u

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is affine in a neighborhood of x or x belongs to a line segment that intersects $\partial\Omega$ at both ends and along which Du is constant. This was proved by Hartman and Nirenberg [11] for $u \in C^2$ and by Kirchheim [15] when $u \in W^{2,\infty}$. Pakzad [16] showed that Kirchheim's proof can be extended to $u \in W^{2,2}$ via a lemma that shows that if $u \in W^{2,2}(\Omega)$ solves the equation $\det D^2u = 0$, then u is C^1. Pogorelov [17] established a similar developability property for functions that are merely C^1, without any kind of condition about $\det D^2u$, but assuming that the image of the gradient map has Lebesgue measure 0. Here we prove an analogous developability result when u is merely a Monge–Ampère function. In this case u need not be C^1, in fact the gradient Du is in general merely a function of bounded variation, so that the statement "Du is constant along a line segment" here means that either every point of the segment is a Lebesgue point, or every point is a jump point of Du, with the same jump at all points on the segment. The theorem thus implies a modest but optimal regularity property: *every* point of Ω is either a Lebesgue point of Du or belongs to the jump set of Du.

A Monge–Ampère function $u\colon \mathbb{R}^n \to \mathbb{R}$ is defined in terms of an n-dimensional integral current in $\mathbb{R}^n \times \mathbb{R}^n$, denoted $[du]$, that can be thought of as a generalized graph of the gradient Du. This current is required to be Lagrangian with respect to the canonical symplectic form, see (2.6); roughly speaking, this means that it is weakly curl-free. It should be noted that in the language of Cartesian currents, (see Giaquinta, Modica, Souček [10]) a Monge–Ampère function u is precisely a function whose gradient supports a *Lagrangian* Cartesian current. Thus, many of our results can be stated as theorems about Cartesian currents. For example, Corollary 4.2 implies that a Lagrangian Cartesian current is uniquely determined by its support function.

The measures associated with minors of the Hessian are defined using this current $[du]$ (see (4.2)); this is motivated by the fact, recalled in (2.4), that if u is smooth, one can recover all minors of the Hessian by integrating suitable n-forms over the graph of the gradient.

1.1 Some Related Work

Fu established a rigidity result for Legendrian currents, as a corollary of his result about Lagrangian currents [7]. This Legendrian version of the theorem has subsequently been used in a number of applications, including works that develop a theory of curvature measures for a number of classes of rather irregular subsets of Euclidean space, including subanalytic sets [9], Lipschitz manifolds [18], and o-minimal sets [3]. These works rely on the notion of a normal cycle, which is a Legendrian current in $\mathbb{R}^n \times S^{n-1}$ that bears roughly the same relation to the graph of the Gauss map as the current $[du]$ associated with a Monge–Ampère function has to the graph of the gradient. The uniqueness theorem for Legendrian cycles has also been used for further developments of general theory related to Legendrian cycles (see for example [2]) and in problems, arising in computational geometry, relating to estimating curvatures in polygonal approximations of smooth surfaces (see [4]). By contrast, the Lagrangian version of the theorem, and corresponding results about Monge–Ampère functions and weak continuity of minors of the Hessian, have to date received less attention. In fact, the authors of several recent papers [6, 12, 14] that discuss the weak continuity of the map $u \mapsto \det D^2u$ seem to be unaware of Fu's earlier work: this is certainly the case for the work of N. Jung and the author [14]. These papers interpret $\det D^2u$ in the sense of distributions, which has the advantage of making it possible to extend the theory to certain functions for which $\det D^2u$ is not a measure. In situations where it is natural to require that $\det D^2u$ be a measure, however, the geometric measure theory framework of Monge–Ampère functions (equivalently, Lagrangian Cartesian currents) yields sharper results.

We finally mention the note [13], which presents some examples and attempts to give an elementary treatment of some aspects of Monge–Ampère functions.

1.2 Organization of This Paper

In Section 2 we recall some background and fix some notation. The definition of Monge–Ampère functions is given in Section 2.5. Section 3 contains some general geometric measure theory results that are used throughout the rest of the paper.

Section 4 contains the proof of our version of Fu's rigidity theorem for Lagrangian currents. This theorem says, heuristically, that a Lagrangian integral current in $\mathbb{R}^n \times \mathbb{R}^n$ with no boundary and with finite mass in sets of the form $K \times \mathbb{R}^n$, K compact, is uniquely determined by its "most horizontal part." We also deduce some corollaries, including results about the weak continuity of the map $u \mapsto \det D^2u$, as well as corresponding results for other minors of the Hessian. In particular, the definition of $\det D^2u$ for a Monge–Ampère function u is given in (4.2); see also Remark 3, where the relation between our notion of $\det D^2u$ and the distributional determinant of the Hessian is discussed.

Section 5 establishes a result that completely characterizes certain 1-dimensional slices of the current $[du]$ associated with a Monge–Amère function u, see Proposition 5.3. This is essentially equivalent to a description of that part of the currrent $[du]$ corresponding to the second derivatives of u, *i.e.,* the 1×1 minors of D^2u (see Proposition 5.1). These rather technical results are used in Section 6 in the proof of our second main result, Theorem 6.1, which shows that if $u \colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a weak solution (in the sense of Monge–Ampère functions) of the equation $\det D^2u = 0$, then the graph of u is a developable surface in the sense described above.

2 Notation and Background

2.1 General Notation

If μ is a Radon measure, we write $\mu \geq 0$ to mean that $\mu(A) \geq 0$ for all measurable A, and we write $\mu_1 \leq \mu_2$ when $\mu_2 - \mu_1 \geq 0$.

We write $I(k, n) := \{\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}^k : 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}$. For $\alpha \in I(k, n)$ we write $\bar{\alpha}$ to denote the element of $I(n-k, n)$ with the property that $(\alpha_1, \ldots, \alpha_k, \bar{\alpha}_1, \ldots, \bar{\alpha}_{n-k})$ is a permutation of $(1, \ldots, n)$. We write $\operatorname{sgn}(\alpha, \bar{\alpha})$ to denote the sign of the permutation. We write 0 to denote the unique element of $I(0, n)$, and $\bar{0} := (1, \ldots, n)$. If $\alpha \in I(k, n)$ is a multiindex, then $|\alpha| = k$.

If $A = (a_{i\,j})_{i,j=1}^n$ is an $n \times n$ matrix and $\alpha, \beta \in I(k, n)$, then $M^{\alpha\beta}(A) = \det[A^{\alpha\beta}]$,

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where $A^{\alpha\beta}$ is the $k \times k$ matrix whose i, j entry is $a_{\alpha_i \beta_j}$. We use the convention $M^{00}(A) = 1$.

We write $B_r(a)$ to denote the *open* ball $\{x : |x - a| < r\}$; the ambient space is normally clear from the context.

We often implicitly sum over repeated indices. However, when we sum over multiindices, we normally indicate the sum explicitly.

2.2 Geometric Measure Theory Notation

We assume some familiarity with basic definitions and facts of geometric measure theory, such as currents, rectifiability, the coarea formula, and properties of functions of bounded variation. Here we recall some notions that will be used often, and we point out some ways in which our notation differs from that found in standard references such as Federer [5].

If $v \colon \Omega \to \mathbb{R}^\ell$ is a BV function, then we write $\mathcal{J}_v$ to denote the jump set of v, and for $x \in \mathcal{J}_v$, we write $v^+(x), v^-(x)$ to denote the approximate limits of v on the two sides of $\mathcal{J}_v$(see [1, Proposition 3.69].)

We say "j-rectifiable", or if no confusion can result, simply "rectifiable" to mean what Federer (see [5, 3.2.14]) calls "countably $(\mathcal{H}^j, j)$ rectifiable."

We follow convention and write $\|T\|$ to denote the total variation measure associated with a current T of locally finite mass.

If Γ is an ℓ-rectifiable subset of some Euclidean space $\mathbb{R}^M$, and $m \colon \Gamma \to (0, \infty)$ and $\tau \colon \Gamma \to \wedge_\ell \mathbb{R}^M$ are $\mathcal{H}^\ell$ measurable, locally integrable functions such that $\tau(x)$ is a simple ℓ-vector associated with the approximate tangent space $T_x\Gamma$ at $\mathcal{H}^\ell$ a.e. $x \in \Gamma$, then we write $\underline{\mathrm{T}}(\Gamma, m, \tau)$ to denote the current defined by

$$\underline{\mathrm{T}}(\Gamma, m, \tau)(\phi) = \int_\Gamma \langle \phi(x), \tau(x) \rangle m(x)\, d\mathcal{H}^\ell(x). \tag{2.1}$$

A current of the form (2.1) is said to be *rectifiable*. If $|\tau| = 1$ and $m(x) \in \mathbb{Z}$ for $\mathcal{H}^\ell$ a.e. $x \in \Gamma$, then we say that $\underline{\mathrm{T}}(\Gamma, m, \tau)$ is *integer multiplicity* rectifiable, abbreviated as *i.m.* rectifiable.

If W is a j-rectifiable subset of some $\mathbb{R}^\ell$, and Z is a k-rectifiable subset of $\mathbb{R}^m$, and $F \colon \mathbb{R}^\ell \to \mathbb{R}^m$ is a Lipschitz map such that $F(x) \in Z$ for $\mathcal{H}^j$ almost every $x \in W$, then we will sometimes write $J_{W \to Z}F$ to denote the Jacobian as appearing in the coarea formula; see [5, 3.2.22]. This might be written by Federer as "$J_k f$, where f is the restriction of F to W." We will normally omit the subscripts when no confusion can result.

2.3 Notation Related to Product Space Structure

The setting for most of our results is a product space, which we will often write as $\Omega_h \times \Omega_v$, with Ω_h an open subset of $\mathbb{R}^n$ and Ω_v an open subset of $\mathbb{R}^m$. We will refer to Ω_h and Ω_v as horizontal and vertical, respectively. Except in Section 3, we require that $m = n$. In cases when Ω_v is a Euclidean space, we will often drop the subscripts and simply write $\Omega \subset \mathbb{R}^n$ for the horizontal space, and $\Omega \times \mathbb{R}^m$ for the product space.

We write $p_h\colon \Omega_h \times \Omega_v \to \Omega_h$ and $p_v\colon \Omega_h \times \Omega_v \to \Omega_v$ to denote the natural projections.

We always write $(x_1, \dots, x_n)$ and $(\xi_1 \dots, \xi_m)$ for coordinates on the horizontal space Ω_h and the vertical space Ω_v, respectively. Thus $dx_1, \dots, dx_n$ will denote horizonal covectors, and $d\xi_1, \dots, d\xi_m$ vertical covectors. We also write $\{e_1, \dots, e_n\}$ to denote the standard basis for the tangent space to Ω_h, and $\{\varepsilon_1, \dots, \varepsilon_m\}$ to denote the basis for the tangent space to Ω_v. The bases for the spaces of vectors and covectors are assumed to be dual in the sense that $\langle dx_\alpha \wedge d\xi_\beta, e_\gamma \wedge \varepsilon_\delta\rangle = 1$ if $\alpha = \gamma$ and $\delta = \beta$ and 0 if not. Here, for example, $dx_\alpha := dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_j}$ if $\alpha = (\alpha_1, \dots, \alpha_j) \in I(j, n)$.

We write $\mathcal{D}^\ell(\Omega_h \times \Omega_v)$ to denote the space of all C^∞ ℓ-forms with compact support in $\Omega_h \times \Omega_v$. For a covector $\phi = \sum_{|\alpha|+|\beta|=j+k} \phi^{\alpha\beta} dx_\alpha \wedge d\xi_\beta$, we will write

$$P^{j,k}\phi = \sum_{\substack{|\alpha|=j\\|\beta|=k}} \phi^{\alpha\beta} dx_\alpha \wedge d\xi_\beta. \tag{2.2}$$

For a differential form $\phi \in \mathcal{D}^{j+k}(\Omega_h \times \Omega_v)$, we define $(P^{j,k}\phi)(x,\xi) = P^{j,k}\phi(x,\xi)$. For a current $T \in \mathcal{D}_{j+k}(\Omega_h \times \Omega_v)$, we define $P_{j,k}T(\phi) = T(P^{j,k}\phi)$. Note that $P_{j,k}\underline{\mathrm{T}}(\Gamma, m, \tau) = \underline{\mathrm{T}}(\Gamma, m, P_{j,k}\tau)$, where $P_{j,k}\tau$ is defined as in (2.2). Given a vector $w \in T_{(x,\xi)}(\Omega_h \times \Omega_v)$, we will often use the notation

$$w_h = \sum_{i=1}^n (w \cdot e_i)e_i = P_{1,0}w, \qquad w_v = \sum_{i=1}^m (w \cdot \varepsilon_i)\varepsilon_i = P_{0,1}w. \tag{2.3}$$

For $v \in C^1(\Omega\,; \mathbb{R}^m)$, following Giaquinta, Modica, and Souček [10] we write G_v to denote the current associated with integration over the graph of v.

$$G_v(\phi) = \int_\Omega \sum_{|\alpha|+|\beta|=n} \phi^{\alpha\beta}(x, v(x))\, \mathrm{sgn}(\alpha, \bar\alpha) M^{\bar\alpha\beta}(Dv(x))\, dx. \tag{2.4}$$

Thus $P_{n-k,k}G_v$ encodes the $k \times k$ minors of Dv. If $v \in W^{1,p}$ for $p \geq \min\{m, n\}$, then the above expression still makes sense and $\partial G_v = 0$ in $\Omega \times \mathbb{R}^m$ (see [10]).

We remark that if T is an ℓ-current of locally finite mass in $\Omega \times \mathbb{R}^m$ such that $\|T\|(K \times \mathbb{R}^m) < \infty$ for every compact $K \subset \Omega$, then $T(\phi)$ is well defined whenever ϕ is a smooth ℓ-form with support in $K \times \mathbb{R}^m$, the point being that compact support is not required in the vertical directions. Indeed, let χ_R be a family of functions such that $\chi_R(x,\xi) = 1$ if $|\xi| \leq R$, $\chi_R(x,\xi) = 0$ if $|\xi| \geq 2R$, and $|\nabla_\xi \chi_R| \leq C/R$ and $\nabla_x \chi_R \equiv 0$. Then it is easy to check that $\lim_{R\to\infty} T(\chi_R\phi)$ exists and is independent of the specific choice of $\{\chi_R\}$; this is how we define $T(\phi)$.

For such T it follows that $p_{h\#}T$ is well defined, where $p_{h\#}T(\phi) = T(p_h^\#\phi)$ for $\phi \in \mathcal{D}^\ell(\Omega)$. One can also check that the standard identity $\partial p_{h\#}T = p_{h\#}\partial T$ still holds, as long as $(\|T\| + \|\partial T\|)(K \times \mathbb{R}^m) < \infty$ for K compact. Indeed, if we let χ_R be as above, then

$$\begin{aligned}\partial p_{h\#}T(\phi) = p_{h\#}T(d\phi) &= \lim_{R\to\infty} T(\chi_R p_h^\# d\phi) = \lim_{R\to\infty} T(\chi_R d p_h^\#\phi)\\ &= \lim_{R\to\infty} [T(d(\chi_R p_h^\#\phi)) - T((d\chi_R) p_h^\#\phi)] = p_{h\#}\partial T.\end{aligned}$$

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2.4 Lagrangian Currents

We write ω to designate the standard symplectic form $\omega = \sum_{i=1}^n dx_i \wedge d\xi_i$ on $\mathbb{R}^n \times \mathbb{R}^n$ or any open subset thereof, so that if v, w are vectors, then

$$\omega(v \wedge w) = \sum_{i=1}^{n} [(v \cdot e_i)(w \cdot \varepsilon_i) - (v \cdot \varepsilon_i)(w \cdot e_i)] = v \cdot \mathbb{J}w,$$

where $\mathbb{J}\varepsilon_i = e_i, \mathbb{J}e_i = -\varepsilon_i$. For $n \geq 2$, an n-plane P in $\mathbb{R}^n \times \mathbb{R}^n$ is said to be *Lagrangian* if $\langle \omega, \tau \wedge \tau' \rangle = 0$ for any two vectors τ, τ' tangent to P. (When $n = 1$ we consider every 1-plane in $\mathbb{R} \times \mathbb{R}$ to be Lagrangian.) A rectifiable n-current in $\mathbb{R}^n \times \mathbb{R}^n$ is Lagrangian if $\mathcal{H}^n$ a.e. approximate tangent plane is Lagrangian.

Note that the current G_v associated as in (2.4) with a smooth map $v \colon \Omega_h \to \mathbb{R}^n$ is Lagrangian if and only if $v^i_{x_j} = v^j_{x_i}$ for all i, j. This is not hard to check. In particular, if $u \colon \Omega_h \to \mathbb{R}$ is a smooth function, then G_{Du} is always Lagrangian.

An alternate definition is sometimes given, whereby if U is an open subset of $\mathbb{R}^n \times \mathbb{R}^n$, $n \geq 2$, then a current $T \in \mathcal{D}_n(U)$ is said to be *Lagrangian* if $T(\omega \wedge \eta) = 0$ for every $\eta \in \mathcal{D}^{n-2}(U)$. It is clear that this condition is preserved under weak convergence.

The alternate definition makes sense for currents that are not necessarily rectifiable, and it agrees with the one we have given for rectifiable currents. We sketch the well-known argument: recall that

$$\langle \omega \wedge \eta, \tau_1 \wedge \cdots \wedge \tau_n \rangle = \sum_{\alpha \in I(2,n)} \operatorname{sgn}(\alpha, \bar{\alpha}) \langle \omega, \tau_{\alpha_1} \wedge \tau_{\alpha_2} \rangle \langle \eta, \tau_{\bar{\alpha}_1} \wedge \cdots \wedge \tau_{\bar{\alpha}_{n-2}} \rangle = 0$$

for every $n-2$-covector η. From this it is not hard to see that an n-plane is Lagrangian if and only if $\langle \omega \wedge \eta, \tau \rangle = 0$ for every $n-2$-covector η and every orienting n-vector τ. Then the equivalence of the two definitions (whenever both make sense) can be verified by rather standard measure theoretic arguments.

2.5 Definition of Monge–Ampère Functions

If Ω is an open subset of $\mathbb{R}^n$, then $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is said to be a Monge–Ampère function if there exists an n-dimensional i.m. rectifiable current $[du]$ in $\Omega \times \mathbb{R}^n$ such that

$$\partial[du] = 0, \tag{2.5}$$

$$[du] \text{ is Lagrangian}, \tag{2.6}$$

$$\|[du]\|(K \times \mathbb{R}^n) < \infty \text{ whenever } K \subset \Omega \text{ is compact}, \tag{2.7}$$

$$[du](\phi\, dx_1 \wedge \cdots \wedge dx_n) = \int \phi(x, Du(x))\, dx, \tag{2.8}$$

for every $\phi \in C^\infty_c(\Omega \times \mathbb{R}^n)$. If u is, for example, C^2, then the current G_{Du} associated with integration over the graph of Du satisfies these conditions. For example, in this

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case (2.8) is an immediate consequence of (2.4). Property (2.5) and rectifiability are well known (see for example [7, 10].

The earlier work of Fu [7, 8] gave a different definition of Monge–Ampère functions in which (2.7) was replaced by the stronger condition that $[du]$ be "locally vertically bounded", which can only hold if u is locally Lipschitz. The terminology here also differs slightly from that used in [13], where we reserved the term *Monge–Ampère* for functions u such that $\mathbf{M}([du]) < \infty$; functions satisfying (2.5)–(2.8) were called *locally Monge–Ampère* functions.

If u is a Monge–Ampère function, then Du has locally bounded variation (see [7] for the proof). Examples of Monge–Ampère functions include convex functions, or more generally functions of the form $\min\{u_1, \dots, u_k\}$ where $u_1, \dots, u_k$ are semiconvex (see again [7]). From these examples it follows that Monge–Ampère functions need not belong to $W^{2,p}$ for any $p \geq 1$.

3 Decomposition of a Stratum of T

In this section we assume Ω_h is an open subset of $\mathbb{R}^n$, Ω_v is an open subset of $\mathbb{R}^m$, and j, k are nonnegative integers with $j \leq n$ and $k \leq m$. We first state a lemma that assembles some results from [5] and fixes some notation that we will use in this section.

Lemma 3.1 *Assume that $T = \underline{T}(\Gamma, m, \tau)$ is a rectifiable $j+k$-current in $\Omega_h \times \Omega_v \subset \mathbb{R}^n \times \mathbb{R}^m$ with $\mathbf{M}(T) < \infty$, and define*

$$\Gamma_h := \{x \in \Omega_h : \mathcal{H}^k(p_h^{-1}(x) \cap \Gamma) > 0\}. \tag{3.1}$$

Then Γ_h is j-rectifiable, and the following hold for $\mathcal{H}^j$ a.e. $x \in \Gamma_h$. First, we can write $\tau(x, \xi)$ in the form

$$\tau(x, \xi) = (\tau_v^1 + \tau_h^1) \wedge \cdots \wedge (\tau_v^j + \tau_h^j) \wedge \tau_v^{j+1} \wedge \cdots \wedge \tau_v^{j+k} \tag{3.2}$$

for $\mathcal{H}^k$ a.e. $\xi \in p_h^{-1}(x) \cap \Gamma$, using the notation (2.3), where $\{\tau_h^i\}_{h=1}^j$ are orthogonal and $\{\tau_v^i\}_{i=j+1}^n$ are orthonormal, and $\tau^i \perp \tau_v^{i'}$ for all $i \leq j$ and $i' > j$. Next, if we write

$$\Gamma_v(x) := \{(x, \xi) \in p_h^{-1}(x) \cap \Gamma : J_{\Gamma \to \Gamma_h} p_h(x, \xi) > 0\}, \tag{3.3}$$

then $\Gamma_v(x)$ is k-rectifiable, and at $\mathcal{H}^k$ a.e. point in $\Gamma_v(x)$,

$$\operatorname{span}\{\tau_h^1, \dots, \tau_h^j\} = T_x\Gamma_h, \qquad \operatorname{span}\{\tau_v^{j+1}, \dots, \tau_v^{j+k}\} = T_{(x,\xi)}\Gamma_v(x). \tag{3.4}$$

Finally,

$$J_{\Gamma \to \Gamma_h} p_h(x, \xi) = |\tau_h^1 \wedge \cdots \wedge \tau_h^j|. \tag{3.5}$$

The proof, consisting mostly of a string of references to Federer [5], is given at the end of this section.

The main result of this section is the following.

Proposition 3.2 *Assume that $T = \underline{T}(\Gamma, m, \tau)$ is an i.m. rectifiable $j + k$-current in $\Omega_h \times \Omega_v$, and that $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$. Further assume that $P_{j+1,k-1}T = 0$.*

Define Γ_h and $\Gamma_v(x)$ as in Lemma 3.1, and let $\tau_h \colon \Gamma_h \to \wedge_j \mathbb{R}^n$ be a measurable map such that $\tau_h(x)$ is for $\mathcal{H}^j$ a.e. x a unit simple j-vector associated with $T_x\Gamma_h$. Then for $\mathcal{H}^j$ a.e. $x \in \Gamma_h$, there exists an i.m. rectifiable k-current V_x in $\Omega_h \times \Omega_v$ carried by $\Gamma_v(x)$, such that if $\alpha \in I(j,n)$, $\beta \in I(k,m)$, and $\phi \in C_c^\infty(\Omega_h \times \Omega_v)$, then the map $x \mapsto V_x(\phi\, d\xi_\beta)$ is $\mathcal{H}^j \llcorner \Gamma_h$ integrable, and

$$T(\phi\, dx_\alpha \wedge d\xi_\beta) = \int_{\Gamma_h} V_x(\phi\, d\xi_\beta)\langle dx_\alpha, \tau_h(x)\rangle\, d\mathcal{H}^j(x). \tag{3.6}$$

If in addition $\partial T = 0$ in $\Omega_h \times \Omega_v$, then $\partial V_x = 0$ in $\Omega_h \times \Omega_v$ for $\mathcal{H}^j$ a.e. $x \in \Gamma_h$. Finally, if A is any Borel subset of $\Omega_h \times \Omega_v$, then

$$\int_{\Gamma_h} \|V_x\|(A)\mathcal{H}^j(dx) \le \|T\|(A). \tag{3.7}$$

If $j = n$, then the condition $P_{j+1,k-1}T = 0$ is automatically satisfied, and for $\mathcal{L}^n$ a.e. $x \in \Omega_h$, the current V_x is just a slice $\langle T, p_h, x\rangle$ of T by a level set of the projection $p_h \colon \Omega_h \times \Omega_v \to \Omega_h$.

Remark 1 One can use the rectifiable slices theorem of B. White [20] to show that $P_{j,k}T$ can be identified with a rectifiable flat j-chain in Ω_h with coefficients in the (normed abelian) group of flat k-chains in Ω_v. In fact this argument can be used to prove Proposition 3.2, although that proof would be more difficult than the one we give here. Note that on a purely formal level, the expression on the right-hand side of (3.6) looks like a rectifiable j-current in Ω_h, carried by the set Γ_h, oriented by $\tau_h(\xi)$, and with "multiplicity" at $x \in \Gamma_h$ given by V_x.

For the proof of the proposition we will need the following.

Lemma 3.3 *Assume that $T = \underline{T}(\Gamma, m, \tau)$ is an i.m. rectifiable $j+k$-current in $\Omega_h \times \Omega_v$, and that $\mathbf{N}(T) := \mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$. Further assume that $P_{j+\ell,k-\ell}T = 0$ for all $\ell \ge 1$. Then $\|T\| = \|T\| \llcorner (\Gamma_h \times \Omega_v)$, for Γ_h as defined in* (3.1) *above.*

The conclusion of the lemma can fail if we do not assume $\mathbf{M}(\partial T) < \infty$ or if for example $P_{j+\ell,k-\ell}T \ne 0$ for some $\ell \ge 1$.

Proof First suppose that $j = 0$, in which case the hypotheses imply that $T = P_{0,k}T$. We may suppose that $k \ge 1$, since the conclusion is clear if $j = k = 0$. It suffices to show that there exists a finite or countable set $\{x^i\} \subset \Omega_h$ such that

$$T = T \llcorner (\cup\{x^i\} \times \Omega_v), \tag{3.8}$$

since then the definition (3.1) implies that $\Gamma_h \subset \{x_i\}$. As remarked by Fu [7], (3.8) follows from Lemma 3.3 in Solomon [19]. We recall the argument for the reader's convenience, and because we will need it later: first, using [5, 4.2.25] and the assumption that T is integer multiplicity, we can write T as a countable sum $T = \sum T_i$ of indecomposable components, with $\mathbf{N}(T) = \sum \mathbf{N}(T_i)$. Solomon's lemma asserts that if f is any Lipschitz real-valued function such that $\langle T, f, r\rangle = 0$ for a.e. $r \in \mathbb{R}$, then each T_i is supported in a level set of f. Let $f_q(x,\xi) = e_q \cdot x = x_q$ for $q \in \{1,\dots,n\}$. Then for every $\phi \in \mathcal{D}^{k-1}(\Omega_h \times \Omega_v)$

$$\int \langle T, f_q, r\rangle(\phi)\, dr = T(dx_q \wedge \phi) = 0,$$

since $P_{0,k}T = T$. It follows that $\langle T, f_q, r\rangle = 0$ for a.e. r, and hence that for each T_i, there exists some $r^i_1,\dots,r^i_n$ such that T_i is supported in $\bigcap_{q=1}^n f_q^{-1}(r^i_q)$. In other words, if we write $x^i = (r^i_1,\dots,r^i_n)$, then $T_i = T_i \llcorner (\{x^i\} \times \Omega_v)$, establishing the lemma when $j = 0$.

Next we prove the lemma for arbitrary positive $j \le n$. For $\alpha \in I(j,n)$, let $p_\alpha(x,\xi) = (x_{\alpha_1},\dots,x_{\alpha_j})$. For every α and a.e. $y \in \mathbb{R}^j$, the slice $\langle T, p_\alpha, y\rangle$ is a k-dimensional rectifiable current with finite mass and finite boundary mass, and it is easy to check that $P_{\ell,k-\ell}\langle T, p_\alpha, y\rangle = 0$ for all $\ell \ge 1$. Thus the $j = 0$ case implies that $\langle T, p_\alpha, y\rangle$ has the form (3.8). In particular the definition (3.1) of Γ_h then implies that $\langle T, p_\alpha, y\rangle$ is supported in $\Gamma_h \times \Omega_v$. Then for $B = (\Omega_h \setminus \Gamma_h) \times \Omega_v$ we have

$$\mathbf{M}(T \llcorner B) \le \sum_{\alpha \in I(j,n)} \int_{\mathbb{R}^j} \mathbf{M}(\langle T, p_\alpha, y\rangle \llcorner B)\, d\mathcal{L}^j(y) = 0.$$

This completes the proof of the lemma. ■

Proof of Proposition 3.2 In this proof we write Jp_h instead of $J_{\Gamma\to\Gamma_h}p_h$. We use notation from Lemma 3.1 throughout.

It is convenient initially to assume that T satisfies the hypotheses of Lemma 3.3, which are stronger than those of the theorem in that we require $P_{j+\ell,k-\ell}T = 0$ for all $\ell \ge 1$, rather than only for $\ell = 1$. This assumption will be relaxed in Step 3.

Step 1. Let us write $\Gamma^* := \{(x,\xi) \in \Gamma : Jp_h(x,\xi) > 0\}$, so that $\Gamma_v(x) = \Gamma^* \cap p_h^{-1}(x)$. Lemma 3.1 implies that $\mathcal{H}^{j+k}$ a.e. in Γ^*,

$$\frac{\tau^1_h \wedge \cdots \wedge \tau^j_h}{|\tau^1_h \wedge \cdots \wedge \tau^j_h|} = \frac{\tau^1_h \wedge \cdots \wedge \tau^j_h}{Jp_h}$$

is a unit simple vector orienting $T_x\Gamma_h$, so there exists some function $\sigma(x,\xi)\colon \Gamma^* \to \pm 1$ such that

$$\frac{\tau^1_h \wedge \cdots \wedge \tau^j_h(x,\xi)}{Jp_h(x,\xi)} = \sigma(x,\xi)\tau_h(x)$$

a.e. in Γ^*. Next, let us write $\tau_v(x,\xi) := \sigma(x,\xi)\tau^{j+1}_v \wedge \cdots \wedge \tau^n_v$. Then if $|\alpha| = j$, $|\beta| = k$,

$$\begin{aligned}(3.9)\qquad \langle dx_\alpha \wedge d\xi_\beta, \tau\rangle &= \langle dx_\alpha \wedge d\xi_\beta, P_{j,k}\tau\rangle \\ &= \langle dx_\alpha \wedge d\xi_\beta, \tau^1_h \wedge \cdots \wedge \tau^j_h \wedge \tau^{j+1}_v \wedge \cdots \wedge \tau^n_v\rangle \\ &= Jp_h(x,\xi)\langle dx_\alpha, \tau_h\rangle\langle d\xi_\beta, \tau_v\rangle.\end{aligned}$$

Lemma 3.3 implies that $\mathcal{H}^{j+k}$ almost all of Γ is contained in $\Gamma_h \times \Omega_v$, so that p_h maps almost all of Γ into Γ_h. Using the coarea formula [5, 3.2.22] and (3.9) we can write, still for $|\alpha| = j, |\beta| = k$,

$$
\begin{aligned}
(3.10) \qquad & T(\phi\, dx_\alpha \wedge d\xi_\beta) \\
&= \int_\Gamma \phi(x,\xi)\langle dx_\alpha \wedge d\xi_\beta, \tau(x,\xi)\rangle m(x,\xi) d\mathcal{H}^{j+k}(x,\xi) \\
&= \int_{\Gamma_h} \int_{\Gamma^* \cap p_h^{-1}(x)} \phi \frac{\langle dx_\alpha \wedge d\xi_\beta, \tau(x,\xi)\rangle}{Jp_h(x,\xi)} m d\mathcal{H}^k(\xi) d\mathcal{H}^j(x) \\
&= \int_{\Gamma_h} \Big(\int_{\Gamma_v(x)} \phi(x,\xi)\langle d\xi_\beta, \tau_v\rangle m(x,\xi)\, d\mathcal{H}^k(\xi) \Big) \langle dx_\alpha, \tau_h\rangle\, d\mathcal{H}^j(x).
\end{aligned}
$$

Recall from Lemma 3.1 that for $\mathcal{H}^j$ a.e. $x \in \Gamma_h$, $\Gamma_v(x)$ is $\mathcal{H}^k$-rectifiable and $\tau_v(x,\xi)$ is a unit simple tangent vector orienting $T_{(x,\xi)}\Gamma_v(x)$ for $\mathcal{H}^k$ a.e. $(x,\xi) \in \Gamma_v(x)$. So the current V_x defined by

$$
(3.11) \qquad V_x(\psi d\xi_\beta) := \int_{\Gamma_v(x)} \psi(x,\xi)\langle d\xi_\beta, \tau_v\rangle m(x,\xi)\, d\mathcal{H}^k(\xi)
$$

is rectifiable for $\mathcal{H}^j$ a.e. x. Combining (3.10) and (3.11), we have proved (3.6). Note that $x \mapsto V_x(\phi\, d\xi_\beta)$ is $\mathcal{H}^j \llcorner \Gamma_h$ -integrable; this too is a consequence of the coarea formula [5, 3.2.22].

Step 2.

Note that T is i.m. rectifiable if and only if $m(x,\xi)$ is an integer $\mathcal{H}^{j+k}$ almost everywhere, and similarly V_x is i.m. rectifiable for $\mathcal{H}^j$ a.e. x if and only if for $\mathcal{H}^j$ a.e. x, $m(x,\xi)$ is an integer for $\mathcal{H}^k$ a.e. ξ. By the coarea formula again, the former condition implies the latter.

Step 3. We now prove that the same conclusions remain valid if we merely assume that $P_{j+1,k-1}T = 0$. If this holds, define $T' = \sum_{\ell \le 0} P_{j+\ell,k-\ell}T$. We claim T' verifies all the hypotheses of Steps 1 and 2 above. It is clear that $P_{j+\ell,k-\ell}T' = 0$ for every $\ell \ge 1$ and that $\mathbf{M}(T') \le \mathbf{M}(T) < \infty$. To see that $\mathbf{M}(\partial T') < \infty$, one checks from the definitions (and using the hypothesis $P_{j+1,k-1}T = 0$ for the third equality below) that if we write $P'\phi = \sum_{\ell \le 0} P^{j+\ell,k-\ell-1}\phi$, then

$$
\partial T'(\phi) = T'(d\phi) = T'(dP'\phi) = T(dP'\phi) = \partial T(P'\phi).
$$

Since $|P'\phi(x)| \le |\phi(x)|$ at every x, this implies that $\mathbf{M}(\partial T') \le \mathbf{M}(\partial T)$. In addition, we prove in Lemma 3.4 below that T' is i.m. rectifiable, so we have checked all the relevant hypotheses.

Let us write Γ' to denote the set that carries T'. Clearly $\Gamma' \subset \Gamma$ up to a set of $\mathcal{H}^{j+k}$ measure zero. Applying Steps 1 and 2 to T', we find that for $\mathcal{H}^j$ a.e. $x \in \Gamma'_h := \{x \in \Omega_h : \mathcal{H}^k(p_h^{-1}(x) \cap \Gamma') > 0\}$ there exists a current V'_x such that (3.6) holds with T, Γ_h, V_x replaced by T', Γ'_h, V'_x.

Since $\Gamma' \subset \Gamma$, it is clear that $\Gamma'_h \subset \Gamma_h$. It is clear from the definition of T' that $P_{j,k}T = P_{j,k}T'$, so if we define

$$V_x = \begin{cases} V'_x & \text{if } x \in \Gamma'_h, \\ 0 & \text{if not,} \end{cases}$$

then (3.6) is satisfied by T.

Step 4. We now prove that

$$\text{if } \partial T = 0 \text{ in } \Omega_h \times \Omega_v, \text{ then } \partial V_x = 0 \text{ in } \Omega_h \times \Omega_v \text{ for } \mathcal{H}^j \text{ a.e. } x. \tag{3.12}$$

To see this, fix $\psi \in \mathcal{D}^{j+k-1}(\Omega_h \times \Omega_v)$ of the form $\psi(x,\xi) = f(x)g(x,\xi)dx_\alpha \wedge d\xi_\beta$ where $|\alpha| = j$ and $|\beta| = k-1$. The fact that $P_{j+1,k-1}T = 0$ and (3.6) imply that

$$\begin{aligned} 0 = \partial T(\psi) = T(d\psi) &= T\Big(\sum_\ell f g_{\xi_\ell} d\xi_\ell \wedge dx_\alpha \wedge d\xi_\beta\Big) \\ &= (-1)^j \int_{\Gamma_h} V_x\Big(\sum_\ell f g_{\xi_\ell} d\xi_\ell \wedge d\xi_\beta\Big) \langle dx_\alpha, \tau_h(x)\rangle d\mathcal{H}^j(x). \end{aligned}$$

Similarly, since V_x is supported in $\{x\} \times \Omega_h$ with $P_{1,k-1}V_x = 0$,

$$V_x\Big(\sum_\ell f g_{\xi_\ell} d\xi_\ell \wedge d\xi_\beta\Big) = f(x)\, \partial V_x(g\, d\xi_\beta).$$

It follows that

$$\int_{\Gamma_h} \partial V_x(g\, d\xi_\beta)\langle dx_\alpha, \tau_h(x)\rangle f(x)\, d\mathcal{H}^j(x) = 0$$

for all f, α as above, and hence that $\partial V_x(g\, d\xi^\beta) = 0$ for $\mathcal{H}^j$ a.e. $x \in \Gamma_h$. Since g, β were arbitrary, linearity and the fact that $V_x = P_{0,k}V_x$ imply that for every $\psi \in \mathcal{D}^{k-1}(\Omega_h \times \Omega_v)$,

$$\partial V_x(\psi) = 0 \text{ for } \mathcal{H}^j \text{ a.e. } x \in \Gamma_h. \tag{3.13}$$

Now let $\{\psi^q\}_{q=1}^\infty$ be a countable dense subset of $\mathcal{D}^{k-1}(\Omega_h \times \Omega_v)$. In view of (3.13), there is a subset of Γ_h of full $\mathcal{H}^j$ measure, in which $\partial V_x(\psi^q) = 0$ for every q. By density, $\partial V_x = 0$ at every x in this set, proving (3.12).

Step 5. Finally, to verify (3.7), note from (3.5) that $Jp_h \le 1$ a.e. in Γ, so that the coarea formula implies that

$$\begin{aligned} \int_{\Gamma_h} \|V_x\|(A)\mathcal{H}^j(dx) &= \int_{\Gamma_h}\int_{\Gamma_v(x)\cap A} m(x,\xi)\mathcal{H}^k(d\xi)\mathcal{H}^j(dx) \\ &= \int_{\Gamma \cap A} m(x,\xi)\, Jp_h(x,\xi)\mathcal{H}^{j+k}(dx\, d\xi) \\ &\le \int_{\Gamma\cap A} m(x,\xi)\mathcal{H}^{j+k}(dx\, d\xi) = \|T\|(A). \end{aligned}$$

This completes the proof of the proposition. ∎

We conclude this section with the proof of Lemma 3.1.

Proof of Lemma 3.1 The rectifiability of Γ_h is a special case of [5, 3.2.31]. It then follows from [5, 3.2.22] that for $\mathcal{H}^j$ a.e. $x \in \Gamma_h$, at $\mathcal{H}^k$ a.e. $\xi \in p_h^{-1}(x) \cap$ a.e.Γ, the approximate tangent space $T_{(x,\xi)}\Gamma$ is mapped by p_h into a set of dimension at most j. The expression (3.2) for τ follows at such points by elementary linear algebra: given a basis $\tilde\tau^1, \dots, \tilde\tau^{j+k}$ for $T_{(x,\xi)}\Gamma$, we can construct a new basis $\{\tau^i\}$ with the property that the horizontal parts $\{\tau_h^i\}$ are orthogonal. Since the horizontal parts span a space of dimension at most j, they can now be nonzero for at most j of the resulting vectors. The remaining vectors, which are necessarily purely vertical, can be taken to be orthogonal by a similar argument, and can further be taken to be orthogonal to $\operatorname{span}(\tau_v^1, \dots, \tau_v^j)$. We obtain (3.2) after relabelling and normalizing suitably.

Next, (3.4) follows by writing conclusions from [5, 3.2.22 (1); 4.3.8 (3)] in terms of the basis appearing on the right-hand side of (3.2).

Lastly, to compute Jp_h, let $\{\tilde\tau^i\}_{i=1}^{j+k}$ be an orthonormal basis for $T_{(x,\xi)}\Gamma$ such that $\tilde\tau^1 \wedge \cdots \wedge \tilde\tau^j = (\tau_h^1 + \tau_v^1) \wedge \cdots \wedge (\tau_h^j + \tau_v^j)$, and with $\tilde\tau^i = \tau^i$ for $i > j$. Then $dp_h(\tilde\tau^i) = \tilde\tau_h^i = 0$ for $i > j$, so the Cauchy–Binet formula yields

$$
\begin{aligned}
Jp_h(x,\xi)^2 &= \sum_{\alpha \in I(j,j+k)} |dp_h(\tilde\tau^{\alpha_1}) \wedge \cdots \wedge dp_h(\tilde\tau^{\alpha_j})|^2 \\
&= |dp_h(\tilde\tau^1) \wedge \cdots \wedge dp_h(\tilde\tau^j)|^2 = |dp_h(\tau^1) \wedge \cdots \wedge dp_h(\tau^j)|^2 \\
&= |\tau_h^1 \wedge \cdots \wedge \tau_h^j|^2
\end{aligned}
$$

which is (3.5). ∎

Finally, we have the following.

Lemma 3.4 *Assume that $T = \underline{\tau}(\Gamma, m, \tau)$ is an i.m. rectifiable $j+k$-current in $\Omega_h \times \Omega_v$, and that $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$. Further assume that $P_{j+1,k-1}T = 0$. Let $T' = \sum_{\ell \le 0} P_{j+\ell,k-\ell}T$. Then T' is i.m. rectifiable.*

Proof *Step 1.* We first claim that if $\eta = \eta^1 \wedge \cdots \wedge \eta^{j+k}$ is any nonzero simple vector such that $P_{j+1,k-1}\eta = 0$, then either $\sum_{\ell \ge 2} P_{j+\ell,k-\ell}\eta = 0$ or $\sum_{\ell \le 0} P_{j+\ell,k-\ell}\eta = 0$.

To prove this, assume toward a contradiction that both $\sum_{\ell \ge 2} P_{j+\ell,k-\ell}\eta \neq 0$ and $\sum_{\ell \le 0} P_{j+\ell,k-\ell}\eta \neq 0$. We write $\eta^i = \eta_h^i + \eta_v^i$, where η_h^i is the horizontal part of η^i. By a Gram–Schmidt argument we can assume that $\eta_h^i \cdot \eta_h^j = 0$ if $i \neq j$. For $\alpha \in I(q, j+k)$ we write $\eta_h^\alpha = \eta_h^{\alpha_1} \wedge \cdots \wedge \eta_h^{\alpha_q}$ and similarly $\eta_v^{\bar\alpha}$. In this notation, $\eta = \sum_{q=0}^{j+k} \sum_{|\alpha|=q} \sigma(\alpha, \bar\alpha)\eta_h^\alpha \wedge \eta_v^{\bar\alpha}$. The orthogonality of $\{\eta_h^i\}$ implies that all nonzero terms in the above sum are linearly independent. In particular, since $P_{j+1,k-1}\eta = 0$, it follows that

$$
\eta_h^\alpha \wedge \eta_v^{\bar\alpha} = 0 \qquad \text{for all } \alpha \in I(j+1, j+k). \tag{3.14}
$$

And because $\sum_{\ell \le 0} P_{j+\ell,k-\ell}\eta \neq 0$, we may assume (after relabeling if necessary) that $\eta_h^1 \wedge \cdots \wedge \eta_h^{j-\ell} \wedge \eta_v^{j-\ell+1} \wedge \cdots \wedge \eta_v^{j+k} \neq 0$ for some $\ell \ge 0$. The fact that

$\sum_{\ell\geq 2} P_{j+\ell,k-\ell}\eta \neq 0$ similarly implies that span$\{\eta_h^i\}$ has dimension at least $j+2$, so after further relabelling, we may assume that $\{\eta_h^1, \dots, \eta_h^{j+2}\}$ are nonzero and hence (by orthogonality) linearly independent. It follows that $\eta_h^1\wedge\cdots\wedge\eta_h^{j+1}\wedge\eta_v^{j+2}\wedge\cdots\wedge\eta_v^{j+k}\neq 0$, contradicting (3.14).

Step 2. The hypothesis that $P_{j+1,k-1}T = 0$ implies that $P_{j+1,k-1}\tau = 0$ $\mathcal{H}^{j+k}$ a.e. in Γ. Thus Step 1 implies that $\sum_{\ell\leq 0} P_{j+\ell,k-\ell}\tau$ equals either τ or 0, a.e. in Γ. From this it is clear that $T' = T \llcorner \Gamma' = \underline{\mathrm{T}}(\Gamma', m, \tau)$, where $\Gamma' \subset \Gamma$ is the set of points at which $\sum_{\ell\leq 0} P_{j+\ell,k-\ell}\tau = \tau$. ■

4 Fu's Theorem Revisited

In this section we prove the following.

Theorem 4.1 *Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and that T is an i.m. rectifiable Lagrangian current in $\Omega\times\mathbb{R}^n$ such that $\partial T = 0$ in $\Omega\times\mathbb{R}^n$, and moreover that $\|T\|(K\times\mathbb{R}^n) < \infty$ for every compact $K\subset\Omega$.*

If $P_{n-k,k}T = 0$, then $P_{n-\ell,\ell}T = 0$ for all $\ell > k$. In particular, if $P_{n,0}T = 0$, then $T = 0$.

Fu [7] proved the $k = 0$ case (which is the main case) of the same result with the stronger hypothesis that T is locally vertically bounded, which means that whenever $K\subset\Omega$ is compact, $T\llcorner(K\times\mathbb{R}^n)$ has compact support in $\mathbb{R}^n\times\mathbb{R}^n$.

The theorem implies that an i.m. rectifiable Lagrangian current in $\Omega\times\mathbb{R}^n$ is determined by its "most horizontal" part. In particular, we have the following.

Corollary 4.2 *Let T_1, T_2 be Lagrangian rectifiable currents in $\Omega\times\mathbb{R}^n$ with no boundary in $\Omega\times\mathbb{R}^n$, and such that $\|T_i\|(K\times\mathbb{R}^n) < \infty$ for every compact $K\subset\Omega$, for $i = 1, 2$.*

If $P_{n,0}T_1 = P_{n,0}T_2$, then $T_1 = T_2$. In particular, if u is a Monge–Ampère function, then there is a unique *current* $[du]$ *satisfying* (2.5)–(2.8)

Proof Apply Theorem 4.1, with $k = 0$, to $T = T_1 - T_2$. Uniqueness of the current $[du]$ for a Monge–Ampère function follows immediately. ■

We have already noted in Section 2.5 that if u is C^2, then

$$[du](\phi dx_\alpha\wedge d\xi_\beta) = \sigma(\alpha,\bar\alpha)\int_\Omega \phi(x, Du(x))M^{\bar\alpha\beta}(D^2u(x))\,dx \tag{4.1}$$

is an i.m. rectifiable current satisfying (2.5)–(2.8), hence the unique such current, in view of the above corollary. An approximation argument then shows that (4.1) continues to hold for $u\in W^{2,n}_{\mathrm{loc}}(\Omega)$. Motivated by (4.1), given a Monge–Ampère function $u\colon\Omega\to\mathbb{R}$ we define signed measures $\mu^{\bar\alpha\beta}(D^2u)$ in $\Omega\times\mathbb{R}^n$ by

$$\int_{\Omega\times\mathbb{R}^n}\phi(x,\xi)\mu^{\bar\alpha\beta}(D^2u)(dx,d\xi) \;=\; \sigma(\alpha,\bar\alpha)[du](\phi\,dx_\alpha\wedge d\xi_\beta). \tag{4.2}$$

In view of (4.1), these measures correspond to the minors of D^2u. In particular $\int\phi\,d\mu^{\bar\alpha\beta}(D^2u) = \int\phi(x,Du)M^{\bar\alpha\beta}(D^2u)\,dx$ whenever $u\in W^{2,n}_{\mathrm{loc}}$. These measures possess good weak continuity properties.

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Corollary 4.3 *If u_k is a sequence of Monge–Ampère functions on a domain $\Omega \subset \mathbb{R}^n$ and if $\|[du_k]\|(K \times \mathbb{R}^n) \le C_K$ for K compact, and $u_k \to u$ in L^1_{loc}, then u is Monge–Ampère with $\|[du]\|(O) \le \limsup_k \|[du_k]\|(O)$ for every open $O \subset \Omega \times \mathbb{R}^n$. Moreover, for every α, β such that $|\alpha| + |\beta| = n$, $\mu^{\bar{\alpha}\beta}(D^2 u_k) \rightharpoonup \mu^{\bar{\alpha}\beta}(D^2 u)$ weakly as measures.*

An analogous result is established in [7] with the additional hypothesis that

$$\|Du_k\|_{L^\infty(K)} \le C_K$$

for K compact.

Proof It follows from the compactness theorem for integral currents (see [10, §2.2.4, Theorem 2] for a version adapted to the present setting) that there exists an i.m. rectifiable current T such that $[du_k] \rightharpoonup T$ weakly in $\mathcal{D}_n(\Omega \times \mathbb{R}^n)$. The fact that $[du_k]$ satisfies (2.5)–(2.8) for every k, together with the assumed local uniform mass bounds on $[du_k]$ and standard facts about weak lower semicontinuity, imply that T satisfies (2.5)–(2.8), with $\|T\|(O) \le \limsup_k \|[du_k]\|(O)$ for every open $O \subset \Omega \times \mathbb{R}^n$. Hence u is Monge–Ampère and $T = [du]$. ■

Remark 2 The corollary implies that if $\{u_k\}$ is a sequence of C^∞ functions on a domain Ω such that $\|M^{\bar{\alpha}\beta}(D^2 u_k)\|_{L^1(K)} \le C_K$ for every α, β and every compact $K \subset \Omega$, then the L^1 limit of any convergent subsequence is Monge–Ampère. It is not known whether every Monge–Ampère function arises in this way.

Remark 3 When $\alpha = \beta = (1, \dots, n)$ we will write $\det D^2 u$ instead of $\mu^{\alpha\beta}(D^2 u)$. Let us temporarily write $\det' D^2 u$ to denote the distributional determinant, when it exists, (see, for example, [6,12]). Note that as we have defined it, $\det D^2 u$ is a measure in the product space $\Omega \times \mathbb{R}^n$, whereas $\det' D^2 u$ is a distribution on Ω.

If u is Monge–Ampère and $\det' D^2 u$ is well defined, it is natural to ask whether $\det' D^2 u$ is a measure and

$$\int_\Omega \phi(x) \det{}' D^2 u(dx) = \int_{\Omega \times \mathbb{R}^n} \phi(x) \det D^2 u(dx, d\xi).$$

This holds if u is smooth, and by approximation if u is a limit of smooth functions in the sense of Corollary 4.3 and in addition u belongs to what is called in [6] an *admissible domain* for $\det' D^2 u$.

Remark 4 As noted in the introduction, Fu [7] deduced from his version of Theorem 4.1 a uniqueness theorem for Legendrian cycles. Going through the same argument but taking our stronger version of Theorem 4.1 as a starting point, we end up with a uniqueness theorem for Legendrian cycles with finite mass, whereas Fu's version of the theorem instead applies to compactly supported Legendrian cycles. The finite mass assumption at first sight appears a bit weaker, but in fact (together with other hypotheses) implies compact support, so here we do not gain any new generality.

Proof of Theorem 4.1 *Step 1.* It clearly suffices to show that if

$$P_{n-k+1,k-1}T = 0 \quad \text{for some } k,\ 1 \le k \le n, \tag{4.3}$$

and if $\Omega' \subset \Omega$ is an open subset with compact closure, then $P_{n-k,k}T = 0$ in $\Omega' \times \mathbb{R}^n$. Replacing Ω by Ω' for convenience, we assume that T is a rectifiable current with finite mass and no boundary in $\Omega \times \mathbb{R}^n$, and that (4.3) holds.

We apply Lemma 3.1 with $j = n-k$, and we use the notation from that lemma. In view of (4.3), Proposition 3.2 asserts that there exists an $n-k$-dimensional rectifiable set $\Gamma_h \subset \Omega$ with unit tangent $n-k$-vectorfield $\tau_h \colon \Gamma_h \to \bigwedge_{n-k}\mathbb{R}^n$, and for $\mathcal{H}^{n-k}$ a.e. $x \in \Gamma_h$, a vertical i.m. rectifiable k-current V_x, such that

$$T(\phi\, dx_\alpha \wedge d\xi_\beta) = \int_{\Gamma_h} V_x(\phi\, d\xi_\beta)\langle dx_\alpha, \tau_h(x)\rangle d\mathcal{H}^j(x). \tag{4.4}$$

Appealing again to Proposition 3.2 and the definition of Lagrangian currents (from Section 2.4), we find that the following hold at $\mathcal{H}^{n-k}$ a.e. $x \in \Gamma_h$:

- $T_x\Gamma_h$ exists;
- V_x is i.m. rectifiable and is carried by $\Gamma_v(x)$ (defined in (3.3)), with $\partial V_x = 0$;
- at $\mathcal{H}^k$ a.e. $(x,\xi) \in \Gamma_v(x)$, (3.2) and (3.4) hold and $T_{(x,\xi)}\Gamma$ is Lagrangian.

Step 2. For x satisfying the above conditions, we claim that either $\mathbf{M}(V_x) = 0$ or $\mathbf{M}(V_x) = +\infty$. We may assume by choosing coordinates suitably that $T_x\Gamma_h = \operatorname{span}\{e_1,\dots,e_{n-k}\}$. The Lagrangian condition implies that at $\mathcal{H}^k$ a.e. $(x,\xi) \in \Gamma_v(x)$, if $i \le n-k$ and $i' > n-k$, then

$$0 = \omega(\tau^i \wedge \tau^{i'}) = \tau^i \cdot \mathbb{J}\tau^{i'} = -\mathbb{J}\tau_h^i \cdot \tau_v^{i'}$$

(since $\tau^{i'} = \tau_v^{i'}$). Then this says that $\tau_v^{i'} \cdot \varepsilon_\ell = 0$ for all $\ell \le n-k$, so that $T_{(x,\xi)}\Gamma_v(x) = \operatorname{span}\{\tau_v^{i'} : i' > n-k\} = \operatorname{span}\{\varepsilon_{n-k+1},\dots,\varepsilon_n\}$ at $\mathcal{H}^k$ a.e. $(x,\xi) \in \Gamma_v(x)$. In particular, the approximate tangent space to $\Gamma_v(x)$ is a.e. constant.

We now demonstrate that this implies that V_x is a union of k-planes with integer multiplicities. Indeed, it follows from the above that $V_x(\phi\, d\xi^\beta) = 0$ for $\beta \in I(k,n)$, unless $\beta = (n-k+1,\dots,n)$. For any $q \le n-k$, let $p_q(x,\xi) = \xi_q$; it follows that $\langle V_x, p_q, s\rangle = 0$ for $\mathcal{L}^1$ a.e. $s \in \mathbb{R}$, and hence (as in the proof of Lemma 3.3) via [19, Lemma 3.3] that the indecomposable components of V_x are contained in level sets of p_q. This holds for every $q \le n-k$, for each indecomposable component $V_{x,i}$ of V_x, so there exists $r_1^i,\dots,r_{n-k}^i$ such that $V_{x,i}$ is supported in the k-plane $P_{x,i} := \{(x,\xi) : \xi_q = r_q^i, q = 1,\dots,n-k\}$. Then recalling that $\partial V_x = 0$, we infer from the constancy theorem that each $V_{x,i}$ corresponds to integration over the k-plane $P_{x,i}$, with an integer multiplicity and suitable orientation.

It follows that either $\mathbf{M}(V_x) = 0$ or $\mathbf{M}(V_x) = +\infty$ as claimed.

Step 3. However, it is clear from (3.7) that for any compact $K \subset \Omega$,

$$\int_{\Gamma_h} \mathbf{M}(V_x)\, d\mathcal{H}^j(x) \ \le\ \|T\|(\mathbb{R}^n) < \infty,$$

so that $\mathbf{M}(V_x) < \infty$ for $\mathcal{H}^j$ a.e. x. Consequently $\mathbf{M}(V_x) = 0$, and hence $V_x = 0$, for $\mathcal{H}^j$ a.e. x. It then follows immediately from (4.4) that $T(\phi dx_\alpha \wedge d\xi_\beta) = 0$, whenever $|\alpha| = n - k, |\beta| = k$. In other words, $P_{n-k,k}T = 0$. ∎

5 Description of $P_{n-1,1}[du]$

The main result of this section has two essentially equivalent forms, the first of which is the following.

Proposition 5.1 *Suppose that u is a Monge–Ampère function on an open set $\Omega \subset \mathbb{R}^n$. Then*

$$(5.1)\quad [du](\phi\, dx_{\bar{i}} \wedge d\xi_j) = \int_{\Omega\setminus\mathcal{J}_{Du}} \phi(x, Du(x))\sigma(\bar{i}, i)u_{x_i x_j}(dx) + \int_{\mathcal{J}_{Du}} \Big(\int_0^1 \phi(x, \theta Du^+ + (1-\theta)Du^-)\, d\theta\Big)\, \sigma(\bar{i}, i)\, u_{x_i x_j}(dx)$$

for every smooth ϕ with compact support in $\Omega \times \mathbb{R}^n$.

Recall that if u is Monge–Ampère, then D^2u is a measure. The Cantor part of D^2u is contained in the first term on the right-hand side. The difficulty in proving an analogous result for $P_{n-k,k}[du]$, $k \geq 2$, lies partly in dealing with the Cantor part of $k \times k$ minors of D^2u.

Remark 5 Recall from (4.2) that we write μ^{ij} to denote the measure in $\Omega \times \mathbb{R}^n$ defined by $\int \phi\, \mu^{ij}(dx, d\xi) = \sigma(\bar{i}, i)[du](\phi dx_{\bar{i}} \wedge d\xi_j)$. The main result of [14] implies that among all matrix-valued measures (ν^{ij}) satisfying

$$\int_\Omega \phi_{x_i}(x, Du(x))\, dx + \int_{\Omega\times\mathbb{R}^n} \phi_{\xi_j}(x, \xi)\nu^{ij}(dx, d\xi) = 0 \quad \forall\, \phi \in C_c^\infty(V\Omega \times \mathbb{R}^n),$$

there is a unique measure with minimal mass. It turns out that the minimizing measure is exactly (μ^{ij}); this follows from combining (5.1) with results from [14]. Note that (μ^{ij}) satisfies the above identity as a consequence of the fact that $\partial[du] = 0$.

Proposition 5.1 is very closely related to Proposition 5.3 below, which gives a description of certain 1-dimensional slices of $[du]$. In order to state the latter result, it is convenient to use the following.

Lemma 5.2 *For an interval $(a, b) \subset \mathbb{R}$, suppose that $v\colon (a, b) \to \mathbb{R}^k$ is a function of bounded variation, with total variation L. Define a 1-current I_v^* in $\mathbb{R}^k$ by*

$$(5.2)\quad I_v^*(\phi^j dy^j) = \int_{(a,b)\setminus\mathcal{J}_v} \phi^j(v(s))v_j'(ds) + \int_{\mathcal{J}_v}\Big(\int_0^1 \phi^j(\theta v^+(s) + (1-\theta)v^-(s))\, d\theta\Big)\, v_j'(ds).$$

Then there exists a Lipschitz curve $\gamma\colon (0,L) \to \mathbb{R}^k$ *such that*

$$I_v^*(\phi^j dy^j) = \int_0^L \phi^j(\gamma(s))\gamma_j'(s)\,ds. \tag{5.3}$$

In particular, I_v^* *is an integral* 1*-current, and*

$$\partial I_v^*(\psi) = \lim_{s\nearrow b}\psi(v(s)) - \lim_{t\searrow a}\psi(v(t)) = \psi(\gamma(L)) - \psi(\gamma(0)).$$

Any integral 1-current can be represented as a sum of terms having the same form as the right-hand side of (5.3); here only one such term is needed.

Note that I_v^* is the current corresponding to the image of v, with jumps "filled in" in the simplest possible way.

Proof If v is C^1, then I_v^* is just the current associated with integration over the image of v, and the conclusions are clear. If not, let $v_\ell\colon (a,b) \to \mathbb{R}^k$ be C^1 functions such that $v_\ell \to v$ strictly in BV, which means that $v_\ell \to v$ in L^1 and

$$L_\ell := \int_{(a,b)} |v_\ell'|dx \to L := \|v'\|((a,b)).$$

It is well known that such sequences exist. It then follows from [14, Theorem 1.1] that $I_{v_\ell}^* \to I_v^*$ weakly. On the other hand, let γ_ℓ be an arclength reparametrization of v_ℓ, so that $\gamma_\ell(s) = v_\ell(\sigma(s))$, where $\sigma\colon (a,b) \to (0,L_\ell)$ is nondecreasing and is chosen so that $|\gamma_\ell'| = 1$ a.e. Let γ be a limit of a uniformly convergent subsequence. Then (5.3) and the assertions about ∂I_v^* follow by passing to limits from the corresponding statements for v_ℓ, γ_ℓ. ∎

Using notation from the lemma, we can state the following.

Proposition 5.3 *Let* $U \subset \mathbb{R}^n$ *be an open subset, and let* $f\colon U \to \mathbb{R}^n$ *be a* $C^{1,1}$ *diffeomorphism onto its image, which we call* V*, with inverse* $g\colon V \to U$*. Also, let* $q(y_1,\dots,y_n) := (y_1,\dots,y_{n-1}) = y'$*.*

If u *is a Monge–Ampère function on* U*, then*

$$\langle [du], q\circ f\circ p_h, y'\rangle = I^*_{w(\,\cdot\,;y')} \quad \textit{for } \mathcal{L}^{n-1} \textit{ a.e. } y' \in \mathbb{R}^{n-1} \tag{5.4}$$

in the notation of Lemma 5.2*, for* $w(y_n;y') := (g(y',y_n), Du(g(y',y_n)))$*.*

For $\mathcal{L}^{n-1}$ a.e. $y' \in \mathbb{R}^{n-1}$, the level set $(q\circ f)^{-1}(y')$ is a Lipschitz curve (or union of Lipschitz curves) in U and $\langle [du], q\circ f\circ p_h, y'\rangle$ is the slice of $[du]$ sitting above this curve. Note also that $y_n \mapsto g(y',y_n)$ is a parametrization of $(q\circ f)^{-1}(y')$, so that the current $I^*_{w(\,\cdot\,;y')}$ on the left-hand side of (5.4) corresponds to the graph of Du above $(q\circ f)^{-1}(y')$, with jumps filled in in the natural way.

We start with the proof of Proposition 5.1, which mostly amounts to establishing Proposition 5.3 in the special case $f(x) = x$.

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Proof of Proposition 5.1 We suppose that (Γ, m, τ) are an n-rectifiable set, integer-valued multiplicity function, and orienting unit tangent n-vectorfield such that $[du] = \underline{\mathrm{T}}(\Gamma, m, \tau)$. By using a partition of unity, we see that it suffices to consider test functions ϕ supported in $B \times \mathbb{R}^n$, where $B \subset \Omega$ is a ball. Thus in fact we can assume that Ω is an open ball and that $\mathbf{M}([du]) < \infty$.

It suffices to prove the proposition for $i = n$. Note that $\sigma(\bar{n}, n) = 1$, so the signs will vanish from our calculations.

Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we will write $x = (x', x_n)$ with $x' = (x_1, \dots, x_{n-1})$. We define $q\colon \Omega \to \mathbb{R}^{n-1}$ by $q(x) = x'$, and $Q(x, \xi) = q(x)$. Throughout the proof we will write JQ to denote the Jacobian $J_{\Gamma \to \mathbb{R}^{n-1}} Q$. Because we have assumed that Ω is a ball, $q^{-1}(x')$ is connected for every x'.

Let $\langle [du], Q, x' \rangle$ denote as usual the slice of $[du]$ by the level set $Q^{-1}(x')$. Recall that

$$[du](\phi\, dx_{\bar{n}} \wedge d\xi_j) = \int_{\mathbb{R}^{n-1}} \langle [du], Q, x' \rangle (\phi\, d\xi_j) \mathcal{L}^{n-1}(dx'). \tag{5.5}$$

So it suffices to describe $\mathcal{L}^{n-1}$ a.e. slice $\langle [du], Q, x' \rangle$. We will do this as follows.

First, we record a number of properties that $\langle [du], Q, x' \rangle$ inherits, for $\mathcal{L}^{n-1}$ a.e. x', from the defining attributes (2.5)–(2.8) of $[du]$. Most important are tangent properties, which follow from the crucial Lagrangian assumption.

Next, we write down a family $\{R_{x'}\}$ of integral 1-currents with the property that the right-hand side of (5.1) is exactly $\int_{\mathbb{R}^{n-1}} R_{x'}(\phi d\xi_j) \mathcal{L}^{n-1}(dx')$. Thus to prove the theorem we must show that $R_{x'} = \langle [du], Q, x' \rangle$ almost everywhere. Toward this end, we deduce a number of properties of a.e. $R_{x'}$, like those already found for the slices of $[du]$.

Finally, we define $S_{x'} = \langle [du], Q, x' \rangle - R_{x'}$, and we argue that $S_{x'} = 0$ for a.e. x'. This is similar in spirit to the proof of the Uniqueness Theorem 4.1.

Step 1. Properties of a.e. slice. For $\mathcal{L}^{n-1}$ a.e. $x' \in \mathbb{R}^{n-1}$, since $\partial [du] = 0$ and $\mathbf{M}([du]) < \infty$, it follows from [5, 4.3.2] that $\partial \langle [du], Q, x' \rangle = 0$ and $\langle [du], Q, x' \rangle$ is i.m. rectifiable, with finite mass. We claim that in addition, for $\mathcal{L}^{n-1}$ a.e. $x' \in \mathbb{R}^{n-1}$,

$$\langle [du], Q, x' \rangle (\phi dx_i) = \delta_{in} \int_{q^{-1}(x')} \phi(x, Du) \mathcal{H}^1\,(dx) \tag{5.6}$$

for all $\phi \in C_c^\infty(\Omega)$. To see this, we use (2.8) to compute

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \langle [du], Q, x' \rangle (\phi dx_i) \psi(x') \mathcal{L}^{n-1}\,(dx') &= [du](Q^{\#}(\psi(\,\cdot\,) dx') \wedge \phi\, dx_i) \\
&= \int_{\mathbb{R}^n} \phi(x, Du)\ \psi \circ q(x)\ dx_{\bar{n}} \wedge dx_i \\
&= \delta_{in} \int_{\mathbb{R}^n} \phi(x, Du)\ \psi(x')\ \mathcal{L}^n(dx) \\
&= \delta_{in} \int_{\mathbb{R}^{n-1}} \left(\int_{q^{-1}(x')} \phi(x, Du)\ \mathcal{H}^1(dx) \right) \psi(x') \mathcal{L}^{n-1}(dx').
\end{aligned}$$

Since this holds for all $\psi \in C_c^\infty(\mathbb{R}^{n-1})$and ϕ as above, it implies (5.6).

Another fact from [5] is that for $\mathcal{L}^{n-1}$ a.e. x', $\langle [du], Q, x' \rangle$ has the explicit representation

$$\langle [du], Q, x' \rangle = \int_{\Gamma(x')} \langle \phi, \zeta \rangle \, m \, d\mathcal{H}^1, \tag{5.7}$$

where

$$\Gamma(x') := Q^{-1}(x') \cap \{(x, \xi) \in \Gamma : JQ > 0\}, \tag{5.8}$$

and for $(x, \xi) \in \Gamma(x')$,

$$\zeta(x, \xi) \text{ is a unit vector in } T_{(x,\xi)}\Gamma \text{ such that } dQ(x, \xi)(\zeta) = 0. \tag{5.9}$$

Recall that $JQ(x, \xi) > 0$ exactly when $dQ(x, \xi) \colon T_{(x,\xi)}\Gamma \to \mathbb{R}^{n-1}$ is of full rank. Thus (5.8) implies that ζ is uniquely specified, up to a sign, by (5.9). Properties (5.7)–(5.9) follow directly from [5, 4.3.8], where the sign of ζ is also specified; we will not need to keep track of this sign in our later arguments.

Following notation in Lemma 3.1 (with $k = 1$, $j = n - 1$), we will write

$$\Gamma_h := \{x \in \Omega : \mathcal{H}^1(p_h^{-1}(x) \cap \Gamma) > 0\},$$
$$\Gamma_h(x') := \{x \in \Omega : \mathcal{H}^1(p_h^{-1}(x) \cap \Gamma(x')) > 0\}.$$

Clearly $\Gamma_h(x') \subset q^{-1}(x') \cap \Gamma_h$, and the fact that Γ_h is $n-1$-rectifiable (see Lemma 3.1) implies that for $\mathcal{L}^{n-1}$ a.e. x', $\Gamma_h(x')$ is at most countable and $T_x\Gamma_h$ exists at every $x \in \Gamma_h(x')$. We finally claim that for $\mathcal{L}^{n-1}$ a.e. x', at every $x \in \Gamma_h(x')$,

$$\zeta(x, \xi) \in \mathbb{J}(T_x\Gamma_h{}^\perp) \quad \text{for } \mathcal{H}^1 \text{ a.e. } \xi \in \Gamma(x') \cap p_h^{-1}(x), \tag{5.10}$$

where $T_x\Gamma_h{}^\perp$ denotes the orthogonal complement in the horizontal directions only, so that $T_x\Omega = T_x\Gamma_h \oplus T_x\Gamma_h{}^\perp$.

Observe that $\zeta(x, \cdot)$ is independent of ξ, up to a sign, for x satisfying (5.10).

To prove (5.10), observe that, for $\mathcal{L}^{n-1}$ a.e. x', the Lagrangian condition (2.6) and Lemma 3.1 imply that at every $x \in \Gamma_h(x')$,

$$\text{for } \mathcal{H}^1 \text{ a.e. } \xi \in p_h^{-1}(x) \cap \Gamma, \text{ (3.2) holds and } T_{(x,\xi)}\Gamma \text{ is Lagrangian}, \tag{5.11}$$

and

$$\text{(3.4) holds (with } j = n - 1, k = 1) \text{ a.e. on the set } \Gamma_v(x), \text{ see (3.3)}. \tag{5.12}$$

So it suffices to prove that (5.10) holds at any x where (5.11), (5.12) are verified.

Fix such an x, and fix also $\xi \in p_h^{-1}(x) \cap \Gamma_h(x')$ at which (3.2) holds and where the tangent n-plane is Lagrangian. Since $Q = q \circ p_h$, the vector $\tau^n = \tau_v^n \in T_{(x,\xi)}\Gamma$ appearing on the right-hand side of (3.2) satisfies $dQ(x, \xi)\tau_v^n = 0$. Because this

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equation defines $\zeta(x,\xi)$ up to a sign, we find that $\zeta(x,\xi) = \pm\tau_\nu^n(x,\xi)$. It follows also that $\tau_h^1,\dots,\tau_h^{n-1}$ appearing on the right-hand side of (3.2) are nonzero (since otherwise $dQ(x,\xi)$ would have a nullspace of dimension at least two) and hence that $(x,\xi) \in \Gamma_\nu(x)$ as defined in (3.3). This in turn implies that (3.4) holds, and then (as in the proof of Theorem 4.1) the Lagrangian condition implies that (5.10) is satisfied.

Step 2. Definition and properties of $R_{x'}$. In this step we appeal to results about one-dimensional sections of BV functions as found for example in [5, §4.5] or [1, Chapter 3].

We henceforth identify Du with its precise representative; see [1, Corollary 3.80] which in particular implies that $Du(x)$ equals its Lebesgue value whenever x is a Lebesgue point of Du. For $x' \in \mathbb{R}^{n-1}$, we define

$$\Omega_{x'} = \{x_n \in \mathbb{R} : (x', x_n) \in \Omega\},$$

We will write $Du(x_n; x') = Du(x', x_n)$, so that we view $Du(\,\cdot\,; x') \colon \Omega_{x'} \to \mathbb{R}^n$ as functions of a single variable, parametrized by $x' \in \mathbb{R}^{n-1}$. Then for $\mathcal{L}^{n-1}$ a.e. x', $Du(\,\cdot\,; x')$ is a BV function on $\Omega_{x'}$. We will write $\partial_{x_n} Du(x')$ to indicate the associated vector-valued derivative measure on $\Omega_{x'}$, and we write $\partial_{x_n} u_{x_j}(x')$, $j = 1,\dots,n$ for the components of $\partial_{x_n} Du(x')$. We define a BV function $\nu(\,\cdot\,; x') \colon \Omega_{x'} \to \mathbb{R}^n \times \mathbb{R}^n$, given by

$$x_n \mapsto (x', x_n, Du(x_n; x')) = \nu(x_n; x')$$

and a 1-current $R_{x'} = I^*_{\nu(\cdot;x')}$, using notation from Lemma 5.2. We will write $\mathcal{J}_{x'}$ to denote the jump set $Du(\,\cdot\,, x')$, which clearly coincindes with the jump set of $\nu(\,\cdot\,; x')$. From the definition in Lemma 5.2 we check that

$$\begin{aligned}
R_{x'}(\phi dx_i) &= \delta_{in} \int_{\Omega_{x'}} \phi(\nu(x_n; x')) \mathcal{L}^1(dx_n),\\
R_{x'}(\phi d\xi_j) &= \int_{\Omega_{x'} \setminus \mathcal{J}_{x'}} \phi(\nu(x_n; x'))\, \partial_{x_n} u_{x_j}(x')(dx_n)\\
&\qquad + \int_{\mathcal{J}_{x'}} \Big(\int_0^1 \phi(\theta\nu^+ + (1-\theta)\nu^-)\, d\theta \Big)\, \partial_{x_n} u_{x_j}(x')\,(dx_n),
\end{aligned}$$

where in the last integral, we write $\nu^\pm := \lim_{s \to x_n^\pm} \nu(s; x')$ for $x_n \in \mathcal{J}_{x'}$.

It follows from Lemma 5.2 that $R_{x'}$ is an integral current with $\partial R_{x'} = 0$ in $\Omega \times \mathbb{R}^n$ and $\mathbf{M}(R_{x'}) < \infty$ for $\mathcal{L}^{n-1}$ a.e. x'. We also claim that

$$\text{the right-hand side of (5.1)} = \int_{\mathbb{R}^{n-1}} R_{x'}(\phi d\xi_j)\, \mathcal{L}^{n-1}(dx'). \tag{5.13}$$

To verify this, note that $(x, Du(x)) = \nu(x_n; x')$, so that upon comparing (5.1) and the formulas given above for $R_{x'}$, we see that (5.13) follows from the facts that when $f, g \colon \Omega \to \mathbb{R}$ are bounded and D^2u-measurable,

$$\int_{\Omega \setminus \mathcal{J}_{Du}} f(x) u_{x_j x_n}(dx) = \int_{\mathbb{R}^{n-1}} \int_{\Omega_{x'} \setminus \mathcal{J}_{x'}} f(x', x_n)\, \partial_{x_n} u_{x_j}(x')(dx_n) \mathcal{L}^{n-1}(dx')$$

and

$$\int_{\mathcal{J}_{Du}} g(x) u_{x_j x_n}(dx) = \int_{\mathbb{R}^{n-1}} \int_{\mathcal{J}_{x'}} g(x', x_n)\, \partial_{x_n} u_{x_j}(x')(dx_n) \mathcal{L}^{n-1}(dx').$$

Proofs of these identities can be found, for example, in [1, Theorems 3.107, 3.108].

We note for future reference that for $\mathcal{L}^{n-1}$ a.e. x', by its definition $R_{x'}$ is supported in $Q^{-1}(x')$, and

(5.14)

$$R_{x'} \llcorner p_h^{-1}(x', x_n) \neq 0 \Leftrightarrow x_n \in \mathcal{J}_{x'} \Leftrightarrow R_{x'} \llcorner p_h^{-1}(x', x_n) = [\nu(x_n^-; x'), \nu(x_n^+; x')]$$

where $[\nu(x_n^-; x'), \nu(x_n^+; x')]$ denotes the oriented line segment joining $\nu(x_n^-; x')$ to $\nu(x_n^+; x')$. Moreover, [1, Theorem 3.108] together with classical results about the rectifiability of the jump set imply that for $\mathcal{L}^{n-1}$ a.e. x',

$$x_n \in \mathcal{J}_{x'} \Rightarrow (x', x_n) \in \mathcal{J}_{Du} \text{ and } T_{(x', x_n)} \mathcal{J}_{Du} \text{ exists,}$$

and moreover, the fact that Du is a gradient implies that the jump in Du across $\mathcal{J}_{Du}$ is normal to $T_{(x',x_n)}\mathcal{J}_{Du}$, if we identify the vertical directions (in which the jump occurs) and the horizontal directions (in which $T_{(x',x_n)}\mathcal{J}_{Du}$ lives). If we do not identify vertical and horizontal, this says that

$$\nu(x_n^+, x') - \nu(x_n^-, x') \in \mathbb{J}[(T_{(x',x_n)}\mathcal{J}_{Du})^\perp], \tag{5.15}$$

where $(T_{(x',x_n)}\mathcal{J}_{Du})^\perp$ denotes the orthogonal complement in the horizontal directions only, as in (5.10).

Step 3. Conclusion of proof. Now we define $S_{x'} = \langle [du], Q, x' \rangle - R_{x'}$. In view of (5.5) and (5.13), it suffices to show that $S_{x'} = 0$ for $\mathcal{L}^{n-1}$ a.e. x'. Note that this is also the conclusion of Proposition 5.3 in the case $f(x) = x$.

The various facts about $\langle [du], Q, x' \rangle$ and $R_{x'}$ assembled above imply that for a.e. x', $S_{x'}$ is an i.m. rectifiable 1-current with finite mass, and such that

$$\partial S_{x'} = 0 \ \text{ in } \Omega \times \mathbb{R}^n, \quad P_{1,0} S_{x'} = 0.$$

If $S_{x'}$ has these properties, then it follows from Lemma 3.3, or from the $j = 0$ case of Proposition 3.2, that there exist points $x^i \in q^{-1}(x')$ and i.m. rectifiable 1-currents V_i (depending on x' in a way not captured by our notation) such that V_i is supported in $\{x^i\} \times \mathbb{R}^n$ and

$$S_{x'} = \sum V_i, \quad \partial V_i = 0,\ \mathbf{M}(V_i) < \infty. \tag{5.16}$$

We fix one of these points x^i. Note that

$$V_i = S_{x'} \llcorner p_h^{-1}(x^i) = [\langle [du], Q, x' \rangle \llcorner p_h^{-1}(x^i)] - [R_{x'} \llcorner p_h^{-1}(x^i)]. \tag{5.17}$$

It suffices to prove that

$$\text{the unit tangent to } V_i \text{ is } \mathcal{H}^1 \text{ a.e. constant, up to a sign} \tag{5.18}$$

(*i.e.,* independent of ξ), up to a sign, since then we can deduce from Solomon's Separation Lemma [19] and the Constancy Theorem that V_i is either zero or has infinite mass, exactly as in Step 2 of the proof of Theorem 4.1. Since $\mathbf{M}(V_i) < \infty$, we conclude that $V_i = 0$, and since this holds for all i, it will follow that $S_{x'} = 0$ as needed.

We define

$$\Sigma_1 := \{x = (x', x_n) \in \Omega : x_n \in \mathcal{J}_{x'}\}, \quad \Sigma_2 := \{x = (x', x_n) \in \Omega : x \in \Gamma_h(x')\}.$$

From what we have said above, $\langle [du], Q, x' \rangle \llcorner p_h^{-1}(x^i) \neq 0$ if and only if $x^i \in \Sigma_2$, and $R_{x'} \llcorner p_h^{-1}(x^i) = 0$ if and only if $x^i \in \Sigma_1$. In particular, $V_i = 0$ unless $x^i \in \Sigma_1 \cup \Sigma_2$.

The unit tangent vector ζ to $\langle [du], Q, x' \rangle \llcorner p_h^{-1}(x^i)$ for $x^i \in \Sigma_2$ is characterized in (5.10). And according to (5.14), for $x^i \in \Sigma_1$ the unit tangent vector to $R_{x'} \llcorner p_h^{-1}(x^i)$ is

$$\frac{\nu(x_n^+, x') - \nu(x_n^-, x')}{|\nu(x_n^+, x') - \nu(x_n^-, x')|} =: \zeta_R(x^i)$$

and is characterized in (5.15). Both ζ and ζ_R are independent of $(x^i, \xi) \in p_h^{-1}(x^i)$ up to a sign, so that (5.18) is immediate unless both terms on the right-hand side of (5.17) are nonzero. When this holds, we must show that $\zeta(x^i, \xi) = \pm\zeta_R(x^i)$ for a.e. ξ, and in view of (5.10), (5.15), it suffices to show that the approximate tangent spaces of Γ_h and of $\mathcal{J}_{Du}$ coincide at x^i.

To do this, we claim that

$$\mathcal{H}^{n-1}(\Sigma_1 \setminus \Sigma_2) = 0,$$

This follows from what we have already said, because if $x = (x', x_n) \in \Sigma_1 \setminus \Sigma_2$, then (unless x' belongs to a set of $\mathcal{L}^{n-1}$ measure zero at which (5.16) fails to hold) (5.16) and (5.17) imply that $S_{x'} = R_{x'} \llcorner p_h^{-1}(x)$ and hence that $\partial R_{x'} \llcorner p_h^{-1}(x) = 0$, and as remarked earlier, it follows that $\mathbf{M}(R_{x'} \llcorner p_h^{-1}(x)) = +\infty$. This is impossible away from a set of $\mathcal{H}^{n-1}$ measure zero.

Now we appeal to the fact that if $\mathcal{H}^k(A \setminus B) = 0$ and A, B are $\mathcal{H}^k$ rectifiable, then $T_x A = T_x B$ at $\mathcal{H}^k$ a.e. point of A. Clearly $\mathcal{H}^{n-1}(\Sigma_1 \setminus \mathcal{J}_{Du}) = 0$, and also $0 = \mathcal{H}^{n-1}(\Sigma_1 \setminus \Sigma_2) = \mathcal{H}^{n-1}(\Sigma_2 \setminus \Gamma_h)$. It follows that $T_x \Sigma_1 = T_x \mathcal{J}_{Du} = T_x \Gamma_h$ at $\mathcal{H}^{n-1}$ a.e. $x \in \Sigma_1$, which proves that (5.18) holds at every x^i, for $\mathcal{L}^{n-1}$ a.e. x'. ■

The proof of Proposition 5.1 has already established Proposition 5.3 when $f(x) = x$. We deduce the general case of Proposition 5.3 from this special case by a change of variables, using a result from [7] that describes the behaviour of Monge–Ampère functions under coordinate transformations.

Proof of Proposition 5.3 We write $F\colon U \times \mathbb{R}^n \to V \times \mathbb{R}^n$ and $G\colon V \times \mathbb{R}^n \to U \times \mathbb{R}^n$ by $F(x, \xi) = (f(x), Dg(f(x))^T \xi)$ and $G(y, \eta) = (g(y), Df(g(y))^T \eta)$. Note that $G = F^{-1}$. According to Fu [7, Proposition 2.5], $u \circ g$ is a Monge–Ampère function on V, and $[d(u \circ g)] = F_\#[du]$. (This is proved for locally Lipschitz Monge–Ampère functions, but the proof remains valid without that restriction.) Thus $[du] = G_\# F_\#[du] = G_\#[d(u \circ g)]$, so that for a.e. $y' \in \mathbb{R}^{n-1}$,

$$\langle [du], q \circ f \circ p_h, y' \rangle = \langle G_\#[d(u \circ g)], q \circ f \circ p_h, y' \rangle = G_\# \langle [d(u \circ g)], q \circ f \circ p_h \circ G, y' \rangle$$

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using basic properties of slices, see Federer [5, 4.3.2 (7)]. From the definitions one checks that $q \circ f \circ p_h \circ G = q \circ p_h = Q$, in the notation of Proposition 5.1, where we have proved that

$$\langle [d(u \circ g)], Q, y' \rangle = I^*_{v(\,\cdot\,;y')}$$

for a.e. y', with $v(y_n; y') := (y', y_n, D(u \circ g)(y', y_n)))$.

To prove the proposition, we must therefore verify that $G_\# I^*_{v(\cdot;y')} = I^*_{w(\cdot;y')}$, where w is defined following (5.4). If $v\colon (0, L) \to V \times \mathbb{R}^n$ is smooth, then $I^*_{v(\cdot;y')} = v_\#[[0, L]]$, where $[[0, L]]$ denotes the 1-current corresponding to integration over $[0, L]$, and so

$$G_\# I^*_{v(\cdot;y')} = G_\# v_\# [[0, L]] = (G \circ v)_\# [[0, L]].$$

It is easy to check that $G \circ v(y_n; y') = w(y_n; y')$, so the corollary follows in this case. This remains valid if v is continuous with bounded variation. If v is merely a function of bounded variation, let us write $\mathcal{J}_v$ for the jump set of v. We split I^*_v into a continuous part and a jump part:

$$I^*_v = v_\#[[0, L] \setminus \mathcal{J}_v] + (I^*_v - v_\#[[0, L] \setminus \mathcal{J}_v]).$$

These are the first and second terms, respectively, on the right-hand side of (5.2). It follows from what we have said that $G_\# v_\#[[0, L] \setminus \mathcal{J}_v] = w_\#[[0, L] \setminus \mathcal{J}_v]$. The other term $(I^*_v - v_\#[[0, L] \setminus \mathcal{J}_v])$ consists of a sum of straight line segments, each having the form $[(y, D(u \circ g)^-(y)), (y, D(u \circ g)^+(y))]$ for some $y \in V$, and from the explicit form of G one can verify that

$$\begin{aligned} G_\# \big[\big(y, D(u \circ g)^-(y) \big), \big(y, D(u \circ g)^+(y) \big) \big] \\ = \big[\big(g(y), Du^-(g(y)) \big), \big(g(y), Du^+(g(y)) \big) \big]. \end{aligned}$$

Combining these, we obtain the desired result. ∎

6 Weak Solutions of a Degenerate Monge–Ampère Equation

In this section we give the proof of the second main result of our paper.

Theorem 6.1 *Suppose that Ω is a bounded open subset of $\mathbb{R}^2$. Assume that u is a Monge–Ampère function on Ω that satisfies*

$$(6.1) \qquad [du](\phi d\xi_1 \wedge d\xi_2) = 0 \quad \text{for all } \phi \in C^\infty_c(\Omega \times \mathbb{R}^2).$$

Then for every $x \in \Omega$, at least one of the following must hold:

(i) *u is affine in an open neighborhood of x.*

(ii) *There exists a line segment ℓ_x, passing through x and meeting $\partial\Omega$ at both endpoints, along which Du is constant in the sense that*

  (a) *every point along ℓ_x is a Lebesgue point of Du, with the same Lebesgue value; or*

(b) *every point on ℓ_x belongs to the jump set $\mathcal{J}_{Du}$ of Du, with same approximate limits on both sides of ℓ_x.*

In particular, every *point in Ω is either a Lebesgue point of Du or belongs to the jump set of Du.*

As mentioned in the introduction, this extends earlier work of Hartman and Nirenberg [11], Kirchheim [15], and Pakzad [16].

The theorem shows that a function u satisfying the hypotheses has the regularity of a BV function of a single variable. This is optimal: if $f\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary function of bounded variation and $u(x, y) = f(x)$, then u satisfies all the hypotheses of the theorem.

Our assumption (6.1) should be compared with the weaker condition:

$$[du](\phi\, d\xi_1 \wedge d\xi_2) = 0 \quad \text{for all } \phi(x, \xi) = \tilde{\phi}(x), \tilde{\phi} \in C_c^\infty(\Omega). \tag{6.2}$$

This weaker condition requires only that the marginal on Ω of the measure in the product space vanishes. The conclusions of the theorem need not hold under assumption (6.2). To see this, let B denote the unit ball in $\mathbb{R}^2$, and suppose that $u\colon B \to \mathbb{R}$ is the restriction to B of a function that is homogeneous of degree 1 and smooth away from the origin. Then one can check that u is Monge–Ampére, and that if $\phi(x, \xi) = \tilde{\phi}(x)$, then $[du](\phi\, d\xi_1 \wedge d\xi_2)$ makes sense, and

$$[du](\phi\, d\xi_1 \wedge d\xi_2) = A\,\tilde{\phi}(0), \qquad A = \frac{1}{2}\int_0^{2\pi} \gamma(\theta) \wedge \gamma'(\theta)\, d\theta, \tag{6.3}$$

where $\gamma(\theta) = Du(\cos\theta, \sin\theta)$. This is proved, for example, in [13, Lemma 4.1, Remark 4]. From this one can check that, given *any* u as above, one can find $c \in R$ such that $A = 0$ for $u_c(x) := u(x) + c|x|$. Such a function u_c satisfies (6.2) but not in general the conclusions of Theorem 6.1.

For functions u as described above, the distributional determinant $\det' D^2u$ exists and is given by $\det' D^2u = A\delta_0$; see Remark 3. Thus the conclusions of the theorem do not hold if we assume that u is a Monge–Ampère function such that $\det' D^2u$ is well defined and vanishes.

Throughout the proof we will write $[du] = \underline{\mathrm{T}}(\Gamma, m, \tau)$. The starting point of the proof is the following.

Lemma 6.2 *Suppose that Ω is a bounded, open subset of $\mathbb{R}^2$, and assume that $u\colon \Omega \to \mathbb{R}$ is a Monge–Ampère function that satisfies* (6.1)*. Then there exists a* 1*-rectifiable set $\Gamma_\nu \subset \mathbb{R}^2$ and an $\mathcal{H}^1$-measurable mapping $\tau_\nu\colon \Gamma_\nu \to \wedge_1\mathbb{R}^2$ such that $|\tau_\nu| = 1$ a.e.,*

$$\text{at } \mathcal{H}^1 \text{ a.e. } \xi \in \Gamma_\nu,\ T_\xi\Gamma_\nu \text{ exists and equals } \operatorname{span}\{\tau_\nu(\xi)\};$$

and for $\mathcal{H}^1$ a.e. $\xi \in \Gamma_\nu$, there exists a horizontal i.m. rectifiable 1*-current H_ξ in $\Omega \times \mathbb{R}^2$ supported in $\Omega \times \{\xi\}$ and satisfying*

$$\partial H_\xi = 0 \quad \text{in } \Omega \times \mathbb{R}^2, \tag{6.4}$$

$$[du](\phi\, dx_i \wedge d\xi_j) = \int_{\Gamma_\nu} H_\xi(\phi\, dx_i)\langle d\xi_j, \tau_\nu(\xi)\rangle \mathcal{H}^1\,(d\xi) \tag{6.5}$$

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for every $\phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$. (In particular, $\xi \in \Gamma_v \mapsto H_\xi(\phi\, dx_i)$ is $\mathcal{H}^1$-measurable for every such ϕ.) Moreover, for every *$\xi \in \Gamma_v$ at which H_ξ is defined, there exists a horizontal unit vector $\tau_h(\xi)$, a collection of nonzero integers $\{m_i(\xi)\}$, and a collection of line segments $\{\ell_i(\xi)\}_i$, each parallel to $\tau_h(\xi)$ and with its endpoints in $\partial\Omega$, such that $p_v^{-1}(\xi) \cap \Gamma = \bigcup_i \ell_i(\xi) \times \{\xi\}$, and such that*

$$(6.6)\quad H_\xi(\phi\, dx_i) = \sum_i \int_{\ell_i(\xi)\times\{\xi\}} \phi(x,\xi)\langle dx_i, \tau_h(\xi)\rangle m_i(\xi)\mathcal{H}^1(dx), \quad H_\xi(\phi\, d\xi_j) = 0$$

for $\phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$. And finally, $\tau_h(\xi) \cdot \mathbb{J}\tau_v(\xi) = 0$ for $\mathcal{H}^1$ a.e. $\xi \in \Gamma_v$.

We will eventually show that for a.e. ξ, each $\ell_i(\xi)$ is contained entirely in either the Lebesgue set of Du or the jump set of Du. Moreover, we will prove that in the former case, $Du \equiv \xi$ along $\ell_i(\xi)$, and in the latter case, $\xi \in [Du^-(x), Du^+(x)]$ for every $x \in \ell_i(\xi)$. A main point will be to show that for a.e. ξ, ξ' and every i, i' either $\ell_i(\xi) = \ell_{i'}(\xi')$ or $\ell_i(\xi) \cap \ell_{i'}(\xi') = \varnothing$. This will be established by carrying out a blowup argument, and then classifying all homogeneous functions satisfying (6.1); these are the key points in the proof of the theorem.

Proof of Lemma 6.2 The proof follows closely that of Theorem 4.1, with the roles of horizontal and vertical reversed. We define

$$\Gamma_v := \{\xi \in \mathbb{R}^2 : \mathcal{H}^1(p_v^{-1}(\xi) \cap \Gamma) > 0\},$$

which in view of Lemma 3.1 is a 1-rectifiable set. Proposition 3.2, with p_h replaced by p_v, implies the existence of a current H_ξ for $\mathcal{H}^1$ a.e. $\xi \in \Gamma_v$, such that (6.4), (6.5) hold. Then as in the proof of Theorem 4.1, the Lagrangian condition (2.6) and facts assembled in Lemma 3.1, again with p_h and p_v switched, imply that for $\mathcal{H}^1$ a.e. ξ, there exists a horizontal $\tau_h(\xi)$, determined up to a sign by the condition $\tau_h(\xi) \cdot \mathbb{J}\tau_v(\xi) = 0$, and such that unit tangent vectors to H_ξ equal $\pm\tau_h(\xi)$, $\mathcal{H}^1$ a.e. Then Solomon's Separation Lemma [19] and the constancy theorem imply that H_ξ is a sum of indecomposables, each of which is supported in $\Omega \times \{\xi\}$ and consists of an oriented integer multiplicity line segment parallel to $\tau_h(\xi)$ and with no boundary in Ω. Each such segment is bounded, since Ω is bounded, and so the endpoints must lie in $\partial\Omega$. These facts are summarized in (6.6). Since (6.5) is insensitive to the behavior of H_ξ on $\mathcal{H}^1$ null sets, we can modify H_ξ on such a set to arrange that (6.6) holds at *every* point where H_ξ is defined. (This is simply for convenience.) ■

Fix $a \in \Omega$ and let $R = \operatorname{dist}(a, \partial\Omega)$. Let $\rho(x,\xi) = |x - a|$. For $r \in (0, R)$, we let $x_r(s) = a + r(\cos\frac{s}{r}, \sin\frac{s}{r})$, $s \in \mathbb{R}/(2\pi r\mathbb{Z})$ be an arclength parametrization of $\partial B_r(a)$. We write $Du(\,\cdot\,; r)$, or simply $Du(r)$, to denote the function $\mathbb{R}/(2\pi r\mathbb{Z}) \to \mathbb{R}^2$ defined by $s \mapsto Du(s; r) = Du(x_r(s))$. Then for $\mathcal{L}^1$ a.e. $r \in (0, R)$, $Du(\,\cdot\,; r)$ is a function of bounded variation; see again [1, Chapter 3]. We will write $\partial_s Du(r)$ to denote the associated vector-valued derivative measure, with components $\partial_s u_{x_j}(r)$, $j = 1, 2$. We also write $\mathcal{J}_r$ to denote the jump set of $Du(r)$.

Note that $\rho, R, x_r, Du(r)$ all depend on a in a way that is not indicated in our notation.

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Lemma 6.3 *For $\mathcal{L}^1$ a.e. $r \in (0,R)$, if we define $v_r\colon \mathbb{R}/(2\pi r\mathbb{Z}) \to \Omega \times \mathbb{R}^2$ by $v_r(s) = (x_r(s), Du(x_r(s)))$, then $\langle [du], \rho, r\rangle = I^*_{v_r}$, using notation from Lemma 5.2. In particular.*

$$(6.7)\quad \langle [du], \rho, r\rangle(\phi\, d\xi_j) = \int_{\mathbb{R}/(2\pi r\mathbb{Z})\setminus \mathcal{J}_r} \phi^j\big(x_r(s), Du(x_r(s))\big)\,\partial_s u_{x_j}(r)(ds)$$
$$+ \int_{\mathcal{J}_r}\Big(\int_0^1 \phi^j\big(x_r(s), \theta Du^+(x_r(s)) + (1-\theta)Du^-(x_r(s))\big)\,d\theta\Big)\,\partial_s u_{x_j}(r)(ds),$$

where as usual $Du^{\pm}(x_r(s)) = \lim_{\sigma\to s^{\pm}} Du(x_r(\sigma))$. Note from Lemma 5.2 that I^*_v is the current associated with integration over a single Lipschitz curve, so that its support is just the image of this curve.

The above lemma is just a special case of Proposition 5.3, with the the right-hand side written out in detail. (Strictly speaking, we need to cover $\Omega \setminus \{a\}$ by simply connected open sets, on each of which $(x_1, x_2) = x_r(s) \mapsto (r, s)$ defines a smooth change of coordinates, and apply Proposition 5.3 on each such open set.)

Proof of Theorem 6.1 *Step 1.* We first claim that

$$(6.8)\qquad \langle [du], \rho, r\rangle(\phi\, d\xi_j) = \int_{\Gamma_v} \langle H_\xi, \rho, r\rangle(\phi)\langle d\xi_j, \tau_v(\xi)\rangle\,\mathcal{H}^1(d\xi)$$

for $\mathcal{L}^1$ a.e. $r \in (0,R)$ and $\phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$. This follows by slicing (6.5). To see this, fix $f \in C_c^\infty(0,R)$; then

$$\begin{aligned}\int_0^R \langle [du], \rho, r\rangle(\phi\, d\xi_j) f(r)\,dr &= [du](\rho^\#(f(\cdot)dr) \wedge \phi\, d\xi_j)\\ &= \int_{\Gamma_v} H_\xi(\phi\, f\circ\rho\, d\rho)\langle d\xi_j, \tau_v(\xi)\rangle\,\mathcal{H}^1(d\xi)\\ &= \int_0^R \int_{\Gamma_v} \langle H_\xi, \rho, r\rangle(\phi)\langle d\xi_j, \tau_v(\xi)\rangle \mathcal{H}^1(d\xi) f(r)\,dr.\end{aligned}$$

Since this holds for all f as above, we deduce (6.8).

Step 2. We now define

$$S_r(a) := \{\xi \in \Gamma_v : H_\xi \text{ is well defined and } \langle H_\xi, \rho, r\rangle \neq 0\}.$$

We will normally write S_r when there is no possibility of confusion. It follows from the explicit description of H_ξ in Lemma 6.2 that

$$(6.9)\qquad S_r = \{\xi \in \Gamma_v : H_\xi \text{ is well defined and } (\bigcup_i \ell_i(\xi)) \cap B_r(a) \neq \varnothing\}.$$

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In particular, it follows that

$$S_{r'} \subset S_r \text{ when } r' < r. \tag{6.10}$$

We will say r is a "good radius" if $Du(r)$ is a function of bounded variation and (6.7), (6.8) hold. For a good radius r, we define

$$Du^*(\partial B_r(a)) := \{Du(x_r(s)) : s \notin \mathcal{J}_r\} \cup \Big(\bigcup_{s\in\mathcal{J}_r} [Du^-(x_r(s)), Du^+(x_r(s))] \Big),$$

where $[p, q]$ denotes the line segment $\{\theta p + (1-\theta)q : 0 \le \theta \le 1\}$. This is just the image via Du of $\partial B_r(a)$, with the jumps filled in as usual. Where no confusion can result, we will often omit the center a and write simply $Du^*(\partial B_r)$. It follows from Lemma 6.3 that $Du^*(\partial B_r) = p_v(\operatorname{supp}\langle[du], \rho, r\rangle)$. Thus (6.8) implies that

$$\mathcal{H}^1(S_r \setminus Du^*(\partial B_r)) + \mathcal{H}^1(Du^*(\partial B_r) \setminus S_r) = 0. \tag{6.11}$$

Note that $Du^*(\partial B_r(a))$ and $S_r(a)$ have complementary good properties. On one hand, $Du^*(\partial B_r(a))$ directly encodes information about the pointwise behavior of the gradient of u. Also, it is not hard to see from the definition that $Du^*(\partial B_r(a))$ is closed and connected; in fact it is the image of a single Lipschitz curve. Note in particular that $\operatorname{diam}(Du^*(\partial B_r)) \le \mathcal{H}^1(Du^*(\partial B_r))$. This need not hold for S_r over which we have little or no control on $\mathcal{H}^1$ null sets. On the other hand, S_r is directly related through (6.9) to the line segments $\ell_i(\xi)$ that we seek to understand.

In any case, (6.11) and (6.10) imply that if r, r' are good, then

$$Du^*(\partial B_{r'}) \subset Du^*(\partial B_r) \text{ when } r' < r. \tag{6.12}$$

Note: From now on, in every assertion we make involving any $Du^*(\partial B_r(a))$, we implicitly assume that r is a good radius for the given center a.

Step 3. It follows from (6.10), (6.11) that $r \mapsto \mathcal{H}^1(Du^*(\partial B_r(a)))$ is almost everywhere nondecreasing, for every a. We will prove that

(6.13) if $\mathcal{H}^1(Du^*(\partial B_r(a)) = 0$ for some $r > 0$, then u is affine in $B_r(a)$,

(6.14) if $\mathcal{H}^1(Du^*(\partial B_r(a))) \to 0$ as $r \to 0$, then a is a Lebesgue point of Du,

(6.15) if $\mathcal{H}^1(Du^*(\partial B_r(a))) \ge \alpha > 0 \;\forall\, r > 0$, then a is a jump point of Du.

These imply in particular that every point of Ω is either a Lebesgue point or a jump point of Du. Conclusions (6.13) and (6.14) follow easily from what we have already said. Indeed, if $\mathcal{H}^1(Du^*(\partial B_r)) = 0$ for some $r \in (0, R)$, then since $Du^*(\partial B_r)$ is connected, it consists of a single point, say $Du^*(\partial B_r) = \{\xi_a\}$. Then (6.12) implies that $Du^*(\partial B_{r'}) = \{\xi_a\}$ for $r < r'$. Thus, in view of the definition of $Du^*(\partial B_r)$, $Du = \xi_a \, \mathcal{L}^2$ a.e. in $B_r(a)$, and so u is affine in $B_r(a)$, with gradient ξ_a.

Similarly, if $\mathcal{H}^1(Du^*(\partial B_r)) \to 0$ as $r \to 0$, then (6.12) implies that there exists a point ξ_a such that $\operatorname{dist}(Du^*(\partial B_r), \xi_a) \to 0$ as $r \to 0$, and then the definition of $Du^*(\partial B_r)$ implies that a is a Lebesgue point of Du, with $Du(a) = \xi_a$.

Step 4. Blowup. In the next two steps we use a blowup argument to prove (6.15). Fix $a \in \Omega$ at which

$$\mathcal{H}^1(S_r) = \mathcal{H}^1(Du^*(\partial B_r)(a)) \geq \alpha > 0 \text{ for all } r > 0. \tag{6.16}$$

We will use a blowup argument to study the behavior of u near such a point. First fix some good r_0. Then (6.12) and the definition of $Du^*(\partial B_r)$ imply that $Du(x) \in Du^*(\partial B_{r_0})$ for $\mathcal{L}^2$ a.e. $x \in B_{r_0}(a)$. Since $Du(r_0)$ has bounded variation, $Du^*(\partial B_{r_0})$ is a bounded set, and so u is Lipschitz in $B_r(a)$.

Now define $u_\varepsilon(x) = \frac{1}{\varepsilon}[u(a+\varepsilon x) - u(a)]$. Note that u_ε is Lipschitz in $B_{r_0/\varepsilon}(0)$, with a uniform Lipschitz constant. We claim that there exists a function $u_0 \colon \mathbb{R}^2 \to \mathbb{R}$, homogeneous of degree 1 and satisfying (6.1), such that

$$u_\varepsilon \to u_0 \text{ locally uniformly and in the sense of Corollary 4.3.} \tag{6.17}$$

The point is that we do not pass to a subsequence.

Step 4.1. Toward this end, we first observe that

$$[du_\varepsilon] = \eta_{\varepsilon\#}[du], \quad \eta_\varepsilon(x,\xi) = \Big(\frac{x-a}{\varepsilon}, \xi\Big).$$

This is checked by verifying that the right-hand side satisfies the defining properties (2.5)–(2.8) of $[du_\varepsilon]$. The above implies in particular that if $\phi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and if ε is small enough that $\phi \circ \eta_e \in C_c^\infty(\Omega \times \mathbb{R}^2)$, then

$$\begin{aligned} [du_\varepsilon](\phi\, dx_i \wedge d\xi_j) = {} & [du]\Big(\phi \circ \eta_\varepsilon \frac{dx_i}{\varepsilon} \wedge d\xi_j\Big) \\ = {} & \int_{\Gamma_v} H_\xi\Big(\phi \circ \eta_\varepsilon \frac{dx_i}{\varepsilon}\Big) \langle d\xi_j, \tau_v(\xi)\rangle \mathcal{H}^1(d\xi) \\ = {} & \int_{\Gamma_v} f_\varepsilon(\xi) \langle d\xi_j, \tau_v(\xi)\rangle \mathcal{H}^1(d\xi), \end{aligned} \tag{6.18}$$

where we write

$$f_\varepsilon(\xi) := H_\xi\Big(\phi \circ \eta_\varepsilon \frac{dx_i}{\varepsilon}\Big).$$

Step 4.2. It follows from the explicit description of H_ξ in Lemma 6.2 that

$$\begin{aligned} f_\varepsilon(\xi) &= \sum_i m^i(\xi) \int_{\ell_i(\xi)} \phi\Big(\frac{x-a}{\varepsilon}, \xi\Big) \Big\langle \frac{dx_i}{\varepsilon}, \tau_h(\xi)\Big\rangle \mathcal{H}^1(dx) \\ &= \sum_i m^i(\xi) \int_{\eta_\varepsilon(\ell_i(\xi))} \phi(x,\xi) \langle dx_i, \tau_h(\xi)\rangle \mathcal{H}^1(dx). \end{aligned}$$

Define $S_0(a) = \bigcap_{r>0} S_r(a)$. It follows from (6.9) that (writing S_0 for short)

$$S_0 = \{\xi \in \Gamma_v : H_\xi \text{ is well defined and } a \in \bigcup \ell_i(\xi)\}. \tag{6.19}$$

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Then by sending $\varepsilon \to 0$, we find that for $\mathcal{H}^1$ a.e. $\xi \in \Gamma_\nu$, $f_\varepsilon(\xi) \to f_0(\xi) = H_{0\,\xi}(\phi\, dx_i)$ as $\varepsilon \to 0$, where

$$H_{0\,\xi}(\phi\, dx_i) := \begin{cases} m(\xi) \int_{\ell(\xi)} \phi(x,\xi)\langle dx_i, \tau_h(\xi)\rangle \mathcal{H}^1(dx) & \text{if } \xi \in S_0, \\ 0 & \text{if not,} \end{cases} \tag{6.20}$$

where $\ell(\xi)$ is the line through the origin with tangent $\tau_h(\xi)$, and $m(\xi) = m^i(\xi)$ for the unique i such that $a \in \ell_i(\xi)$.

Step 4.3. We now show that $|f_\varepsilon(\,\cdot\,)|$ is dominated by a locally integrable function. We fix M such that ϕ is supported in $\{(x,\xi) : |x| \le M\}$, and we define

$$F_\varepsilon(\xi) := \frac{1}{\varepsilon} \sum_i m^i(\xi)\, \mathcal{H}^1\left(\ell_i(\xi) \cap B_{\varepsilon M}(a)\right) = \frac{1}{\varepsilon} \|H_\xi\|(B_{\varepsilon M}(a) \times \mathbb{R}^2).$$

For any straight line ℓ, it is clear that $\varepsilon \mapsto \frac{1}{\varepsilon}\mathcal{H}^1(\ell \cap B_{\varepsilon M}(a))$ is a nondecreasing function. Thus if $\varepsilon_0 \le r_0/M$, and $\varepsilon < \varepsilon_0$, then

$$|f_\varepsilon(\xi)| \le \|\phi\|_\infty F_\varepsilon(\xi) \le \|\phi\|_\infty F_{\varepsilon_0}(\xi).$$

Also, $F_{\varepsilon_0}(\,\cdot\,)$ is nonnegative, and it follows from (3.7) that

$$\int_{\Gamma_\nu} F_{\varepsilon_0}(\xi)\mathcal{H}^1(d\xi) < \infty. \tag{6.21}$$

Hence we deduce from the dominated convergence theorem and (6.18), (6.20)

$$[du_\varepsilon](\phi dx_i \wedge d\xi_j) \to \int_{S_0} H_{0\,\xi}(\phi dx_i)\mathcal{H}^1(d\xi)\langle d\xi_j, \tau_\nu(\xi)\rangle \tag{6.22}$$

as $\varepsilon \to 0$. Note from (6.10) and (6.16) that $\mathcal{H}^1(S_0) > 0$.

Step 4.4. It is easy to check that $[du_\varepsilon](\phi d\xi_i \wedge d\xi_j) = 0$ for $\phi \in C_c^\infty(\eta_\varepsilon(\Omega) \times \mathbb{R}^2)$, so (6.21) and (6.18) imply that $\|[du_\varepsilon]\|(K \times \mathbb{R}^2) \le C(K) < \infty$ for every compact K. Since the Lipschitz constants of $\{u_\varepsilon\}$ are locally uniformly bounded, we can pass to a subsequence $\{u_{\varepsilon_n}\}$ that converges to a limit locally uniformly and in the sense of Corollary 4.3. In order to show that the whole sequence converges, we must show that there is a unique such limit $u_0 \colon \mathbb{R}^2 \to \mathbb{R}$. Thus, suppose we have a different subsequence $\{u_{\varepsilon_n'}\}$ converging to a limit u_1. We claim that $u_0 = u_1$. Since $u_\varepsilon(0) = 0$ for every ε, clearly $u_0(0) = u_1(0) = 0$. And since ϕ was arbitrary in Step 4.3, we deduce that

$$[du_0](\phi dx_i \wedge d\xi_j) = \int_{S_0} H_{0\,\xi}(\phi dx_i)\langle d\xi_j, \tau_\nu(\xi)\rangle \mathcal{H}^1(d\xi) \tag{6.23}$$

for all $\phi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and all i, j. Clearly the same holds for u_1. It follows that $\langle [du_0], \rho, r\rangle = \langle [du_1], \rho, r\rangle$ for $\mathcal{L}^1$ a.e. r. Fix r such that (6.7) holds for both $[du_0]$

and $[du_1]$. By inspection of (6.7), we then infer that the $\mathbb{R}^n$-valued measures $\partial_s Du_0(r)$ and $\partial_s Du_1(r)$ on $\mathbb{R}/2\pi r\mathbb{Z}$ are equal, which in particular implies that $Du_0(x_r(\cdot))$ and $Du_1(x_r(\,\cdot\,))$ have the same jump sets, and moreover that $Du_0^{\pm}(x_r(s)) = Du_1^{\pm}(x_r(s))$ at points s in the jump set. We further read off from (6.7) that $Du_0(x_r(s)) = Du_1(x_r(s))$ for $\|\partial_s Du_0(r)\|$ a.e. $s \in \mathbb{R}/2\pi r\mathbb{Z}$ away from the jump set. Also, from (6.22) one can check that $\phi \mapsto \langle [du_0], \rho, r\rangle(\phi d\xi_j)$ is nonzero, which implies that $\|\partial_s Du_0(r)\|$ is a nonzero measure on $\mathbb{R}/2\pi r\mathbb{Z}$. These facts together imply that $Du_0(x_r(s)) = Du_1(x_r(s))$ for $\mathcal{L}^1$ a.e. s. Since this is true for a.e. r, we deduce that $Du_0 = Du_1$, $\mathcal{L}^2$ a.e. Since $u_0(0) = u_1(0) = 0$, it follows that $u_0 = u_1$ as claimed.

Step 4.5. To finish the proof of (6.17) we only need to show that u_0 is homogeneous of degree 1. This, however, follows from the fact that $u_\varepsilon \to u_0$ locally uniformly, since

$$u_0(\lambda x) = \lim_{\varepsilon\to 0} u_\varepsilon(\lambda x) = \lambda \lim_{\varepsilon\to 0} u_{\lambda\varepsilon}(x) = \lambda u_0(x).$$

Step 5. Description of homogeneous solutions. In this step we prove that there exist vectors $p^+, p^- \in \mathbb{R}^2$ such that the blowup limit u_0 found above satisfies

$$Du_0(x) = \begin{cases} p^+ & \text{if } x\cdot(p^+ - p^-) > 0,\\ p^- & \text{if } x\cdot(p^+ - p^-) < 0.\end{cases} \tag{6.24}$$

This amounts essentially to a classification of homogenous, degree 1 solutions of the equation $\det D^2 u = 0$ in the sense of (6.1).

Step 5.1. We first show that

$$\text{there exists } x_0 \neq 0 \text{ such that } u_0(x_0) = -u_0(-x_0). \tag{6.25}$$

Toward this goal, we first notice that

$$\langle [du_0], \rho_0, r\rangle(\phi d\xi_j) = \int_{S_0} \langle H_{0\,\xi}, \rho_0, r\rangle(\phi)\langle d\xi_j, \tau_\nu(\xi)\rangle$$

for $r > 0$. This follows by slicing (6.23) in exactly the same way that we deduced (6.8) from (6.5). (Note that for a homogeneous function, every r is a good radius.) It follows that

$$\operatorname{supp}\langle H_{0\,\xi}, \rho_0, r\rangle \subset \operatorname{supp}\langle [du_0], \rho_0, r\rangle(\phi d\xi_j) \quad \text{for } \mathcal{H}^1 \text{ a.e. } \xi \in S_0.$$

From the explicit form of $H_{0\,\xi}$, we see that $\operatorname{supp}\langle H_{0\,\xi}, \rho_0, r\rangle = \{(\pm x_0, \xi)\}$, where $\{\pm x_0\} = \partial B_r(0) \cap \ell(\xi)$. Therefore, to prove (6.25) it suffices to show that

$$\text{if } (x,\xi) \in \operatorname{supp}\langle [du_0], \rho, r\rangle, \text{ then } u(x) = x\cdot\xi. \tag{6.26}$$

This follows essentially from the description of $\langle [du_0], \rho_0, r\rangle$ in Lemma 6.3, which implies that

$$\begin{aligned}\operatorname{supp}\langle [du_0], \rho_0, r\rangle = \{(x_r(s), Du_0(x_r(s)) : s \notin \mathcal{J}_r\}\\ \cup \Big(\bigcup_{s\in\mathcal{J}_r} \{x_r(s)\} \times [Du_0^-(x_r(s)), Du_0^+(x_r(s))]\Big).\end{aligned} \tag{6.27}$$

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The homogeneity of u_0 implies that

$$u_0(x) = x \cdot Du_0(x) \text{ whenever } x \text{ is a Lebesgue point of } Du. \tag{6.28}$$

Similarly, if $x \neq 0$ is a jump point of Du_0, then since u_0 is continuous, we can pass to limits in (6.28) along sequences of Lebesgue points approaching x from both sides of the jump, to find that

$$x\cdot Du_0^+(x) = x\cdot Du_0^-(x) = u_0(x) \text{ and hence } x\cdot\xi = u_0(x) \text{ for } \xi \in [Du_0^+, Du_0^-]. \tag{6.29}$$

Now (6.26) follows from (6.27), (6.28), and (6.29).

Step 5.2. Now define $v_0(x) := u_0(x) - (x \cdot \frac{x_0}{|x_0|^2})u_0(x_0)$, and note that

$$v_0(\lambda x) = \lambda v_0(x) \text{ for } \lambda > 0, \quad v_0(x_0) = v_0(-x_0) = 0, \quad \det D^2 v_0 = 0 \tag{6.30}$$

in the sense of (6.1). Let us write $v_0(r\cos\theta, r\sin\theta) = rf(\theta)$. We will prove that

$$\int_0^{2\pi} (f^2 - f'^2)d\theta = 0. \tag{6.31}$$

We first prove this under the assumption that v_0 is smooth away from the origin. Then the condition $\det D^2 v_0 = 0$ implies that the constant A in (6.3) must equal zero, where we recall the definition:

$$A = A(v_0) = \frac{1}{2}\int_0^{2\pi} \gamma(\theta) \wedge \gamma'(\theta)d\theta, \quad \gamma(\theta) = Dv_0(\cos\theta, \sin\theta). \tag{6.32}$$

We will temporarily use the notation $n(\theta) = (\cos\theta, \sin\theta)$ and $t(\theta) = n'(\theta)$, so that $\gamma(\theta) = n(\theta)f(\theta) + t(\theta)f'(\theta)$. Then we easily compute that $\gamma \wedge \gamma' = f(f + f'')$, so that (6.31) follows from the identity $A = 0$ via integration by parts.

If v_0 is not smooth, then let $v_k(r\cos\theta, r\sin\theta) = rf_k(\theta)$ for a sequence of smooth 2π-periodic functions f_k converging to f in $W^{1,2}$ and such that $\int |f_k''|d\theta \leq C$. Such a sequence exists, since $Dv_0 \in BV$. This convergence implies that $v_k \to v$ in the sense of Corollary 4.3 (this is proved in [13, Proposition 4.1]). If we define $A_k = A(v_k)$ as in (6.32), then it follows that

$$0 = \lim_{k\to\infty} 2A_k = \lim_{k\to\infty}\int_0^{2\pi} (f_k^2 - f_k'^2)d\theta = \int_0^{2\pi}(f^2 - f'^2)d\theta,$$

which proves (6.31) in the general case.

Step 5.3. In view of (6.30), we see that there exists $\alpha \in [0, \pi)$ such that $f(\alpha) = f(\alpha + \pi) = 0$. Then Poincaré's inequality implies that

$$\int_\alpha^{\alpha+\pi} (f^2 - f'^2)\, d\theta \leq 0, \quad \int_{\alpha-\pi}^{\alpha} (f^2 - f'^2)\, d\theta \leq 0.$$

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In view of (6.31), equality holds in both integrals, and so the optimality conditions in Poincaré's inequality imply that there exist a, b such that $f(\theta) = a\sin(\theta - \alpha)$ for $\theta \in (\alpha, \alpha + \pi)$ and $f(\theta) = b\sin(\theta - \alpha)$ for $\theta \in (\alpha - \pi, \alpha)$. In other words,

$$v_0(x) = \begin{cases} ax \cdot t(\alpha) & \text{if } x \cdot t(\alpha) > 0, \\ bx \cdot t(\alpha) & \text{if } x \cdot t(\alpha) < 0. \end{cases}$$

Since u_0 is the sum of v_0 and a linear function, (6.24) follows.

Step 6. It follows from Steps 4 and 5 that if a is such that (6.16) holds, then $a \in \mathcal{J}_{Du}$. It also follows from Step 3 that if (6.16) does not hold, then $a \notin \mathcal{J}_{Du}$.

We now prove, continuing to assume (6.16), that there is a line segment ℓ_a passing through a and meeting $\partial\Omega$ at both endpoints, such that

$$\ell_a \subset \mathcal{J}_{Du}, \quad [Du^-(a), Du^+(a)] = [Du^-(b), Du^+(b)]\ \forall b \in \ell_a. \tag{6.33}$$

Step 6.1. To do this, we will show below that S_0 as defined in (6.19)

$$S_0 = [Du^-(a), Du^+(a)] \text{ up to a set of } \mathcal{H}^1 \text{ measure } 0 \tag{6.34}$$

and

$$\text{for } \mathcal{H}^1 \text{ a.e. } \xi \in S_0,\ \tau_h(\xi) \text{ is tangent to } \mathcal{J}_{Du} \text{ at } a. \tag{6.35}$$

First, we demonstrate that these will prove (6.33). To do this, let ℓ_a be the line segment passing through a, tangent to $\mathcal{J}_{Du}$ at a, and terminating when it meets $\partial\Omega$. Then (6.35) implies that for $\mathcal{H}^1$ a.e. $\xi \in S_0$, the associated line segments $\ell_i(\xi)$ passing through a all coincide with ℓ_a. Let b denote any other point on ℓ_a. Then $b \in \ell_i(\xi)$ for $\mathcal{H}^1$ a.e. $\xi \in S_0(a)$. It follows that $S_0(a) \subset S_0(b) \subset S_r(b)$ for $r > 0$, and hence that b satisfies (6.16). Thus $b \in \mathcal{J}_{Du}$, and (6.34) implies that $[Du^-(a), Du^+(a)] \subset [Du^-(b), Du^+(b)]$. Reversing the roles of a and b establishes the opposite inclusion and so will prove (6.33), once we have proved (6.34), (6.35).

Step 6.2. We now prove (6.34), (6.35). Since all information about S_0 and $\tau_h(\xi), \xi \in S_0$ is recorded in the blowup limit u_0, we may argue with u_0, about which we know everything, instead of u.

It is convenient to assume that $\mathcal{J}_{Du_0}$ is the x_2-axis. This can be achieved by a change of coordinates. Then there exist numbers p_1^+, p_1^-, p_2, such that

$$Du_0(x_1, x_2) = \begin{cases} (p_1^-, p_2) & \text{if } x_1 < 0, \\ (p_1^+, p_2) & \text{if } x_1 > 0. \end{cases}$$

Moreover, $u_{0,x_i x_j} = 0$ unless $i = j = 1$, and $u_{0,x_1 x_1}$ is a 1-dimensional Hausdorff measure restricted to the x_2 axis, multiplied by the constant $p_1^+ - p_1^-$.

By combining (5.1), and (6.24), we arrive at

$$[du_0](\phi dx_i \wedge d\xi_j) = \begin{cases} -\displaystyle\int_{\{(x_1,x_2)\in\mathbb{R}^2 : x_1=0\}} \bar{\phi}(x)\mathcal{H}^1(dx) & \text{if } i=2,\ j=1, \\ 0 & \text{if not,} \end{cases}$$

where $\bar{\phi}(x) = \int_{p_1^-}^{p_1^+} \phi(x,(\xi_1,p_2))\mathcal{L}^1(d\xi_1)$ for $x \in \{(x_1,x_2)\in\mathbb{R}^2 : x_1 = 0\}$. By Fubini's Theorem,

$$[du_0](\phi\, dx_2 \wedge d\xi_1) = \int_{p_1^-}^{p_1^+}\int_{\mathbb{R}} \phi((0,x_2),(\xi_1,p_2))\, dx_2 d\xi_1.$$

On the other hand, from (6.23) and (6.20) we write

$$[du_0](\phi dx_i \wedge d\xi_j) = \int_{S_0} m(\xi)\int_{\ell(\xi)} \phi(x,\xi)\langle dx_i, \tau_h(\xi)\rangle \mathcal{H}^1(dx)\langle d\xi_j, \tau_v(\xi)\rangle\mathcal{H}^1(d\xi).$$

From the fact that $[du_0](\phi dx_i \wedge d\xi_j) = 0$ for $(i,j) \neq (2,1)$, we conclude that $\tau_h(\xi) = \pm e_2, \tau_v(\xi) = \pm\varepsilon_1$ for $\mathcal{H}^1$ a.e. $\xi \in S_0$, which includes the claim (6.35). And by comparing the above two identities for $[du_0](\phi dx_2 \wedge d\xi_1)$, we deduce (6.34), and as already noted, (6.33) follows.

Step 7. Let $\mathcal{A} := \{x \in \Omega : u$ is affine in some neighborhood of $x\}$. To complete the proof, it remains to show that if $a \notin (\mathcal{A} \cup \mathcal{J}_{Du})$, then there exists a line segment ℓ_a passing through a and meeting $\partial\Omega$ at both endpoints, such that every point of ℓ_a is a Lebesgue point for Du, and $Du(x) = Du(a)$ for all $x \in \ell_a$.

Step 7.1. We first prove that there is a subset of Γ_v of full $\mathcal{H}^1$ measure, say Γ_v^0, such that

(6.36) $$\text{if } \xi \in \Gamma_v^0, \text{ then } Du(x) = \xi \text{ for every } x \in (\cup_i \ell_i(\xi)) \setminus \mathcal{J}_{Du}.$$

We will use the notation $f_{\phi,i}(\xi) = H_\xi(\phi dx_i)$. Recall that $f_{\phi,i}\colon \Gamma_v \to \mathbb{R}$ is $\mathcal{H}^1$-measurable for $i = 1,2$ and every $\phi \in C_0(\Omega\times\mathbb{R}^2)$, which denotes as usual the sup-norm closure of $C_c^\infty(\Omega\times\mathbb{R}^2)$. Let $\{\phi_k\}_{k=1}^\infty \subset C_c^\infty(\Omega\times\mathbb{R}^2)$ be a countable dense subset of $C_0(\Omega\times\mathbb{R}^2)$. We fix $\epsilon > 0$, and for each $k = 1,2,\dots$, we apply Lusin's Theorem to find that there exists a set $E_k \subset \Gamma_v$ such that $\mathcal{H}^1(E_k) \le \epsilon 2^{-k}$, and such that the restriction to $\Gamma_v \setminus E_k$ of $f_{\phi_k,i}$ is continuous for $i = 1,2$. Let $E_\epsilon := \bigcup E_k$, and let $\Gamma_v^\epsilon := \Gamma_v \setminus E_\epsilon$. It follows that the restriction to Γ_v^ϵ of $\xi \mapsto H_\xi(\phi\, dx_i)$ is continuous for every $\phi \in C_0(\Omega\times\mathbb{R}^2)$ and $i = 1,2$, and also that $\mathcal{H}^1(E_\epsilon) = \mathcal{H}^1(\Gamma_v \setminus \Gamma_v^\epsilon) \le \epsilon$. By discarding a set of $\mathcal{H}^1$ measure 0 from Γ_v^ϵ if necessary, we can arrange that

(6.37) $$\mathcal{H}^1((\Gamma_v^\epsilon) \cap B_s(\xi)) > 0 \quad \forall s > 0$$

for every $\xi \in \Gamma_v^\epsilon$, while preserving the condition $\mathcal{H}^1(\Gamma_v \setminus \Gamma_v^\epsilon) \le \epsilon$. We claim that

(6.38) $$\text{if } \xi \in \Gamma_v^\epsilon, \text{ then } Du(a) = \xi \text{ for every } a \in (\cup_i \ell_i(\xi)) \setminus \mathcal{J}_{Du}.$$

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This will prove (6.36), since if we define $\Gamma_\nu^0 = \bigcup_k \Gamma_\nu^{\epsilon_k}$ for some sequence $\epsilon_k \to 0$, then (6.38) implies that Γ_ν^0 has the properties required in (6.36).

To prove (6.38), fix $\xi \in \Gamma_\nu^\epsilon$ and $a \in (\bigcup_i \ell_i(\xi)) \setminus \mathcal{J}_{Du}$. Step 3 shows that in order to prove that $Du(a) = \xi$, it suffices to show that $\xi \in Du^*(\partial B_r)(a)$ for every good $r < \operatorname{dist}(a, \partial\Omega)$. Fix some such r, and fix a smooth $\phi\, dx_i$ with compact support in $B_r(a) \times \mathbb{R}^2$ such that $H_\xi(\phi\, dx_i) \neq 0$. This is possible because $a \in \ell_i(\xi)$ for some i. Since $f_{\phi,i}$ is continuous in Γ_ν^ϵ, we deduce that $H_{\xi'}(\phi\, dx_i) \neq 0$ for all $\xi' \in \Gamma_\nu^\epsilon$ sufficiently close to ξ. For such ξ', it follows that $\bigcup \ell_i(\xi')$ must intersect $B_r(a)$, or equivalently that $\xi' \in S_r(a)$. Thus (6.37) implies that every neighborhood of ξ intersects $S_r(a)$ in a set of positive measure, and hence (in view of (6.11)) every neighborhood of ξ contains points of $Du^*(\partial B_r)(a)$. But $Du^*(\partial B_r)(a)$ is closed, so $\xi \in Du^*(\partial B_r)(a)$, and (6.38) follows.

Step 7.2. Note that if $\xi \in \Gamma_\nu^0$ then for every i, either $\ell_i(\xi) \subset \mathcal{J}_{Du}$ or $\ell_i(\xi) \cap \mathcal{J}_{Du} = \varnothing$. Indeed, if $\ell_i(\xi) \not\subset \mathcal{J}_{Du}$, then since $\ell_i(\xi)$ is a line segment, and $\mathcal{J}_{Du}$ is also a union of line segments, $\ell_i(\xi)$ and $\mathcal{J}_{Du}$ can only intersect transversally. But this would imply that Du jumps where $\ell_i(\xi)$ crosses $\mathcal{J}_{Du}$, which is impossible, since $Du \equiv \xi$ on $\ell_i(\xi) \setminus \mathcal{J}_{Du}$.

Similarly, if $\xi, \xi' \in \Gamma_\nu^0$ and $\ell_i(\xi), \ell_j(\xi')$ do not intersect $\mathcal{J}_{Du}$, then $\ell_i(\xi) \cap \ell_i(\xi') = \varnothing$, since if $a \in \ell_i(\xi) \cap \ell_j(\xi')$, then a is a Lebesgue point of Du with $Du(a) = \xi = \xi'$, which is clearly impossible. Thus if $\xi, \xi' \in \Gamma_\nu^0$, then $\ell_i(\xi)$ and $\ell_{i'}(\xi')$ either coincide or are disjoint.

Step 7.3. Now fix $a \notin \mathcal{J}_{Du} \cup \mathcal{A}$. In view of (6.13), (6.14),

$$0 < \mathcal{H}^1(Du^*(\partial B_r(a))) = \mathcal{H}^1(S_r(a)) \to 0 \quad \text{as } r \to 0,$$

Let $S_r^*(a) = S_r(a) \cap Du^*(\partial B_r)(a) \cap \Gamma_\nu^0$, and let

$$\mathcal{L}_r = \{\ell_i(\xi) : \xi \in S_r^*(a), \ell_i(\xi) \cap B_r(a) \neq \varnothing\}.$$

For $\ell \in \mathcal{L}_r$, let τ_ℓ denote a unit tangent vector. We have just argued that these segments are pairwise disjoint. This implies that if r is small enough, then the signs of these tangent vectors can be chosen so that $\tau_\ell \cdot \tau_{\ell'} > \frac{1}{2}$ for all $\ell, \ell' \in \mathcal{L}_r$, and then it further follows from disjointness that there exists some unit vector τ such that

$$\sup_{\ell \in \mathcal{L}_r} |\tau_\ell - \tau| \to 0 \quad \text{as } r \to 0.$$

Let ℓ_a denote the segment passing through a with unit tangent τ. Note that ℓ_a is not a subset of $\mathcal{J}_{Du}$, since by assumption $a \notin \mathcal{J}_{Du}$. So ℓ_a can only intersect $\mathcal{J}_{Du}$ transversally. If this occurs, then there must exist some $\ell \in \mathcal{L}_r$ that intersects $\mathcal{J}_{Du}$ transversally, which is impossible. So every point of ℓ_a is a Lebesgue point of Du.

We finally argue that $Du(x) = Du(a)$ for every $x \in \ell_a$. To see this, recall first from Step 3 that

$$\sup_{\xi \in S_r^*(a)} |\xi - Du(a)| \le \sup_{\xi \in Du^*(\partial B_r)(a)} |\xi - Du(a)| \to 0 \quad \text{as } r \to 0.$$

Fix $x \in \ell_a$ and, given $\varepsilon > 0$, fix $\rho > 0$ so small that $|\xi - Du(x)| < \varepsilon$ for all $\xi \in Du^*(\partial B_\rho(x))$. Since the tangents to segments in $\mathcal{L}_r$ converge to τ as $r \to 0$, it is clear that for r sufficiently small, every ℓ in $\mathcal{L}_r$ must intersect $B_\rho(x)$, which implies that $S_r^*(a) \subset S_\rho(x)$ for r sufficiently small. Then (6.11) implies that $Du^*(\partial B_\rho(x)) \cap S_r^*(a) \neq \varnothing$ for r sufficiently small. Then

$$\begin{aligned} |Du(a) - Du(x)| &\leq \sup_{\xi \in S_r^*(a) \cap Du^*(\partial B_\rho(x))} \big[|Du(a) - \xi| + |\xi - Du(x)|\big] \\ &\leq \varepsilon + \sup_{\xi \in S_r^*(a)} |Du(a) - \xi| \to \varepsilon \quad \text{as } r \to 0. \end{aligned}$$

Since ε is arbitrary, the conclusion follows. ∎

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