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ON THE NUMBER OF NORMAL SUBGROUPS OF AN UNCOUNTABLE GROUP

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Abstract

In this paper two theorems are proved that give a partial answer to a question posed by G. Behrendt and P. Neumann. Firstly, the existence of a group of cardinality $\aleph_1$ with exactly $\aleph_1$ normal subgroups, yet having a subgroup of index 2 with $2^{\aleph_1}$ normal subgroups, is consistent with ZFC (the Zermelo-Fraenkel axioms for set theory together with the Axiom of Choice). Secondly, the statement "Every metabelian-by-finite group of cardinality $\aleph_1$ has $2^{\aleph_1}$ normal subgroups" is consistent with ZFC.

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1. Introduction

We are interested here in the following question.

QUESTION 1. Given any uncountable cardinal $\mathfrak{m}$, does there exist a group of cardinality $\mathfrak{m}$ with at most $\mathfrak{m}$ normal subgroups, having a finite-index subgroup with $2^{\mathfrak{m}}$ normal subgroups?

The analogous question for the countable case was answered affirmatively by G. Behrendt and P. M. Neumann in their paper [1], which concludes with the remark that they were unable to generalize their method to the situation of uncountable cardinals. Here we prove two, somewhat complementary, consistency results, which at least eliminate some of the directions in which one might be tempted to proceed in trying to settle Question 1. Partly for ease of presentation we shall throughout take $\mathfrak{m} = \aleph_1$, the first uncountable cardinal, although our

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results hold for certain other uncountable cardinals as well. Our first main result shows that Question 1 cannot have a negative answer within the system ZFC.

THEOREM 1. *The existence of a group of cardinality* $\aleph_1$ *with exactly* $\aleph_1$ *normal subgroups, yet having a subgroup of index* 2 *with* $2^{\aleph_1}$ *normal subgroups, is consistent with* ZFC (*the Zermelo-Fraenkel axioms for set theory together with the Axiom of Choice*).

(In fact, our construction yields the consistency with ZFC of the existence of such a group having only $\aleph_0$ normal subgroups.)

The countable example of Behrendt and Neumann in [1] is, at least in its simplest form, abelian-by-finite. Our second main result shows that an uncountable example with the property given in Question 1, if there is one, will certainly have a more complex structure.

THEOREM 2. *The statement "Every metabelian-by-finite group of cardinality* $\aleph_1$ *has* $2^{\aleph_1}$ *normal subgroups" is consistent with* ZFC.

The key to both theorems is the known consistency with ZFC of the statement that $2^{\aleph_0} = 2^{\aleph_1}$ (which holds for certain uncountable cardinals other than $\aleph_1$ also; see [6, Section 19]). In fact in proving Theorem 1 we exploit the same properties of the quasi-cyclic group $\mathbb{Z}_{p^\infty}$ as do Neumann and Behrendt [1], namely that while it has only $\aleph_0$ subgroups, it has $2^{\aleph_0}$ endomorphisms, so that its direct square $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ has $2^{\aleph_0}$ subgroups (one for each $\eta \in \operatorname{End} \mathbb{Z}_{p^\infty}$, namely $S_\eta = \{(x, x\eta) \mid x \in \mathbb{Z}_{p^\infty}\}$). Thus in seeking an answer to Question 1, it is natural to ask for a module of uncountable cardinality with the analogous property.

QUESTION 2. Does there exist a module of cardinality $\aleph_1$ with only $\aleph_1$ submodules but with $2^{\aleph_1}$ endomorphisms? (Note that by [3, Theorem 1] the underlying ring of such a module cannot be (right) Noetherian of cardinality less than $\aleph_1$.)

In the course of proving Theorem 1 it emerges that Question 2 cannot have a negative answer. (However this consistency result of course gets us no further with Question 1.)

PROPOSITION 1. *An affirmative answer to Question* 2 *is consistent with* ZFC. (*In fact it is consistent with* ZFC *that there exist such a module with only* $\aleph_0$ *submodules.*)

In connexion with Question 2 the following simple result may also be of interest.

PROPOSITION 2. *A module of cardinality* $\aleph_1$ *with fewer than* $\aleph_1$ *submodules has at most* $2^{\aleph_0}$ *endomorphisms* (*in fact at most* $\aleph_1$ *endomorphisms if it has only finitely many submodules*).

If Question 2 should have an affirmative answer (i.e. within the system ZFC) then with an eye to the original Question 1, we might ask more specifically

QUESTION 3. Is there a module of cardinality $\aleph_1$ over a countable group ring $\mathbb{Z}[G]$, with only $\aleph_1$ submodules but with $2^{\aleph_1}$ endomorphisms?

(Note that in view of [3, Theorem 3] one cannot hope for an abelian group G in an affirmative answer to this question.)

We thank Juris Steprāns for some helpful remarks.

2. Proofs of Theorem 1 and the propositions

PROOF OF THEOREM 1. Denote by D the direct sum

$$D = \bigoplus \{A_\alpha | \alpha \in \aleph_1\},$$

where each $A_\alpha \cong \mathbb{Z}_{p^\infty}$ for some (single) fixed odd prime p, and where $\aleph_1$ is to be thought of as the least ordinal of cardinality $\aleph_1$ (and an ordinal as the set of its predecessors). For each $\alpha \in \aleph_1$ present A_α additively as follows:

$$A_\alpha = \langle a_{\alpha,1}, a_{\alpha,2}, \ldots \mid pa_{\alpha,1} = 0,\ pa_{\alpha,i+1} = a_{\alpha,i},\ i = 1, 2, \ldots \rangle.$$

Let B denote the discrete alternating group of permutations of the index set $\aleph_1$, i.e. the group of all even permutations of $\aleph_1$ fixing all but finitely many elements; then an action of B on D is defined in the obvious way by

$$(a_{\alpha,i})g = a_{\alpha g,i}, \qquad \alpha \in \aleph_1, \qquad i = 1, 2, \ldots, \qquad g \in B,$$

so that B merely permutes the A_α as wholes. For each $i = 1, 2, \ldots,$ write L_i for the ith "layer" of D, i.e. for the subgroup of D generated by all $a_{\alpha,i}$, $\alpha \in \aleph_1$; thus L_i is a direct sum of $\aleph_1$ p^i-cycles. It is clear that each L_i is B-invariant. Furthermore any B-invariant subgroup S of L_i not contained in L_{i-1} (where L_0 is defined as the trivial subgroup) will contain all elements of the form $a_{\mu,i} - a_{\nu,i}$ $\mu, \nu \in \aleph_1$, $\mu \neq \nu$. To see this observe first that for some $\alpha \in \aleph_1$, S must contain an element of the form $a_{\alpha,i} + x$, where $x \in L_i$ is a (finite) sum of multiples of elements $a_{\beta,j}$ with $\beta \neq \alpha$, $j \leqslant i$. By applying to this element any permutation from B fixing all these β and sending α to some $\gamma \in \aleph_1$ different from α, we see that also $a_{\gamma,i} + x \in S$, whence $a_{\alpha,i} - a_{\gamma,i} \in S$, and then the application of any permutation in B sending α to μ, γ to ν, yields finally $a_{\mu,i} - a_{\nu,i} \in S$.

Write M_i for the subgroup of L_i generated by all $a_{\mu,i} - a_{\nu,i}$; it is clear that M_i has index p^i in L_i, and also that $M_i > M_j$ for $i > j$. It follows that each non-trivial B-invariant subgroup S of D is the union of a finite or countably infinite chain $S_1 < S_2 < \cdots$ of non-trivial subgroups, where each S_i $(= S \cap L_i)$ is either L_i or M_i or $M_i + L_j$ for some $0 < j < i$; moreover, if $S_i = M_i$ or $M_i + L_j$, then $S_k = M_k$ or $M_k + L_j$, respectively, for all subsequent S_k in the chain $(k > i)$. Hence D contains only $\aleph_0$ B-invariant subgroups (whence, iciden-tally, since B is simple, it follows easily that the semi-direct product DB, a "permutational wreath product", has only $\aleph_0$ normal subgroups).

Now each endomorphism θ of

$$\mathbf{Z}_{p^\infty} = A = \langle a_1, a_2, \ldots \mid pa_1 = 0,\ pa_{i+1} = a_i,\ i = 1, 2, \ldots \rangle$$

determines an endomorphism θ_D of D, defined by

$$\theta_D\colon a_{\alpha,i} \mapsto a_i\theta\phi_\alpha,\ \alpha \in \aleph_1, \qquad i = 1, 2, \ldots,$$

where $\phi_\alpha\colon A \to A_\alpha$ is the obvious isomorphism; thus θ_D maps the summands A_α of D "uniformly", and is easily seen to commute with the action of B on D. Hence the $\mathbf{Z}[B]$-module D has at least $2^{\aleph_0}$ endomorphisms, since card(End $\mathbf{Z}_{p^\infty}$) $= 2^{\aleph_0}$. Imitating [1], we now observe that for each $\theta_D \in \operatorname{End} D$ the set $\{(x, x\theta_D) \mid x \in D\}$ forms a submodule of $D \oplus D$, and that distinct elements of End D give rise in this way to distinct submodules of $D \oplus D$. Hence $D \oplus D$ has at least $2^{\aleph_0}$ B-invariant subgroups. Continuing as in [1], we let T be a 2-cycle whose generator t acts on $D \oplus D$ according to the rule $(x, y)t = (x, -y)$. This action turns $D \oplus D$ into a $\mathbf{Z}[B \times T]$-module, which since p is odd is easily seen to have only $\aleph_0$ submodules. From this and from the simplicity of B it follows readily that the semi-direct product $R = (D \oplus D)(B \times T)$ is a group of cardinal-ity $\aleph_1$ with $\aleph_0$ normal subgroups, but with a subgroup of index 2, namely $U = (D \oplus D)B$, with $2^{\aleph_0}$ normal subgroups.

Finally, to obtain the full theorem we need (somewhat artificially) to boost the number of normal subgroups to $\aleph_1$. To this end let K be a field of characteristic 2 and cardinality $\aleph_1$, write $P = \mathrm{PSL}(2, K)$, and consider the $\mathbf{Z}[P]$-module $K^2 \oplus K^2$, where P acts naturally on each $K^2 = K \oplus K$. Then (cf. the conclusion of [1]) the semi-direct product $Q = (K^2 \oplus K^2)P$ has precisely $\aleph_1$ normal sub-groups, and by using the simplicity of $\mathrm{PSL}(2, K)$ and the fact that $p \neq 2$, it follows (see below) that $R \times Q$, of cardinality $\aleph_1$, has exactly $\aleph_1$ normal subgroups, yet has a subgroup of index 2, namely $U \times Q$, with at least $2^{\aleph_0}$ normal subgroups. The theorem now follows from the aforementioned con-sistency with ZFC of the statement that $2^{\aleph_0} = 2^{\aleph_1}$.

To see that $R \times Q$ has exactly $\aleph_1$ normal subgroups, consider it as the semi-direct product WG of $W = (D \oplus D) \oplus (K^2 \oplus K^2)$ by $G = (B \times T) \times P$, where the actions of $B \times T$ on $D \oplus D$ and of P on $K^2 \oplus K^2$ are as defined above, and where $B \times T$ and P act trivially on $K^2 \oplus K^2$ and $D \oplus D$, respectively. For each $N \trianglelefteq WG$, write N_W for its projection on W and N_G for its projection on G. Then, as noted in the concluding paragraph of [3], N_W is a normal subgroup of WG (contained in W), N_G is a normal subgroup of G, and N is determined by the groups N_W, N_G, and $N \cap W$, together with the map f_N: $N_G \to W/N \cap W$ defined by f_N: $g \mapsto w(N \cap W)$, where $gw \in N$. It is not difficult to see that f_N is well-defined and in fact a homomorphism, so that $N_G/\mathrm{Ker} f_N$ is either trivial or a 2-cycle (since P and B are non-abelian simple, while W is abelian). Hence, for each choice of N_G, there are at most $\aleph_1$ possibilities for f_N. Since there are exactly $\aleph_1$ choices for N_W and $N \cap W$ (as normal subgroups of WG contained in W; here we are using the fact that p is odd while $\mathrm{char}\, K = 2$), and since there are only finitely many choices for N_G, it follows that indeed $WG = R \times Q$ has exactly $\aleph_1$ normal subgroups N.

Proof of Proposition 1. Consider the $\mathbb{Z}[B \times P]$-module $D \oplus (K^2 \oplus K^2)$, where the actions of B on D and of P on $K^2 \oplus K^2$ are as defined in the preceding proof, and where B and P act trivially on $K^2 \oplus K^2$ and D, respectively. It is straightforward to see that since the module D has only $\aleph_0$ submodules and at least $2^{\aleph_0}$ endomorphisms, while the module $K^2 \oplus K^2$ has exactly $\aleph_1$ submodules, and also since $p \neq 2$, the module $D \oplus (K^2 \oplus K^2)$ has exactly $\aleph_1$ submodules and at least $2^{\aleph_0}$ endomorphisms. Proposition 1 now follows by invoking as before the consistency with ZFC of the statement that $2^{\aleph_0} = 2^{\aleph_1}$.

Proof of Proposition 2 (cf. the conclusion of [1]). Let M be a module of cardinality $\aleph_1$ with fewer thatn $\aleph_1$ submodules. Then, as in the proof of the theorem, $M \oplus M$ has at least card(End M) submodules. Consider the automorphisms a, b of $M \oplus M$ defined by

$$(x, y)a = (x + y, -y), \quad (x, y)b = (y, x).$$

It is a matter of direct calculation to see that the subgroup C of $\mathrm{Aut}(M \oplus M)$ generated by a and b is finite (since $a^2 = b^2 = 1$ and ab has order at most 6). If N is any C-invariant submodule of $M \oplus M$, and if $(x, y) \in N$, then also

$$(x, y) + (x, y)a - (x, y)ba = (x, x) \in N,$$

and then further

$$(x, x) - (x, y) = (0, x - y) \in N.$$

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Hence each C-invariant submodule of $M \oplus M$ is determined by its intersection with the diagonal of $M \oplus M$, together with its intersection with either summand, so that $M \oplus M$ has fewer than $\aleph_1$ C-invariant submodules (say $\mathfrak{k}$), since M has fewer than $\aleph_1$ submodules. (Note that, moreover, if M has only finitely many submodules, then $\mathfrak{k}$ will also be finite.) By the Theorem and Corollary 1.4 of [4], this implies that $M \oplus M$ can have no chain of more than $(\operatorname{card} C) \times \mathfrak{k} \leqslant 12\mathfrak{k}$ (not necessarily C-invariant) submodules. Hence each submodule of $M \oplus M$ is generated by at most $12\mathfrak{k}$ elements, so that $M \oplus M$ can have at most $\aleph_1^{12\mathfrak{k}}$ submodules. Since $\mathfrak{k} < \aleph_1$ by hypothesis, either $\mathfrak{k}$ is finite or $\mathfrak{k} = \aleph_0$; in the former case $\aleph_1^{12\mathfrak{k}} = \aleph_1$, and in the latter $2^{\aleph_0}$. Since $M \oplus M$ has at least $\operatorname{card}(\operatorname{End} M)$ submodules, the proposition follows.

3. Proof of Theorem 2

Theorem 2 follows readily (as shown below) from the following lemma (and from the known consistency with ZFC of the statement $2^{\aleph_0} = 2^{\aleph_1}$). For the purpose of this lemma we assume that our rings all have a multiplicative identity, and that all modules are unital. A ring T is said to be a *finite normalizing extension* of a subring R if there exist elements $a_1, \dots, a_k \in T$ (where we may suppose that $a_1 = 1$) such that $T = \sum_{j=1}^{k} a_j R$, and such that $a_j R = R a_j$ for $j = 1, 2, \dots, k$.

LEMMA. *Let T be a finite normalizing extension of a commutative subring R of cardinality $\leqslant \aleph_0$. Then every uncountable T-module $M = M_T$ contains at least $2^{\aleph_0}$ T-submodules.*

(We note that the case $k = 1$, i.e. $T = R$, of this lemma is just Theorem 4.2 of [7], and also that Theorem C of [5] places strong limitations on the possibility of weakening the hypothesis that T be a finite normalizing extension of R.)

To see how Theorem 2 follows from this lemma, suppose, as in that theorem, that G is a group of cardinality $\aleph_1$ with a metabelian normal subgroup H of finite index, and let H' denote the commutator subgroup of H. Then the integral group ring $\mathbb{Z}[G/H']$ is a finite normalizing extension of $\mathbb{Z}[H/H']$, and taking, in the notation of the lemma, $M = H'$, $T = \mathbb{Z}[G/H']$, $R = \mathbb{Z}[H/H']$ (with the appropriate actions on H'), we infer from the lemma that if $\operatorname{card}(H/H') \leqslant \aleph_0$, then the $\mathbb{Z}[G/H']$-module H' has at least $2^{\aleph_0}$ submodules. In the group context this translates into the existence of at least $2^{\aleph_0}$ normal subgroups of G contained in H', and then the desired conclusion follows from the consistency of the statement $2^{\aleph_0} = 2^{\aleph_1}$ with ZFC. If, on the other hand, $\operatorname{card}(H/H') = \aleph_1$, then by

[3, Theorem 1 and Note added in proof] G/H' has $2^{\aleph_1}$ normal subgroups, whence so does G.

Proof of Lemma. It is easily seen that it suffices to show that the module $M = M_T$ contains a countably infinite "semi-independent" set $\{s_1, s_2, \dots\}$ of elements, i.e. having the property that, for each $i = 1, 2, \dots,$ the element s_i does not belong to the submodule of M_T generated by all $s_j \neq s_i$.

This in turn will follow if we can show that M_T must contain a "large" proper submodule, i.e. a module of cardinality $\geqslant \aleph_1$. For, assuming this, we can define, in imitation of the proof of Lemma 4.1 of [7], a descending chain $M_1 \supset M_2 \supset \cdots$, of T-submodules of M, and a sequence of elements $s_1, s_2, \dots,$ such that, for all $n = 1, 2, \dots,$ we have

$$\text{(1)} \qquad \operatorname{card} M_n \geqslant \aleph_1, \qquad s_{n+1} \in M_n,\ s_{n+1} \notin s_1T + \cdots + s_nT + M_{n+1},$$

and clearly a set $\{s_1, s_2, \dots\}$ with these properties is semi-independent. To see how the presence in modules like M_T of uncountable proper submodules allows the construction of such M_i, s_i, first set $M_1 = M$ and choose s_1 to be any non-zero element of M_1. Suppose inductively that $M_1 \supset \cdots \supset M_k$ and $s_1, \dots, s_k$ have been defined (satisfying (1)). Writing $L = s_1T + \cdots + s_kT$, we clearly have $\operatorname{card} L \leqslant \aleph_0$, whence

$$\operatorname{card}\big((M_k + L)/L\big) = \operatorname{card} M_k \geqslant \aleph_1.$$

Hence, by our assumption, the T-module $(M_k + L)/L$ has a proper submodule of cardinality $\geqslant \aleph_1$. Let $H \subset M_k + L$ be the complete preimage of that module under the natural map $M_k + L \to (M_k + L)/L$. Then $H = (M_k \cap H) + L$, so that $M_k \cap H$ is a proper submodule of M_k of cardinality $\geqslant \aleph_1$. Define $M_{k+1} = M_k \cap H$ and choose $s_{k+1} \in M_k \setminus H$ arbitrarily. It is then easily verified that s_{k+1} and M_{k+1} satisfy (1).

Thus we now have only to show that M_T must contain a proper submodule of cardinality $\geqslant \aleph_1$. Assume the contrary. We may suppose without loss of generality that M_T is faithful, since the property of being a finite normalizing extension is preserved under homomorphic images. Let $m \in M$ be any non-zero element of M and let B be a T-submodule which is maximal with respect to the property that m and B together generate the direct sum $mT \oplus B$. Then $mT \oplus B$ is "essential" in M_T, i.e. intersects non-trivially every non-zero T-submodule of M. By our assumption, B, and therefore also $mT \oplus B$, has cardinality $\leqslant \aleph_0$. Now in a similar manner take C to be an R-submodule of M_R such that $[(mT \oplus B) \oplus C]_R$ is essential in M_R. We claim that $mT \oplus B \oplus C \neq M$. To see this, suppose the contrary, and consider the map

$$\psi\colon C \to (mT \oplus B)^k, \qquad c \mapsto (ca_1{\downarrow}, \dots, ca_k{\downarrow}),$$

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from C to the Cartesian product of k copies of $mT \oplus B$, where, as above, $T = \sum_{i=1}^{k} a_i R$, and where $ca_i \downarrow$ denotes the projection of $ca_i (\in M = mT \oplus B \oplus C$ by assumption) onto $mT \oplus B$. Since $\operatorname{card}(mT \oplus B) \leqslant \aleph_0$ and $\operatorname{card} M > \aleph_0$, so that we must have $\operatorname{card} C > \aleph_0$ for $M = mT \oplus B \oplus C$ to hold, there must certainly be two distinct elements c_1, c_2 of C such that $\psi(c_1) = \psi(c_2)$. However, since $\psi(c_1 - c_2) = \psi(c_1) - \psi(c_2)$, we then have $\psi(c_1 - c_2) = 0$, whence $(c_1 - c_2)a_i \in C$ for $i = 1, \ldots, k$. It follows from this that the T-submodule $(c_1 - c_2)T$ is contained in C, which contradicts the essentiality of the T-submodule $mT \oplus B$. Hence the essential R-submodule $[mT \oplus B \oplus C]_R$ must be proper in M_R, as claimed.

By [2, Lemma 1.2] the largest T-submodule $N_T = N$ contained in $mT \oplus B \oplus C$ has the property that N_R is essential in M_R. As N_T is a proper submodule of M_T, we must have $\operatorname{card} N \leqslant \aleph_0$ by assumption. For each $a \in M$, define the ideal $E(a)$ to be $\{r \in R \mid ar \in N\}$, and then, for each $a \in M$, $j = 1, \ldots, k$, define the ideal $E(a, j)$ by

$$E(a, j) = \left\{r \in R \middle| \exists s \in E(a) \text{ such that } sa_j = a_j r\right\}.$$

It follows from the essentiality of N_R in M_R that each $E(a)$ is an essential ideal of R (i.e. intersects non-trivially every non-zero R-ideal), and thence that the $E(a, j)$, and therefore also the intersections $E^*(a) = \bigcap_{j=1}^{k} E(a, j)$, are all essential R-ideals. The ideals $E^*(a)$ have the further property that, for each $r \in E^*(a)$, and for each $j = 1, \ldots, k$, there is an element $s_j \in E(a)$ such that $s_j a_j = a_j r$. We now define a map γ: $M \to R^k$ (where R^k denotes the Cartesian product of k copies of R) as follows: for each $a \in M$ choose a nonzero element $b \in E^*(a)$, then choose $b_1, b_2, \ldots, b_k \in E(a)$ such that $b_j a_j = a_j b$, and then set $\gamma(a) = (b_1, \ldots, b_k)$. (Note that since $a_1 = 1$, we have $b_1 = b$.) Since $\operatorname{card} R \leqslant \aleph_0$, and since M is uncountable, there is an uncountable subset S_0 of M such that $\gamma(s_1) = \gamma(s_2)$ for all $s_1, s_2 \in S_0$. Define a map δ: $S_0 \to N^k$ by

$$s \mapsto (sb_1, \ldots, sb_k), \quad \text{where } (b_1, \ldots, b_k) = \gamma(s).$$

Once again, since n is countable while S_0 is not, there must be an uncountable subset S of S_0 such that $\delta(s_1) = \delta(s_2)$ for all $s_1, s_2 \in S$. Hence the T-submodule K of M generated by the set $\{x - y \mid x, y \in S\}$ will also be uncountable. We now complete the proof by showing that K is in fact a proper submodule. This we do by finding a non-zero element r of R which annihilates it; it will then follow that $K \neq M$, since M is faithful by assumption. For all $s \in S$, $\gamma(s)$ is constant, say $\gamma(s) = (b_1, \ldots, b_k)$. Take $r = b_1$. Then for any x, $y \in S$, and for any element $\sum_{j=1}^{k} a_j r_j \in T$, we have

$$(x - y)\left(\sum_{j=1}^{k} a_j r_j\right) b_1 = \sum_{j=1}^{k} (x - y) a_j b_1 r_j = \sum_{j=1}^{k} (x - y) b_j a_j r_j = 0,$$

where the last equality follows from the fact that $\delta(x) = \delta(y)$.

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