Appendix D

Computation of the holographic stress tensor

In this appendix we give some details of the computation of the holographic stress tensor for the fluid metric discussed in Section 7.2. The basic tool is the relation (5.51) between the boundary theory stress tensor and the curvature of the bulk metric whose derivation we reviewed in Section 5.3.2, and which we repeat here for convenience:

$$\langle T^{\mu\nu}\rangle = \lim_{z\to 0} \frac{1}{8\pi G_N} \frac{R^{d+2}}{z^{d+2}} \left(K^{\mu\nu} - g^{\mu\nu} K - \frac{d-1}{R} g^{\mu\nu} \right) , \tag{D.1}$$

where $g_{\mu\nu}$ is the induced metric on a constant-z hypersurface Σ_z. We will denote its inverse by $g^{\mu\nu}$. We shall henceforth denote $\langle T^{\mu\nu}\rangle$ by just $T^{\mu\nu}$ as we have done in Chapter 7 and as is standard in the hydrodynamic literature. In this appendix we shall consider a bulk metric of the general form

$$ds^2 = N^2 dz^2 + g_{\mu\nu}(dx^\mu + N^\mu dz)(dx^\nu + N^\nu dx^\nu) \tag{D.2}$$

where N and N^μ are functions that specify the explicit form of the metric. The extrinsic curvature of a hypersurface of constant z is given by

$$K_{\mu\nu} = \frac{1}{2N} \left(\partial_z g_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu \right) \tag{D.3}$$

where $N_\mu = g_{\mu\nu} N^\mu$ and D_μ is the covariant derivative associated with $g_{\mu\nu}$.

D.1 Holographic stress tensor for the AdS black brane

Before considering the fluid metric of Section 7.2, let us first consider a simpler example as a warmup. Consider a diagonal metric with $N^\mu = 0$ and

$$g_{\mu\nu} = \frac{R^2}{z^2} h_{\mu\nu}(x^\mu, z), \qquad N^2 = \frac{R^2}{z^2} \mathfrak{n}^2 , \tag{D.4}$$

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meaning that the metric is now specified by the functions $h_{\mu\nu}$ and $\mathfrak{n}$. For a metric with this form,

$$K_{\mu\nu} = \frac{R}{z}\frac{1}{\mathfrak{n}}\left(\frac{1}{2}\partial_z h_{\mu\nu} - \frac{1}{z}h_{\mu\nu}\right), \qquad K = \frac{z}{R}\frac{1}{\mathfrak{n}}\left(\frac{1}{2}h^{\mu\nu}\partial_z h_{\mu\nu} - \frac{d}{z}\right) \tag{D.5}$$

and thus

$$T_{\mu\nu} = \frac{R^{d-1}}{8\pi G_N}\lim_{z\to 0}\frac{1}{z^{d-1}}\frac{1}{\mathfrak{n}}\left(\frac{1}{2}\partial_z h_{\mu\nu} - \frac{1}{2}h_{\mu\nu}h^{\lambda\rho}\partial_z h_{\lambda\rho} + \frac{d-1}{z}h_{\mu\nu}(1-\mathfrak{n})\right). \tag{D.6}$$

The AdS black brane metric dual to plasma at rest in thermal equilibrium with temperature T is given by

$$ds^2 = \frac{R^2}{z^2}\left(-f\,dt^2 + \frac{1}{f}dz^2 + d\vec{x}^2\right) \tag{D.7}$$

with $f(z) = 1 - \frac{z^d}{z_0^d}$ and where z_0 is related to the temperature by $T = \frac{d}{4\pi z_0}$. This metric is therefore an instance of the general form that we have introduced above, with

$$\mathfrak{n} = \frac{1}{\sqrt{f}} = 1 + \frac{z^d}{2z_0^d} + \cdots, \qquad h_{\mu\nu} = \eta_{\mu\nu} + \frac{z^d}{z_0^d}\delta_{\mu 0}\delta_{\nu 0}\,. \tag{D.8}$$

We thus find that

$$T_{\mu\nu} = \frac{R^{d-1}}{8\pi G_N}\frac{1}{2z_0^d}\left(\eta_{\mu\nu} + d\,\delta_{\mu 0}\delta_{\nu 0}\right) = \frac{R^{d-1}}{16\pi G_N}\left(\frac{4\pi T}{d}\right)^d\left(\eta_{\mu\nu} + d\,\delta_{\mu 0}\delta_{\nu 0}\right)\,, \tag{D.9}$$

which is indeed the stress tensor for the strongly coupled plasma at rest, in thermal equilibrium, which we have derived for $d = 4$ in Eqs. (6.6) in Section 6.1. (Recall from (5.12) that for $d = 4$ we have $R^3/G_N = 2N_c^2/\pi$.)

D.2 Computation of the holographic stress tensor for the fluid metric

We now compute the stress tensor corresponding to the metric (7.26) discussed in Section 7.2.1 that describes the hydrodynamic fluid in motion. Henceforth, we specialize to $d = 4$. When we write the metric (7.26), in terms of the standard representation (D.2) we find

$$N^2 = \frac{R^2}{z^2}\mathfrak{n}^2, \quad \mathfrak{n}^2 \equiv -u_\mu h^{\mu\nu}u_\nu, \quad g_{\mu\nu} = \frac{R^2}{z^2}h_{\mu\nu}, \quad N_\mu = \frac{R^2}{z^2}u_\mu, \quad N^\mu = h^{\mu\nu}u_\nu, \tag{D.10}$$

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where $h^{\mu\nu}$ is the inverse of $h_{\mu\nu}$. Recall that in our convention $u^\mu = \eta^{\mu\nu}u_\nu$. Note that D_μ is also the covariant derivative associated with $h_{\mu\nu}$. From (D.3) and (D.10),

$$K_{\mu\nu} = \frac{R}{z}k_{\mu\nu}, \qquad k_{\mu\nu} = \frac{1}{2\mathfrak{n}}\left(\partial_z h_{\mu\nu} - \frac{2}{z}h_{\mu\nu} + D_\mu u_\nu + D_\nu u_\mu\right) . \tag{D.11}$$

We then find that

$$T_{\mu\nu} = \frac{R^3}{8\pi G_5}t_{\mu\nu}, \qquad t_{\mu\nu} = \lim_{z\to 0}\frac{z^{-3}}{\mathfrak{n}}\left(\frac{1}{2}\partial_z h_{\mu\nu} + D_{(\mu}u_{\nu)} - Ah_{\mu\nu}\right) , \tag{D.12}$$

where

$$A = \frac{1}{2}h^{\mu\nu}\partial_z h_{\mu\nu} + D_\mu u^\mu + \frac{3}{z}(\mathfrak{n}-1) . \tag{D.13}$$

In the discussion of Section 7.2, we write $h_{\mu\nu}$ in a derivative expansion as

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + \cdots \tag{D.14}$$

with $h^{(0)}_{\mu\nu}$ and its inverse given by

$$h^{(0)}_{\mu\nu} = -fu_\mu u_\nu + \Delta_{\mu\nu}, \qquad h^{\mu\nu}_{(0)} = -f^{-1}u^\mu u^\nu + \Delta^{\mu\nu} , \tag{D.15}$$

and $h^{(1)}_{\mu\nu}$ given by the expression (7.58). The inverse $h^{\mu\nu}$ has the expansion

$$h^{\mu\nu} = h^{\mu\nu}_{(0)} - \epsilon h^{\mu\nu}_{(1)} + \cdots , \tag{D.16}$$

where $h^{\mu\nu}_{(1)}$ is obtained from $h^{(1)}_{\mu\nu}$ by raising the indices using $h^{\mu\nu}_{(0)}$. We can then write $t_{\mu\nu}$ in a derivative expansion as

$$t_{\mu\nu} = t^{(0)}_{\mu\nu} + \epsilon\, t^{(1)}_{\mu\nu} + \cdots . \tag{D.17}$$

Upon evaluating (D.12) up to zeroth order (i.e. no derivatives) we find

$$t^{(0)}_{\mu\nu} = \lim_{z\to 0}\frac{z^{-3}}{\mathfrak{n}^{(0)}}\left(\frac{1}{2}\partial_z h^{(0)}_{\mu\nu} - A^{(0)}h^{(0)}_{\mu\nu}\right), \quad A^{(0)} = \frac{1}{2}h^{\mu\nu}_{(0)}\partial_z h^{(0)}_{\mu\nu} + \frac{3}{z}(\mathfrak{n}^{(0)} - 1) . \tag{D.18}$$

Using (D.15), we have

$$\mathfrak{n}^{(0)} = f^{-\frac{1}{2}}, \quad \partial_z h^{(0)}_{\mu\nu} = -\partial_z f u_\mu u_\nu = 4(\pi T)^4 z^3 u_\mu u_\nu + \mathcal{O}(z^4), \tag{D.19}$$

and from these expressions we obtain

$$A^{(0)} = -\frac{1}{2}(\pi T)^4 z^3 + \mathcal{O}(z^4) \tag{D.20}$$

which then yields

$$t^{(0)}_{\mu\nu} = 2(\pi T)^4 u_\mu u_\nu + \frac{(\pi T)^4}{2}\eta_{\mu\nu} . \tag{D.21}$$

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This is the zeroth order stress tensor describing a fluid in motion, which we stated as Eqs. (7.35) and (7.36) in Section 7.2. For a fluid at rest this reproduces the stress tensor (D.9).

Now let us consider the contributions to the stress tensor that are first order in derivatives. In the iterative procedure described in Section 7.2, there are two types of contributions to $t^{(1)}_{\mu\nu}$. One type comes from the expansion to higher order of terms that are already present at zeroth order, i.e. terms that arise in $t^{(0)}_{\mu\nu}(T, u_\mu)$ if we take $T = T^{(0)} + \epsilon T^{(1)} + \cdots$ and $u_\mu = u^{(0)}_\mu + \epsilon u^{(1)}_\mu + \cdots$ and which can therefore be absorbed into a redefinition of T and u_μ. It is straightforward to derive the contributions of this type, and as they do not affect the structure of $t_{\mu\nu}$ they are not what is of interest to us here. The second type of contribution gives new first derivative terms which are not present in $t^{(0)}_{\mu\nu}$. We will concentrate on these contributions, which can be written as

$$t^{(1)}_{\mu\nu} = \lim_{z\to 0} \frac{z^{-3}}{\mathfrak{n}^{(0)}} \left(\frac{1}{2}\partial_z h^{(1)}_{\mu\nu} + D^{(0)}_{(\mu} u^{(0)}_{\nu)} - A^{(0)} h^{(1)}_{\mu\nu} - A^{(1)} h^{(0)}_{\mu\nu} \right) - \lim_{z\to 0} \frac{\mathfrak{n}^{(1)}}{\mathfrak{n}^{(0)}} t^{(0)}_{\mu\nu}, \quad \text{(D.22)}$$

where

$$A^{(1)} = \frac{1}{2}(h^{\mu\nu}\partial_z h_{\mu\nu})_{(1)} + D^{(0)}_\mu u^\mu_{(0)} + \frac{3}{z}\mathfrak{n}^{(1)} \, . \quad \text{(D.23)}$$

In these terms the differences between T, u_μ and $T^{(0)}, u^{(0)}_\mu$ can be neglected since these differences only contribute at higher order. For notational simplicity below we can drop all superscripts on these variables. After some algebra we then find that

$$t^{(1)}_{\mu\nu} = -\frac{(\pi T)^3}{2}\sigma_{\mu\nu} \, , \quad \text{(D.24)}$$

where $\sigma_{\mu\nu}$ was defined in (7.3), which yields the result (7.59).

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