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SINGULAR INTEGRALS ON ULTRASPHERICAL SERIES

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1. Introduction. One of the main uses of harmonic analysis on the sphere is to discover new theorems about series of ultraspherical (Gegenbauer) polynomials. In this paper, we will construct singular integral operators from scalar functions on the sphere to vector functions. These operators when restricted to zonal functions give L^p-bounded $(1 < p < \infty)$ operators on ultraspherical series.

We will use [**7**, Chapter 9] as our main reference. Let G denote a compact group, with identity e, and $\hat{G}$ its dual, the set of equivalence classes of continuous irreducible unitary representations of G. Choose $T_\alpha \in \alpha$, where $\alpha \in \hat{G}$; then T_α is a continuous homomorphism of G into $U(n_\alpha)$, the unitary group on complex n_α-space. For $1 \leqq i, j \leqq n_\alpha$, the function

$$T_{\alpha ij}: x \mapsto T_\alpha(x)_{ij} \ (x \in G)$$

is the matrix entry function in T_α. Define the character χ_α of α by $\chi_\alpha = \sum^{n_\alpha}{}_{i=1} T_{\alpha ii}$. Then each integrable (with respect to the normalized Haar measure m_G of G) function f has the *Fourier series*

$$f \sim \sum_{\alpha \in \hat{G}} n_\alpha \chi_\alpha * f.$$

Henceforth, *representation* means a continuous unitary finite dimensional representation.

Let H be a closed subgroup of G; then put $G/H = \{Hx : x \in G\}$, the space of right cosets of H, a compact homogeneous space. Functions on G/H are identified with the functions on G which satisfy the condition:

$$f(hx) = f(x) \quad (h \in H, x \in G). \tag{1-1}$$

Now let (τ, V) be a representation of H (here, τ is the homomorphism, V is the vector space). We will consider various linear spaces of functions of G into V satisfying the following condition:

$$f(hx) = \tau(h) f(x) \quad (h \in H, x \in G). \tag{1-2}$$

Further, G acts on such spaces by *right translation* R, where $R(x)f(y) = f(yx)$ $(x, y \in G)$. A function f on G is said to be *zonal* if $R(h)f = f$ $(h \in H)$.

Observe for each $\alpha \in \hat{G}$, that $T_\alpha|H$ splits into a direct sum of irreducible representations of H.

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PROPOSITION 1. *Suppose that there is an* $\alpha \in \hat{G}$ *such that* $T_\alpha|H = 1 \oplus \tau \oplus \sigma$, *where* τ *is irreducible,* $\tau \not\simeq 1$ (*the representation* $H \to \{1\}$), *and* σ *is a representation of* H *which does not involve* 1 *or* τ *in its decomposition. Then there exists a nonzero, unique* (*up to multiplication by a scalar*) *zonal function satisfying* (1-2), *whose Fourier series has only an* α-*term.*

Proof. Choose an orthonormal basis $\{v_i\}^n{}_{i=1}$ for V (where τ acts on V, an n-dimensional space); then denote the matrix entries of $\tau(h)$ by $\tau(h)_{ij}$ ($h \in H$; $1 \leqq i, j \leqq n$). For a continuous function $f: G \to V$ we write $f = \sum^n{}_{i=1} f_i v_i$, with f_i scalar-valued, and let

$$||f||_2 = \left(\int_G \left(\sum_{i=1}^n |f_i|^2\right) dm_G\right)^{1/2}.$$

Now choose a matrix representation for T_α so that $T_\alpha(h)_{00} = 1$, $T_\alpha(h)_{ij} = \tau(h)_{ij}$ for $1 \leqq i, j \leqq n$, and $T_\alpha(h)_{ij} = 0$ if

(i) $i = 0, j > 0$,

(ii) $i > 0, j = 0$,

(iii) $1 \leqq i \leqq n, j > n$,

(iv) $i > n, 1 \leqq j \leqq n$,

for all $h \in H$. Now let $\phi_{\alpha\tau} = \sum^n{}_{i=1} T_{\alpha i0} v_i$. It is easy to check that $\phi_{\alpha\tau}$ is the required function. Further, it is uniquely determined (up to a constant of absolute value 1) by the additional hypothesis that

$$||\phi_{\alpha\tau}||_2 = (n/n_\alpha)^{\frac{1}{2}}.$$

Definition. A *trig polynomial* is a (possibly vector-valued) function on G which has a terminating Fourier series. For a representation τ of H, let $C_f(\tau)$ denote the space of trig polynomials which satisfy condition (1-2). In particular, $C_f(1)$ is the algebra (under pointwise operations) of trig polynomials on G/H, and each $C_f(\tau)$ is a $C_f(1)$ module.

We will consider G-operators (linear maps which commute with each $R(x)$, $x \in G$) from $C_f(1)$ to $C_f(\tau)$. Note that each $C_f(\tau)$ is dense in the appropriate L^p-space, $1 \leqq p < \infty$. If f is a trig polynomial on G, then $f \in C_f(1)$ if and only if $m_H * f = f$ (where m_H is the normalized Haar measure of H). Thus, each $f \in C_f(1)$ has the Fourier series $\sum_{\alpha \in \hat{G}} n_\alpha \phi_\alpha * f$, where $\phi_\alpha = \chi_\alpha * m_H$ (a spherical function).

PROPOSITION 2. *If the pair* (G, H) *has the property that for* $\alpha \in G$, $T_\alpha|H$ *never contains two copies of the same irreducible representation of* H, *and, further, if* J *is a* G-*operator:* $C_f(1) \to C_f(\tau)$, *with* τ *irreducible, then there exists complex numbers* $j_\alpha (\alpha \in \hat{G})$ *such that*

$$Jf = \sum_{\alpha \in \hat{G}} n_\alpha j_\alpha \phi_{\alpha\tau} * f \qquad (f \in C_f(1)).$$

Proof. Let $f \in C_f(1)$; then $Jf = \sum n_\alpha (J\phi_\alpha) * f$, since J commutes with right convolution. Further, $J\phi_\alpha$ is zonal. The rest is straightforward. Note that $\phi_\alpha = 0$ whenever $T_\alpha|H$ does not contain 1, and $\phi_{\alpha\tau} = 0$ unless $T_\alpha|H$ contains both 1 and τ.

LEMMA 3: *Let* ρ_0 *be a linear map:* $C_f(\tau) \to V'$ *such that* $\rho_0(R(h)f) = \tau'(h)\rho_0(f)$ $(f \in C_f(\tau), h \in H)$, *where* (τ', V') *is a representation of H. Then there exists a unique G-operator* $\rho: C_f(\tau) \to C_f(\tau')$ *such that* $\rho_0(f) = \rho f(e)$ $(f \in C_f(\tau))$, *and* ρ *is defined by* $\rho f(x) = \rho_0(R(x)f)$ $(x \in G)$.

2. The rotation group and ultraspherical polynomials. The rotation group is denoted by $SO(n)$. For technical reasons, we require $n \geqq 4$, but the case $n = 3$ will be discussed later. The unit sphere

$$S^{n-1} = \{s \in \mathbf{R}^n : |s| = (\sum s_j^2)^{\frac{1}{2}} = 1\}$$

is expressed as $SO(n)/H$, by choosing $p = (1, 0, \ldots 0) \in S^{n-1}$ and letting $H = \{g \in SO(n) : pg = p\}$; that is, $H = \{g \in SO(n) : g_{11} = 1\} \cong SO(n-1)$. The irreducible representations of $SO(n)$ realized on $C_f(1)$ (trig polynomials on S^{n-1}), are those equivalent to right translation acting on $\mathscr{H}_m^n$, the space of harmonic homogeneous polynomials, in n real variables, of degree m, for $m = 0, 1, 2, \ldots$. The degree of the representation on $\mathscr{H}_m^n$ is denoted

$$D_m^n = \binom{n+m-3}{m}\left(\frac{2m}{n-2} + 1\right).$$

Further, each $f \in C_f(1)$ has the Fourier series

$$\sum_{m=0}^{\infty} D_m^n \phi_m * f, \text{ where } \phi_m(g) = P_m^{(n-2)/2}(g_{11}).$$

Here, P_m^s is the ultraspherical polynomial of degree k and index $s > 0$, and is normalized by $P_k^s(1) = 1$. A generating function for these is given by

$$(1 - 2rt + r^2)^{-s} = \sum_{m=0}^{\infty} \frac{\Gamma(2s+m)}{m!\Gamma(2s)} r^m P_m^s(t).$$

For later use we state the identity (see [**8**, p. 141]) (with $k = 0, 1, 2 \ldots$, and $n > 3$):

$$t^k = \sum_{j=0}^{[k/2]} a_{kj} P_{k-2j}^{(n-3)/2}(t), \tag{2-1}$$

where

$$a_{kj} = \frac{(2k - 4j + n - 3)(n + k - 2j - 4)!k!}{2^k((n-1)/2)_{k-j}(n-3)!(k-2j)!j!}$$

Here, $[u]$ is the largest integer $\leqq u$, and $(u)_s = u(u+1)\ldots(u+s-1)$, for $s = 1, 2, \ldots$.

We will use the following representations of H: for $k = 0, 1, \ldots, \tau_k$ is right translation of H acting on $\mathscr{H}_k^{n-1}$; for convenience, we write the elements of $\mathscr{H}_m^{n-1}$ as functions of points like $x = (x_2, \ldots, x_n) \in \mathbf{R}^{n-1}$, since $H = \{g \in SO(n) : g_{11} = 1\}$. The space $\mathscr{H}_k^{n-1}$ is furnished with the inner product

$$[p, q] = \int_{|x|=1} p(x)\overline{q(x)}d\omega(x),$$

where ω is the normalized H-invariant measure on the unit sphere S^{n-2}. An element of $C_f(\tau_k)$ has the form $f(g, x)$ with

$$f(hg, x) = f(g, xh) \quad (h \in H, g \in \mathrm{SO}(n), x \in \mathbf{R}^{n-1}),$$

and for fixed g, $x \mapsto f(g, x)$ is in $\mathscr{H}_k^{n-1}$.

Our next aim is to find the function ϕ_{mk}, the zonal function in $C_f(\tau_k)$ with only an m-term in its Fourier series. By the Branching Theorem [**4**], the pair $(\mathrm{SO}(n), H)$ has the property described in Proposition 2, so we will construct a differential $\mathrm{SO}(n)$-operator: $C_f(\tau_0) \to C_f(\tau_k)$ (note that $\tau_0 = 1$) and use Propositions 1 and 2 to compute ϕ_{mk}. The Branching Theorem shows that $\phi_{mk} = 0$, unless $m \geqq k$.

Definition. Let $1 \leqq p < q \leqq n$, $-\pi < \theta < \pi$, and let $r^{pq}(\theta) \in \mathrm{SO}(n)$ be defined by

$$[r^{pq}(\theta)]_{ij} = \delta_{ij}(1 + (\delta_{ip} + \delta_{iq})(\cos\theta - 1)) + (\sin\theta)(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) \quad (1 \leqq i, j \leqq n).$$

For a trig polynomial f on $\mathrm{SO}(n)$, define

$$R_{pq}f(g) = (d/d\theta)f(gr^{pq}(\theta))|_{\theta=0} \quad (g \in \mathrm{SO}(n)).$$

Observe that

$$R_{pq}(R(g)f) = \textstyle\sum_{1\leqq i<j\leqq n}(g_{pi}g_{qj} - g_{qi}g_{pj})R(g)R_{ij}f.$$

Let τ_k' be the representation of H on $\mathscr{P}_k^{n-1}$, the homogeneous polynomials of degree k, in $x_2, \ldots, x_n$.

PROPOSITION 4. *Let* $f \in C_f(\tau_k')$ $(k = 0, 1, \ldots)$ *and define* ∂f *by*

$$\partial f(g, x) = \sum_{i=2}^{n} x_i R_{1i}(R(g)f)(e, x).$$

Then ∂ *is an* $\mathrm{SO}(n)$*-operator:* $C_f(\tau_k') \to C_f(\tau'_{k+1})$.

Proof. By Lemma 3, it suffices to show that $\partial f(h, x) = \partial f(e, xh)$ $(x \in \mathbf{R}^{n-1}, h \in H)$. Now,

$$\begin{aligned}\partial f(h, x) &= \sum_{i=2}^{n} x_i R_{1i}(R(h)f)(e, x)\\ &= \sum_{i,j=2}^{n} x_i h_{ij} R(h) R_{ij} f(e, x)\\ &= \sum_{j=2}^{n} (xh)_j R_{1j} f(h, x)\\ &= \sum_{j=2}^{n} (xh)_j R_{1j} f(e, xh)\\ &= \partial f(e, xh).\end{aligned}$$

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Thus, the map ∂^k is an SO(n)-operator: $C_f(\tau_0)$ to $C_f(\tau_k')$. There is a canonical H-projection π_k of τ_k' onto τ_k (which can be described as a convolution operator over S^{n-2}).

Definition. Let ∇_k be the map $\pi_k \circ \partial^k$ of $C_f(\tau_0)$ to $C_f(\tau_k)$; then ∇_k is an SO(n)-operator, and is a differential operator of order k.

LEMMA 5. *Let* $y = (y_2, \ldots, y_n) \in \mathbf{R}^{n-1}$ *be fixed,* $k = 1, 2, \ldots,$ *and let* $p(x) = (\sum^n{}_{i=2} x_i y_i)^k$. *Then* $p \in \mathscr{P}_k^{n-1}$ *and*

$$\pi_k p(x) = |x|^k |y|^k a_{k0} P_k^{(n-3)/2}\left(\sum_{i=2}^n x_i y_i / |x||y|\right),$$

where a_{k0} *is described in* (2-1). *Denote* $\pi_k p(x)$ *by* $\psi_k(x, y)$.

For $g \in$ SO(n), let g_{*1} denote the vector $(g_{21}, g_{31}, \ldots, g_{n1}) \in \mathbf{R}^{n-1}$; then $|g_{*1}| = (1 - g^2{}_{11})^{1/2}$.

THEOREM 6. *For* $k = 1, 2, \ldots, m = 1, 2, \ldots,$

$$\nabla_k \phi_m(g, x) = A_{km} \psi_k(x, g_{*1}) P_{m-k}^{k+(n-2)/2}(g_{11}),$$

and

$$\phi_{mk}(g, x) = \frac{A_{km}}{C_{km}} \psi_k(x, g_{*1}) P_{m-k}^{k+(n-2)/2}(g_{11}),$$

where

$$C_{km} = \frac{k!}{2^k((n-1)/2)_k} \left(\frac{m!(n+k+m-3)!}{(m-k)!(n+m-3)!}\right)^{1/2} \sim m^k$$

as $m \to \infty$ ($a_m \sim b_m$ *as* $m \to \infty$ *means* $a_m/b_m \to$ *some constant as* $m \to \infty$) *and*

$$A_{km} = \frac{m!(n+k+m-3)!}{(m-k)!(n+m-3)!} \frac{1}{2^k((n-1)/2)_k},$$

determined by

$$\left(\frac{d}{dt}\right)^k P_m^{(n-2)/2}(t) = A_{km} P_{m-k}^{k+(n-2)/2}(t).$$

Proof. First, we compute $\partial u(g, x)$ where u is a function of g_{11} only, obtaining $\partial u(g, x) = (\sum^n{}_{i=2} g_{i1} x_i) u'(g_{11})$.

Let $v(g, x) = \sum^n{}_{i=2} x_i g_{i1}$, and note that $\partial v(g, x) = -g_{11}|x|^2$; then we claim that $\partial_k u(g, x) = [v(g, x)]^k u^{(k)}(g_{11}) + |x|^2$ {terms composed of lower powers of v, g_{11}, lower order derivatives of u, and powers of $|x|^2$}. The expression in { } is a polynomial in x homogeneous of degree $k - 2$. To prove the claim, observe that $\partial^r u$ is a sum of terms of the form $v^m|x|^{r-m} f(g_{11})$ ($r - m$ even), since

$$\partial(v^m|x|^{r-m} f(g_{11})) = -mv^{m-1}|x|^{r-m+2} g_{11} f(g_{11}) + v^{m+1}|x|^{r-m} f'(g_{11}).$$

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The only term in $\partial^k u$ which does not contain a nonzero power of $|x|^2$ is $v^k u^{(k)}$; thus,

$$\pi_k \partial_0{}^k u = \pi_k(v^k u^{(k)}) = \psi_k(x, g_{*1}) u^{(k)}(g_{11})$$

(see Lemma 5). The result for $\nabla_k \phi_m$ follows by setting $u = P_m{}^{(n-2)/2}$.

The L^2-norm on $C_f(\tau_k)$ is

$$||f||_2 = \left\{ \int_{SO(n)} \int_{|x|=1} |f(g, x)|^2 d\omega(x) dg \right\}^{1/2},$$

and $||\nabla_k \phi_m||_2{}^2 = |C_{km}|^2 D_k{}^{n-1}/D_m{}^n$. We choose $C_{km} > 0$, and obtain the stated value. The computation involves

$$\int_{-1}^{1} (P_r{}^s(t))^2 (1 - t^2)^{s-1/2} dt,$$

for various r, s.

3. Particular operators.

Definition. For $f \in C_f(\tau_k)$, $1 \leqq p < \infty$, define the L^p-norm by

$$||f||_p = \left\{ \int_{SO(n)} \left(\int_{|x|=1} |f(g, x)|^2 d\omega(x) \right)^{p/2} dg \right\}^{1/p}.$$

Then $L^p(\tau_k)$ is the completion of $C_f(\tau_k)$ under the norm $||\cdot||_p$.

Definition. For $\lambda > 0$, $1 \leqq p < \infty$, let $L_\lambda{}^p(-1, 1)$ be the space of measurable functions u on $(-1, 1)$ such that

$$\int_{-1}^{1} |u(t)|^p (1 - t^2)^{\lambda - 1/2} dt < \infty.$$

Let

$$||u||_p = \left[K_\lambda \int_{-1}^{1} |u(t)|^p (1 - t^2)^{\lambda - 1/2} dt \right]^{1/p} \qquad (1 \leqq p < \infty),$$

where

$$K_\lambda = \left[\int_{-1}^{1} (1 - t^2)^{\lambda - 1/2} dt \right]^{-1}.$$

PROPOSITION 7. *Let $\lambda = n/2 - 1$, $k = 0, 1, 2, \ldots$, $1 \leqq p < \infty$, and let u be measurable on $(-1, 1)$ such that $u(t)(1 - t^2)^{k/2} \in L_\lambda{}^p(-1, 1)$; then there exists an element $U_k u \in L^p(\tau_k)$, such that*

$$||U_k u||_p = ||u(t)(1 - t^2)^{k/2}||_p.$$

The map U_k is linear, one-to-one, and onto the zonal functions in $L^p(\tau_k)$, and is given by

$$U_k u(g, x) = \frac{(D_k{}^{n-1})^{1/2}}{a_{k0}} u(g_{11}) \psi_k(x, g_{*1}).$$

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Proof.

$$||U_k u||_p^p = \int_{SO(n)} dg \left\{ \int_{|x|=1} \frac{D_k^{n-1}}{a_{k0}} |u(g_{11})|^2 |\psi_k(x, g_{*1})|^2 d\omega(x) \right\}^{p/2}$$

$$= \int_{SO(n)} |u(g_{11})|^p (1 - g_{11}^2)^{kp/2} dg$$

$$= K_\lambda \int_{-1}^{1} |u(t)(1 - t^2)^{k/2}|^p (1 - t^2)^{\lambda - 1/2} dt$$

$$= ||u(t)(1 - t^2)^{k/2}||_p^p.$$

Thus, $U_k u \in L^p(\tau_k)$. As u runs through finite linear combinations of $\{P_{m-k}^{(n+2k-2)/2}(t): m \geqq k\}$, $U_k u$ runs through finite linear combinations of $\{\phi_{mk}: m \geqq k\}$. These two sets are dense in $L_\lambda^p(-1, 1)$ and $\{f \in L^p(\tau_k): f$ is zonal$\}$, respectively; thus, U_k is onto.

PROPOSITION 8. *Let $u \in \{f \in L^1(\tau_k): f$ is zonal$\}$; thus, u has a Fourier series $\sum_{m=k}^{\infty} D_m^n \hat{u}_m \phi_{mk}$ ($\hat{u}_m$ scalar), and if $f \in L^p(\tau_0)$, $1 \leqq p < \infty$, then $u * f \in L^p(\tau_k)$,*

$$||u * f||_p \leqq ||u||_1 ||f||_p,$$

and

$$u * f \sim \sum_{m=k}^{\infty} D_m^n \hat{u}_m \phi_{mk} * f.$$

Proof. The inequality is a standard convolution inequality.

For the subsequent theorems we need information about some special series given in the work of Askey and Wainger [3].

LEMMA 9. *Let $N = 3, 4, \ldots, 1 \leqq r \leqq N - 1$, $\{a_m\}$ be a sequence of complex numbers such that*

$$a_m = \sum_{j=r}^{N-1} \alpha_j m^{-j} + a_m',$$

$a_m' = O(m^{-N})$ as $m \to \infty$, $\alpha_r, \ldots, \alpha_{N-1}$ fixed. Then there exists

$$u \in L^1_{(N-2)/2}(-1, 1)$$

such that

$$u \sim \sum_{m=0}^{\infty} D_m^N a_m P_m^{(N-2)/2}$$

(this is the ultraspherical expansion of u),

$$a_m = \hat{u}_m = K_{(N-2)/2} \int_{-1}^{1} u(t) P_m^{(N-2)/2}(t)(1 - t^2)^{(N-3)/2} dt),$$

and

$$u(t) = \sum_{j=0}^{N-r-2} \beta_j \theta^{j+(r+1-N)} + \gamma \log \theta + E(\theta),$$

where $\cos \theta = t$, $0 \leqq \theta \leqq \pi$, β_j, γ *are constants,* $E(\theta)$ *is continuous on* $[0, \pi]$.

Askey and Wainger's result deals directly with series of the form $\sum D_m{}^N m^{-j} P_m{}^{(N-2)/2}$, and the series $\sum D_m{}^N a_m{}' P_m{}^{(N-2)/2}$ converges absolutely.

The Laplacian Δ is defined by $\sum_{i<j} R^2{}_{ij}$; then for $f \in C_f(\tau_0)$,

$$\Delta f = -\sum_{m=1}^{\infty} D_m{}^n m(m+n-2)(\phi_m * f).$$

Definition. Let Λ be the SO(n)-operator on $C_f(\tau_0)$ defined by

$$\Lambda f = \sum_{m=1}^{\infty} D_m{}^n (m(m+n-2))^{-1/2} \phi_m * f.$$

Note that $\Delta\Lambda^2 f = f_0 - f$, where

$$f_0 = \int_{\mathrm{SO}(n)} f.$$

For $k = 1, 2, \ldots, f \in C_f(\tau_0)$ we obtain

$$\nabla_k \Lambda f = \sum_{m=k}^{\infty} D_m{}^n C_{km} (m(m+n-2))^{-k/2} \phi_{mk} * f$$

(by Theorem 6). We will now show that $\nabla_k \Lambda^k$ is L^p-bounded, $1 < p < \infty$, and is a singular integral SO(n)-operator.

THEOREM 10. *For each $k = 1, 2, \ldots$, there exists a measurable function F_k on $(-1, 1)$ such that*

$$\nabla_k \Lambda^k f = (F_k(g_{11}) \psi_k(x, g_{*1})) * f,$$

where the convolution integral is a principal value (to be defined in the proof), and is defined for $f \in L^p(\tau_0)$, $1 < p < \infty$, with $\|\nabla_k \Lambda^k f\|_p \leqq B_{kp} \|f\|_p$, B_{kp} a constant depending only on k and p ($f \in C_f(\tau_0)$).

Proof. Formally, we write

$$\nabla_k \Lambda^k f \sim \left\{ \sum_{m=k}^{\infty} D_m{}^n (m(m+n-2))^{-k/2} A_{km} \psi_k(x, g_{*1}) P_{m-k}^{k+(n-2)/2}(g_{11}) \right\} * f.$$

Let

$$a_m = [(m+k)(m+k+n-2)]^{-k/2} A_{k,m+k} \frac{D^n{}_{m+k}}{D_m{}^{n+2k}};$$

then $\{a_m\}$ satisfies the hypotheses of Lemma 9 with $N = n + 2k$, $r = k$; thus, there exists $F_k \in L^1{}_{(N=2)/2}(-1, 1)$ such that

$$F_k(t) \sim \sum_{m=0}^{\infty} D_m{}^{n+2k} a_m P_m{}^{(n+2k-2)/2}(t)$$

and

$$F_k(t) \sim \beta_0 (1-t)^{-(n+k-1)/2}$$

as $t \to 1_-$ (since $\theta \sim [2(1-t)]^{1/2}$ as $\theta \to 0$, $t = \cos\theta$). For $0 < \epsilon < 1$, let

$$K_\epsilon(t) = \begin{cases} 1 & -1 \leqq t \leqq 1 - \epsilon \\ 0 & 1 - \epsilon < t \leqq 1; \end{cases}$$

then

$$K_\epsilon(g_{11})F_k(g_{11})\psi_k(x, g_{*1}) \in L^1(\tau_k)$$

and $K_\epsilon F_k \psi_k * f$ is defined for all $\epsilon > 0, f \in C_f(\tau_0)$ and

$$K_\epsilon F_k \psi_k * f(g, x) = \int_{y_1 \leqq 1-\epsilon} \psi_k(x, y) F_k(y_1) f(s) d\omega_n(s),$$

where

$$y_i = \sum_{j=1}^{n} g_{ij} s_j \qquad (i = 1, \ldots, n)$$

(let $s = pg'$; then

$$(gg'^{-1})_{i1} = \sum_j g_{ij} g'_{1j} = \sum_j g_{ij} s_j = y_i,$$

where ω_n is the normalized $\mathrm{SO}(n)$-invariant measure on S^{n-1}; see [**7**, Chapter 9] for expressing $\mathrm{SO}(n)$-convolutions as integrals over S^{n-1}). The integrand has a singularity at $s = pg(y_1 = 1)$ of order $(1 - y_1)^{-(n-1)/2}$, and since the great circle distance between pg and s is $\arccos y_1 \sim 2(1 - y_1)^{1/2}$, this is $(\text{distance})^{-(n-1)}$. Further, the integral of the kernel with respect to s around any $(n - 2)$-sphere centred at pg is easily seen to be zero; take any $s \neq pg$ and the required sphere through s is $\{sg^{-1}hg: h \in H\}$, since $(pg)\cdot(sg^{-1}hg) = p\cdot(sg^{-1}h) = (pg)\cdot s$, and if $u \in L^1(\tau_k)$, then

$$\begin{aligned}\int_H u(g(g'g^{-1}hg)^{-1}, x) dm_H(h) &= \int_H u(hgg'^{-1}, x) dm_H(h)\\ &= \int_H u(gg'^{-1}, xh) dm_H(h)\\ &= 0,\end{aligned}$$

for $pg' = s, k = 1, 2, 3, \ldots$ (note that $p\cdot q = \sum^n{}_{i=1} p_i q_i$).

Now, by a local transfer argument similar to that used by Seeley in [**11**], it follows that the Calderón–Zygmund inequality holds locally. But S^{n-1} is compact, so we can conclude

$$\|K_\epsilon F_k \psi_k * f\|_p \leqq B_{kp} \|f\|_p \quad (1 < p < \infty),$$

where B_{kp} is independent of ϵ, and $\lim_{\epsilon \to 0_+} K_\epsilon F_k \psi_k * f$ exists in L^p. Thus, $\nabla_k \Lambda^k$ extends to a bounded $\mathrm{SO}(n)$-operator: $L^p(\tau_0) \to L^p(\tau_k)$.

THEOREM 11. *Let $\{a_m: m \geqq k\}$ be a sequence of complex numbers such that*

$$a_m = \sum_{j=0}^{n+k-1} \alpha_j (m - k)^{-j} + a_m',$$

$a_m' = O(m^{-k-n})$ as $m \to \infty$, and let $f \in L^p(\tau_0)$, $1 < p < \infty$; then the map

$$J: f \mapsto \sum_{m=k}^{\infty} D_m{}^n a_m \phi_{mk} * f$$

is bounded in L^p, and

$$Jf(g,x) = \alpha_0 \frac{2^k((n-1)/2)_k}{k!} \nabla_k \Lambda^k f(g,x) + (F(\cdot,x) * f)(g),$$

where $F \in L^1(\tau_k)$ and F is zonal.

Proof. By Lemma 9, there exists $u \in L^1{}_{k+(n-2)/2}(-1,1)$ $(N = n + 2k, r = 1)$, such that

$$\left[\alpha_0 \frac{2^k((n-1)/2)_k}{k!} F_k(g_{11}) + u(g_{11})\right]\psi_k(x, g_{*1}) \sim \sum_{m=k}^{\infty} D_m{}^n a_m \phi_{mk},$$

and

$$u(t) \sim (1-t)^{-(n+k-2)/2}.$$

Then

$$||u\psi_k||_1 = c\int_{-1}^{1} |u(t)|(1-t^2)^{(k+n-3)/2}dt < \infty$$

(c some constant; see Proposition 7), and

$$||u\psi_k * f||_p \leqq ||u\psi_k||_1||f||_p \quad (1 < p < \infty).$$

THEOREM 12. *Let $\{a_m\}$ be as above and $f \in L^p{}_{(n-2)/2}(-1,1)$ $(1 < p < \infty)$; then there is a linear map*

$$J_0: f \mapsto J_0 f \in L^1{}_{k+(n-2)/2}(-1,1)$$

such that

$$J_0 f \sim \sum_{m=k}^{\infty} D_m{}^n a_m \hat{f}_m \frac{A_{km}}{C_{km}} P_{m-k}^{k+(n-2)/2},$$

and

$$J_0 f(t)(1-t^2)^{k/2} \in L^p{}_{(n-2)/2}(-1,1)$$

with

$$||J_0 f(t)(1-t^2)^{k/2}||_p \leqq B'_{kp}||f||_p \quad (1 < p < \infty).$$

Proof. Let J be defined as above; then

$$J(U_0 f) \in L^p(\tau_k), \quad ||J(U_0 f)||_p \leqq B_{kp}||f||_p,$$

and

$$J(U_0 f) \sim \sum_{m=k}^{\infty} D_m{}^n a_m \hat{f}_m \phi_{mk}$$

$$\sim \sum_{m=k}^{\infty} D_m{}^n a_m \frac{A_{km}}{C_{km}} \hat{f}_m P_{m-k}^{(k+(n-2)/2}\psi_k.$$

Then

$$J_0 f = \frac{a_{k0}}{(D_k{}^{n-1})^{1/2}} U_k{}^{-1} J U_0 f$$

is the required map. Hölder's inequality shows that $J_0 f \in L^1_{k+(n-2)/2}(-1, 1)$, which justifies the series

$$J_0 f(t) \sim \sum_{m=k}^{\infty} D_m{}^n a_m \hat{f}_m \frac{A_{km}}{C_{km}} P_{m-k}^{k+(n-2)/2}(t).$$

Remark. For

$$k = 1, a_m = \frac{1}{n-1}\left\{1 + \frac{n-2}{m}\right\}^{-1/2},$$

the conjugate series theorem of Stein and Muckenhoupt [**10**] is obtained. This theorem is a "transplantation" theorem. For results dealing with transplantation between Fourier and ultraspherical series, see Askey and Wainger [**2**].

4. Remarks.

The case $n = 3$. The propositions and theorems of § 3 are still valid when $n = 3$. Note that the polynomials $P_k{}^0$ are the Tchebyshev polynomials given by $P_k{}^0(\cos\theta) = \cos k\theta$ $(k = 0, 1, \ldots)$. The main change in § 2 is that τ_k is no longer irreducible for $k \geqq 1$, but breaks up into 2 one-dimensional components. So $D_0{}^2 = 1$, and $D_k{}^2 = 2$, for $k = 1, 2, \ldots$. In the expression (2-2) given for a_{kj}, the limit as $n \to 3_+$, is found to be

$$a_{kj} = \binom{k}{j} \frac{1}{2^k} b_{k-2j},$$

where $b_p = 2$ for $p > 0$ and $b_0 = 1$.

Vector bundles. Some of the results obtained could be phrased in the language of vector bundles. For example, one may construct singular integrals on $C_f(\tau_0)$ of any desired symbol (a symbol here is essentially a "smooth" function u on $SO(n) \times S^{n-2}$ such that

$$u(hg, x) = u(g, xh) \quad (g \in SO(n), h \in H, x \in S^{n-2}).$$

Now let

$$Jf(g) = \int_{|x|=1} \sum_{k=0}^{\infty} c_k \nabla_k \Lambda^k f(g, x) u(g, x) d\omega(x),$$

for suitable constants c_k, independent of f and u (see [**6**]), where $\nabla_0 \Lambda^0$ is the identity map. Then J has the symbol u. By replacing $\nabla_k \Lambda^k$ in the above formula by $\nabla_k \Delta^j$, various j, one may construct differential operators with any specified symbol (note that $\nabla_k \Delta^j$ is a differential $SO(n)$-operator of order $k + 2j$ of $C_f(\tau_0)$ into $C_f(\tau_k)$).

Calderón and Zygmund [**5**] first constructed singular integrals on $\mathbf{R}^n$. Seeley [**11**; **12**] extended the theory to vector bundles over manifolds. Levine [**9**] has also investigated singular integrals on spheres.

References

1. M. Abramowitz and I. Stegun, *Handbook of mathematical functions*, National Bureau of Standards (Dover, New York, 1964).
2. R. Askey and S. Wainger, *A transplantation theorem between ultraspherical series*, Illinois J. Math. *10* (1966), 322–344.
3. ——— *On the behaviour of special classes of ultraspherical expansions.* I, J. d'Anal. Math. *15* (1965), 193–220.
4. H. Boerner, *Representations of groups* (North-Holland, Amsterdam, 1963).
5. A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. *78* (1956), 289–309.
6. C. Dunkl, *Linear functionals on homogeneous polynomials*, Can. Math. Bull. *11* (1968), 465–468.
7. C. Dunkl and D. Ramirez, *Topics in harmonic analysis* (Appleton-Century-Crofts, New York, 1971).
8. L. K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, Amer. Math. Soc. Translation (Amer. Math. Soc., Providence, R.I., 1963).
9. D. Levine, *Systems of singular integral operators on spheres*, Trans. Amer. Math. Soc. *144* (1969), 493–522.
10. B. Muckenhoupt and E. M. Stein, *Classical expansions and their relation to conjugate harmonic functions*, Trans. Amer. Math. Soc. *118* (1965), 17–92.
11. R. T. Seeley, *Singular integrals on compact manifolds*, Amer. J. Math. *81* (1959), 658–690.
12. ——— *Integro-differential operators on vector bundles*, Trans. Amer. Math. Soc. *117* (1965), 167–204.

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