DECOMPOSITION OF JORDAN AUTOMORPHISMS OF TRIANGULAR MATRIX ALGEBRA
OVER COMMUTATIVE RINGS

XING TAO WANG and YUAN MIN LI
Department of Mathematics, Harbin Institute of Technology, Harbin 150001, P.R. China
e-mail: xingtao@hit.edu.cn

(Received 28 April 2009; revised 12 November 2009; accepted 21 February 2010)

Abstract. Let $T_{n+1}(R)$ be the algebra of all upper triangular $n+1$ by $n+1$ matrices over a 2-torsionfree commutative ring $R$ with identity. In this paper, we give a complete description of the Jordan automorphisms of $T_{n+1}(R)$, proving that every Jordan automorphism of $T_{n+1}(R)$ can be written in a unique way as a product of a graph automorphism, an inner automorphism and a diagonal automorphism for $n \geq 1$.

2010 Mathematics Subject Classification. 17C30, 17C50, 13C10.

1. Introduction. Let $M_{n+1}(R)$ be the $R$-algebra of all square matrices of order $n+1$ over a commutative ring $R$ with the identity 1. Jordan multiplication is defined by $x \circ y = xy + yx$ for any $x, y \in M_{n+1}(R)$. Obviously $x \circ y = y \circ x$. If an $R$-module automorphism $\varphi$ of $M_{n+1}(R)$ satisfies $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$, then $\varphi$ is called Jordan automorphism of $M_{n+1}(R)$. It is well known that an $R$-algebra automorphism, which is a ring automorphism and also an $R$-module automorphism of $M_{n+1}(R)$, must be a Jordan automorphism. However, there are Jordan automorphisms which are neither $R$-algebra automorphisms nor $R$-algebra anti-automorphisms [3]. Let $A$ and $B$ be subsets of $M_{n+1}(R)$. We denote Jordan multiplication of $A$ and $B$ by $A \circ B = \{x \circ y | x \in A, y \in B\}$. Let us consider the sub-algebra of $M_{n+1}(R)$ denoted by $T_{n+1}(R)$, which consists of all upper triangular matrices of $M_{n+1}(R)$. Jordan isomorphisms of associative algebras have been studied by many authors for several decades [1–4, 6, 7, 10, 11, 12]. The algebra of all triangular matrices is an interesting topic for many researchers. Many papers are concerned with the study of automorphisms and Lie automorphisms [5, 8, 9, 13]. On the basis of these papers, we consider the problem on decomposition of Jordan automorphism of upper triangular matrix algebra into some standard automorphisms.

Throughout this paper, $R$ denotes a 2-torsionfree commutative ring with the identity 1. The main results are as follows:

Theorem 1.1. For any Jordan automorphism $\varphi$ of $T_{n+1}(R)(n \geq 1)$, there exist unique graph, inner and diagonal automorphisms, respectively, $\zeta, \theta$ and $\lambda, d$ of $T_{n+1}(R)$ such that

$$\varphi = \zeta \circ \theta \circ \lambda, d.$$ 

Theorem 1.2. Let $G$, $I$ and $D$ be the graph, inner and diagonal automorphism group, respectively. When $n \geq 1$, then

$$\text{Aut}(n_0) = G \ltimes (I \ltimes D).$$
2. Preliminaries. Let $e_g$ denote the matrix unit of $M_{n+1}(R)$ and $e$ the identity matrix of $M_{n+1}(R)$. The matrix set \{\(e_{i,i+k}\)|\(i = 1, \ldots, n-k+1\), \(k = 0, 1, \ldots, n\}\} is a basis of $T_{n+1}(R)$. For any $x \in T_{n+1}(R)$, it can be expressed $x = \sum_{k=0}^{n} \sum_{i=1}^{n-k+1} a_{i,i+k} e_{i,i+k}$ for some $a_{i,i+k} \in R$. Let $n_1$ be the sub-algebra of all strictly upper matrices of $T_{n+1}(R)$. The matrix set \{\(e_{i,i+k}\)|\(i = 1, \ldots, n-k+1\), \(k = 1, \ldots, n\}\} is a basis of $n_1$. Let $n_0 = T_{n+1}(R)$ and $\text{Aut}(n_k)$, $k=0,1$ denote the Jordan automorphism group of $n_k$, respectively. If $R$ is 2-torsionfree, then a Jordan automorphism of $M_{n+1}(R)$ coincides with the semi-automorphism of $M_{n+1}(R)$ such that $\varphi(x^2) = [\varphi(x)]^2$ and $\varphi(xy) = \varphi(x)\varphi(y)\varphi(x)$ for any $x, y \in M_{n+1}(R)$.

**Lemma 2.1.** Let $\varphi$ be an $R$-module automorphism of $n_1$. The following two statements are equivalent:

(i) $\varphi$ is in $\text{Aut}(n_1)$;

(ii) For any $e_{i,i+k} \in n_1$, $\varphi(e_{i,i+k}) = \varphi(e_{i,i+m}) \circ \varphi(e_{i+m,i+k})$ for $l \leq m < k$ and $\varphi(e_g) \circ \varphi(e_{m,k}) = 0$ for $j \neq m$ and $i \neq k$.

**Proof.** See [12, Lemma 2.1].

**Lemma 2.2.** Let $\varphi$ be a Jordan automorphism of $n_1$. The following two statements are equivalent:

(i) $\varphi$ is in $\text{Aut}(n_0)$;

(ii) For any $e_{i,i+k} \in n_1$, $[\varphi(e_{i,j})]^2 = \varphi(e_{ii})$, $\varphi(e_{i,i+k}) = \varphi(e_{ii}) \circ \varphi(e_{i,i+k})$, $\varphi(e_{i,i+k}) = \varphi(e_{i,i+k}) \circ \varphi(e_{i+k,i+k})$, $\varphi(e_{ij}) \circ \varphi(e_{ii}) = 0$ and $\varphi(e_{ij}) \circ \varphi(e_{i,i+k}) = 0$.

**Proof.** By Lemma 2.1 it is not difficult to prove Lemma 2.2.

Lemma 2.2 implies that the set \{\(\varphi(e_{11})\), $\varphi(e_{i+1,i+1})$, $\varphi(e_{i,i+1})$\}|\(i = 1, \ldots, n\}\) generates $T_{n+1}(R)$. So we will investigate $\varphi(e_{11})$, $\varphi(e_{i+1,i+1})$, $\varphi(e_{i,i+1})$, $i = 1, \ldots, n$.

**Lemma 2.3.** Let $\varphi$ be in $\text{Aut}(n_0)$. For any $x \in n_0$ and $y, e_{ij} \in n_1$, then $[\varphi(e_{ij})]^2 = 0$, $\varphi(e_{ij}) \varphi(e_{ij}) = 0$ and $e_{ij} ye_{ij} = 0$.

**Proof.** For any $e_{ij} \in n_1$, clearly $(e_{ij})^2 = 0$ so that $[\varphi(e_{ij})]^2 = 0$. It is easy to check that for $e_{mk} \in n_0$, $e_{g} e_{mk} e_{g} = 0$ so that $e_{g} x e_{g} = 0$ for any $x \in n_0$. Therefore $e_{g} \varphi^{-1}(x) e_{g} = 0$ then $\varphi(e_{ij}) \varphi(e_{ij}) = 0$. Similarly, for $e_{mk} \in n_1$, $e_{ii} e_{mk} e_{ii} = 0$ leads to $e_{ii} ye_{ii} = 0$.

**Lemma 2.4.** Let $\varphi$ be in $\text{Aut}(n_0)$. Then $\varphi(n_1) = n_1$.

**Proof.** We express $\varphi(e_{ii})$ and $\varphi(e_{i,i+1})$, respectively, as
\[
\varphi(e_{ii}) = \sum_{k=1}^{n+1} a_{ik}^{(i)} e_{kk} \text{ mod } n_1, \quad i = 1, 2, \ldots, n+1,
\]
\[
\varphi(e_{i,i+1}) = \sum_{k=1}^{n+1} b_{ik}^{(i)} e_{kk} \text{ mod } n_1, \quad i = 1, \ldots, n.
\]
Then, we have
\[
\varphi(e_{ii}) = [\varphi(e_{ii})]^2 = \sum_{k=1}^{n+1} (a_{ik}^{(i)})^2 e_{kk} \text{ mod } n_1, \quad i = 1, 2, \ldots, n+1.
\]
This implies that \( m = 1 \) and \( n = 0 \). Therefore \( \varphi^{-1}(n_1) \subset n_1 \), that is, \( n_1 = \varphi \varphi^{-1}(n_1) \subset \varphi(n_1) \).

**Lemma 2.5.** Let \( \varphi \in \text{Aut}(n_0) \). Then
\[
\varphi(e_{11}) = a_{11}^{(1)} e_{11} + a_{n+1, n+1}^{(1)} e_{n+1, n+1} + t_1
\]
where \( a_{11}^{(1)} + a_{n+1, n+1}^{(1)} = 1 \) and \( a_{11}^{(1)} \) is an idempotent of \( R \).

**Proof.** We express \( \varphi(e_{11}) \) as \( \varphi(e_{11}) = \sum_{k=1}^{n+1} a_{kk}^{(1)} e_{kk} + t_1 \). Let \( e_{1m} \in n_1 \). By Lemma 2.4 \( \varphi^{-1}(e_{1m}) \in n_1 \). By Lemma 2.3 \( e_{11} \varphi^{-1}(e_{1m}) e_{11} = 0 \). Consequently,
\[
\varphi(e_{11}) e_{1m} \varphi(e_{11}) = a_{11}^{(1)} e_{1m} + t_m, m = 2, \ldots, n + 1.
\]

Let \( e_{m, n+1} \in n_1 \). Similarly,
\[
\varphi(e_{11}) e_{m, n+1} \varphi(e_{11}) = a_{mn}^{(1)} e_{n+1, n+1} + t_{n-m+2} = 0, m = 1, \ldots, n.
\]

So \( a_{11}^{(1)} a_{mn}^{(1)} = 0 \) and \( a_{mn}^{(1)} e_{n+1, n+1} = 0, m = 2, \ldots, n \). From \( \varphi(e_{n+1}) = b e_{1, n+1} \), \( b \in R^* \), we have
\[
\varphi(e_{n+1}) = \varphi(e_{11}) \circ \varphi(e_{n+1}) = (a_{11}^{(1)} + a_{n+1, n+1}^{(1)}) b e_{1, n+1},
\]
then \( a_{11}^{(1)} + a_{n+1, n+1}^{(1)} = 1 \). So \( a_{mn}^{(1)} = a_{mn}^{(1)} (a_{11}^{(1)} + a_{n+1, n+1}^{(1)}) = 0, m = 2, \ldots, n \). From the process of proving Lemma 2.4 we know \( (a_{11}^{(1)})^2 = a_{11}^{(1)} \).

Now let us introduce standard Jordan automorphisms of \( T_{n+1}(R) \).

(i) Let \( \varepsilon \) be an idempotent of \( R \). Then \( \varepsilon, 1 - \varepsilon \) are orthogonal idempotents, that is, \( \varepsilon (1 - \varepsilon) = 0 \). Let \( e_0 = \sum e_{i, n+2} \). We define a map \( \xi_\varepsilon: x \mapsto \varepsilon x + (1 - \varepsilon)(e_0 x e_0) \tau \), where \( \tau \) denotes the transpose of a matrix. If both \( \varepsilon \) and \( \bar{\varepsilon} \) are idempotents of \( R \), then \( 1 - (\varepsilon - \bar{\varepsilon})^2 \) is also an idempotent of \( R \) and \( \xi_\varepsilon \xi_{\bar{\varepsilon}} = \xi_{\varepsilon - \bar{\varepsilon}} \xi_{\bar{\varepsilon}} \). This implies that \( \xi_{\varepsilon^{-1}} \xi_\varepsilon = \xi_\varepsilon \xi_{\varepsilon^{-1}} \) and \( \xi_\varepsilon \) is an \( R \)-module automorphism of \( T_{n+1}(R) \).
Obviously, \( \zeta_1 \) is the identity automorphism of \( T_{n+1}(R) \) and \( \xi_\epsilon = \epsilon \zeta_1 + (1 - \epsilon)\zeta_0 \). From \( \zeta_\epsilon(x \circ y) = \zeta_\epsilon(x) \circ \zeta_\epsilon(y) \) for any \( x, y \in T_{n+1}(R) \), we know that \( \zeta_\epsilon \) is a Jordan automorphism of \( T_{n+1}(R) \). We call \( \zeta_\epsilon \) a graph automorphism. If \( \epsilon \) is non-trivial, the graph automorphism \( \zeta_\epsilon \) is neither an \( R \)-algebra automorphism nor an \( R \)-algebra anti-automorphism of \( T_{n+1}(R) \), unless one of the ideals \( \epsilon T_{n+1}(R) \) or \( (1 - \epsilon)T_{n+1}(R) \) of \( T_{n+1}(R) \) is commutative. The graph automorphism \( \zeta_\epsilon \) on the basis of \( T_{n+1}(R) \) acts as \( \zeta_\epsilon(e_{ij}) = \epsilon e_{ij} + (1 - \epsilon)e_{i-j+n+1} \), \( 1 \leq k \leq \left[ \frac{n+1}{2} \right] \), \( k \leq j \leq n-k+1 \), \( \zeta_\epsilon(e_{n-k+1}) = e_{n-k+2} \) \( (1 \leq k \leq \left[ \frac{n}{2} \right] + 1) \) and \( \zeta_\epsilon(e_{n-j+n+1} = (1 - \epsilon)e_{ij} + \epsilon e_{n-j+n+1} \), \( 1 \leq k \leq \left[ \frac{n+1}{2} \right] \), \( k \leq j \leq n-k+1 \), where \( \left[ \frac{n+1}{2} \right] \) is the integer part of \( \frac{n+1}{2} \). The set of all graph automorphisms of \( T_{n+1}(R) \) is a subgroup of \( \operatorname{Aut}(n_0) \), which is denoted by \( \mathcal{G} \).

(ii) For any \( y \in n_1 \), let \( h = e + y \). The map \( \theta_y : x \mapsto hxh^{-1} \) is called an inner automorphism which is an \( R \)-algebra automorphism of \( T_{n+1}(R) \). If \( h = h_y(a) = e + ae_y(i < j) \) with some \( a \in R \), then \( \theta_y(a) \) is called the ‘simple’ form. Using \( [h_y(a)]^{-1} = h_y(-a) \) we know that \( \theta_y(a)(e_{ii}) = e_{ii} - ae_y \), \( \theta_y(a)(e_{ij}) = e_{ij} + ae_y \) for \( i < j \) and \( \theta_y(a)(e_{kk}) = ek_k \) for \( k \neq i, j \) and that \( \theta_y(a)(e_{ij+1}) = e_{i+1,j+1} + ae_{m, j+1} \) and \( \theta_y(a)(e_{ij+1}) = e_{i+1,j+1} - ae_y \) also \( \theta_y(a)(e_{kk+1}) = e_{k,k+1} \) and \( \theta_y(a)(e_{kk}) = e_{k,k+1} \) for \( k \neq m, j \). It is easy to see that \( \theta_y(a) = \theta_y(-a) \). The set of all the ‘simple’ inner automorphisms of \( T_{n+1}(R) \) generates a subgroup of \( \operatorname{Aut}(n_0) \), which is denoted by \( \mathcal{H} \).

(iii) Let \( d = \sum n_{i+1} d_e e_{ii} \) where \( d_i \in R^*, 1, 2, \ldots, n + 1 \). The map \( \lambda_{ij} : x \mapsto dx^{-1} \) is called a diagonal automorphism which is an \( R \)-algebra automorphism of \( T_{n+1}(R) \). It is obvious that \( \lambda_{ij}^{-1} = \lambda_{ij} \). A diagonal automorphism on the basis of \( T_{n+1}(R) \) yields that \( \lambda_{ij}(e_{ii}) = e_{ii} \) and \( \lambda_{ij}(e_{i+k}) = \prod_{m=n}^{k} e_{i+m}^{c_{i+m}}e_{i+k} \) for \( d_i = 1, d_i = \prod_{m=n}^{k} e_{i+m}^{c_{i+m}}e_{i+k} \) \( e_{i+1,j+1} + e_{m,j+1} \) and \( \lambda_{ij}(e_{i+k}) = e_{i+k} \) for \( k \neq m, j \). The set of all diagonal automorphisms of \( T_{n+1}(R) \) is a subgroup of \( \operatorname{Aut}(n_0) \), which is denoted by \( \mathcal{D} \).

3. Lemmas for main results. In order to achieve our goal, we also need other lemmas.

**Lemma 3.1.** Let \( \varphi \) be in \( \operatorname{Aut}(n_0) \). There exists a graph automorphism \( \xi_\epsilon \) such that \( \xi_\epsilon(\varphi(e_{11})) = e_{11} + t_1 \).

**Proof.** By Lemma 2.5, \( \varphi(e_{11}) = a_{11}^{(1)} e_{11} + a_{n+1,n+1}^{(1)} e_{n+1,n+1} + t_1 \). Take \( \epsilon = a_{11}^{(1)} \), then \( \xi_\epsilon(\varphi(e_{11})) = a_{11}^{(1)} \xi_\epsilon(a_{11}^{(1)} e_{11}) + a_{n+1,n+1}^{(1)} \xi_\epsilon(a_{n+1,n+1}^{(1)} e_{n+1,n+1}) + \xi_\epsilon(t_1) \)

\[
= a_{11}^{(1)}[a_{11}^{(1)} e_{11} + (1 - a_{11}^{(1)}) e_{n+1,n+1}] + (1 - a_{11}^{(1)})[a_{11}^{(1)} e_{n+1,n+1} + (1 - a_{11}^{(1)}) e_{11}] + t_1
\]

\[
= (a_{11}^{(1)})^2 e_{11} + (1 - a_{11}^{(1)}) e_{11} + t_1 = e_{11} + t_1.
\]

This completes the proof. \( \square \)

**Lemma 3.2.** Let \( \varphi \) be in \( \operatorname{Aut}(n_0) \). If \( \varphi(e_{11}) = e_{11} + t_1 \), then \( \varphi(e_{ii}) = e_{ii} + t_1 \), \( i = 1, 2, \ldots, n + 1 \) and \( \varphi(e_{i+1,j}) = b_{i+1,j}^{(1)} e_{i+1,j} + t_2 \), \( i = 1, \ldots, n \) where \( b_{i+1,j}^{(1)} \in R^* \).

**Proof.** If \( e_{ij} \in n_1 \), then \( \varphi^{-1}(e_{ij}) \in n_1 \). By Lemma 2.3 we have \( e_{ij} \varphi^{-1}(e_{ij}) = 0 \) then \( \varphi(e_{ij}) \varphi(e_{ij}) = 0 \). Therefore,

\[
\varphi(e_{ij}) e_{im} \varphi(e_{ij}) = a_{ii}^{(1)} a_{mm}^{(i)} e_{im} + t_m e_{i+m+1} = 0, \ i = 1, \ldots, m - 1 \ (m \geq 2);
\]

\[
\varphi(e_{ij}) e_{mi} \varphi(e_{ij}) = a_{ii}^{(1)} a_{mm}^{j} e_{ki} + t_{i+m+1} = 0, \ i = m + 1, \ldots, n + 1 \ (m \leq n),
\]
so \( a_{ii}^{(i)} a_{mm}^{(i)} = 0, \ i \neq m \). When \( i \neq j \), \( \varphi(e_{ii}) \circ \varphi(e_{jj}) = \sum_{k=1}^{n-1} a_{kk}^{(i)} a_{kk}^{(j)} e_{kk} + t_1 = 0 \), so \( a_{kk}^{(i)} a_{kk}^{(j)} = 0, \ i \neq j \). Let us express \( \varphi(e_{i+1}) \) as \( \varphi(e_{i+1}) = \sum_{k=1}^{n} b_{k,k+1}^{(i)} e_{k,k+1} + t_2 \). Therefore \( \varphi(e_{12}) = \varphi(e_{11}) \circ \varphi(e_{12}) = b_{12}^{(i)} e_{12} + t_2 \). From \( \varphi^{-1}(e_{11}) = \varphi^{-1}(e_{11}) + t_1 \), we have \( \varphi^{-1}(e_{11}) = e_{11} + t_1 \). Then \( \varphi^{-1}(e_{12}) = b_{12}^{(i)} e_{12} + t_2 \). Furthermore, \( e_{12} = \varphi^{-1}(e_{12}) = b_{12}^{(i)} b_{12}^{(i)} e_{12} + t_2 \), then \( b_{12}^{(i)} b_{12}^{(i)} = 1 \), that is, \( b_{12}^{(i)} \in R^* \). Also we have \( \varphi(e_{12}) = \varphi(e_{12}) \circ \varphi(e_{22}) = (a_{11}^{(2)} + a_{22}^{(2)}) b_{12}^{(i)} e_{12} + t_2 \). Then \( a_{11}^{(2)} + a_{22}^{(2)} = 1 \). From \( a_{11}^{(2)} a_{11}^{(2)} = 0 \), we know \( a_{22}^{(2)} = 0 \), that is, \( \varphi(e_{mn}) = e_{mn} + t_2 \). From

\[
\varphi(e_{m,m+1}) = \varphi(e_{mm}) \circ \varphi(e_{m,m+1}) = b_{m,m+1}^{(m)} e_{m,m+1} + b_{m,m+1}^{(m)} e_{m,m+1} + t_2.
\]

we have \( b_{k,k+1}^{(m)} = 0, \ k \neq m - 1, m \). From

\[
\varphi(e_{m-1,m}) \circ \varphi(e_{m,m+1}) = b_{m-1,m}^{(m)} e_{m-1,m} + t_2 = 0,
\]

we have \( b_{m-1,m}^{(m)} = 0 \), that is, \( \varphi(e_{m,m+1}) = b_{m,m+1}^{(m)} e_{m,m+1} + t_2 \). In the same way, we know \( b_{m,m+1}^{(m)} \in R^* \). Furthermore,

\[
\varphi(e_{m,m+1}) = \varphi(e_{m,m+1}) \circ \varphi(e_{m,m+1}) = (a_{mm}^{(m+1)} + a_{m+1,m}^{(m+1)}) b_{m,m+1}^{(m)} e_{m,m+1} + t_2.
\]

Then \( a_{mm}^{(m+1)} + a_{m+1,m}^{(m+1)} = 1 \). So \( a_{m+1,m}^{(m+1)} = 1 \). When \( m = n \), the proof is completed. □

**Lemma 3.3.** Let \( \varphi \) be in Aut(\( n \)). If \( \varphi(e_{ii}) = e_{ii} + t_1, i = 1, 2, \ldots, n + 1, \) then

\[
\varphi(e_{ii}) = e_{ii} + a_{ii}^{(1)} e_{ii} + t_2,
\]

\[
\varphi(e_{ii}) = e_{ii} + a_{ii}^{(1)} e_{ii} + a_{ii}^{(1)} e_{ii} + t_2, \quad i = 2, \ldots, (n(n + 2)) \]

\[
\varphi(e_{n+1,n+1}) = e_{n+1,n+1} + a_{n,n+1}^{(n)} e_{n,n+1} + t_2.
\]

**Proof.** We write \( \varphi(e_{ii}) \) as

\[
\varphi(e_{ii}) = e_{ii} + \sum_{k=1}^{n} a_{k,k+1}^{(i)} e_{k,k+1} + t_2, \quad i = 1, 2, \ldots, n + 1.
\]

From \( \varphi(e_{ij}) = [\varphi(e_{ii})]^2 \) we have

\[
\varphi(e_{ii}) = e_{ii} + a_{ii}^{(1)} e_{ii} + t_2,
\]

\[
\varphi(e_{ii}) = e_{ii} + a_{ii}^{(1)} e_{ii} + a_{ii}^{(1)} e_{ii} + t_2, \quad i = 2, \ldots, n \]

\[
\varphi(e_{n+1,n+1}) = e_{n+1,n+1} + a_{n,n+1}^{(n+1)} e_{n,n+1} + t_2.
\]

Then

\[
\varphi(e_{ii}) \circ \varphi(e_{i+1,i+1}) = (a_{i,i+1}^{(i)} + a_{i,i+1}^{(i+1)}) e_{i,i+1} + t_2 = 0, \quad i = 1, \ldots, n.
\]

So \( a_{i,i+1}^{(i)} = -a_{i,i+1}^{(i+1)}, \ i = 1, \ldots, n \). □
LEMMA 3.4. Let $\varphi$ be in $\text{Aut}(\mathfrak{n}_0)$. If $\varphi(e_{ii}) = e_{ii} + t_1$, $i = 1, 2, \ldots, n + 1$, we take that

$$\theta = \prod_{i=1}^{n} \theta_{h_{i+1}(a_{i+1}^{0})}.$$  

Then

$$\theta \varphi(e_{ii}) = e_{ii} + t_2, \ i = 1, 2, \ldots, n + 1.$$  

Proof. From $\theta_{h_{i+1}(a_{i+1}^{0})}(e_{ii}) = e_{ii} - d_{i+1}^{(i)} e_{i,i+1}$ and $\theta_{h_{i+1}(a_{i+1}^{0})}(e_{i+1,i+1}) = e_{i+1,i+1} + d_{i+1}^{(i)} e_{i,i+1}$ and then by Lemma 3.3 it is not difficult to complete the proof. □

LEMMA 3.5. Let $\varphi$ be in $\text{Aut}(\mathfrak{n}_0)$. If $\varphi(e_{ii}) = e_{ii} + t_{m-1}$, $i = 1, 2, \ldots, n + 1$, then

$$\varphi(e_{ii}) = e_{ii} + d_{i+1,m-1}^{(i)} e_{i,i+m-1} + t_m, \ 1 \leq i \leq \min(m - 1, n - m + 2),$$  

$$\varphi(e_{ii}) = e_{ii} + d_{i+1,m-1}^{(i)} e_{i,i+m-1} - d_{i,m-1,i}^{(i-m+1)} e_{i-m+1,i} + t_m,$$

$$m \leq i \leq n - m + 2 \left( m \leq \left\lceil \frac{n+1}{2} \right\rceil \right),$$

$$\varphi(e_{ii}) = e_{ii} + t_m,$$

$$n - m + 3 \leq i \leq m \left( m \geq \left\lceil \frac{n+1}{2} \right\rceil + 1 \text{ or when } n \text{ is odd, } m > \left\lceil \frac{n+1}{2} \right\rceil + 1 \right),$$

$$\varphi(e_{ii}) = e_{ii} - d_{i,m-1,i}^{(i-m+1)} e_{i-m+1,i} + t_m, \ \max(n - m + 3, m) \leq i \leq n + 1.$$  

Proof. It is the case in Lemma 3.3 if $m = 2$. Using the method of proving Lemma 3.3 we may verify the consequence. □

LEMMA 3.6. Let $\varphi$ be in $\text{Aut}(\mathfrak{n}_0)$. If $\varphi(e_{ii}) = e_{ii} + t_{m-1}$, $i = 1, 2, \ldots, n + 1$, we take that

$$\theta = \prod_{i=1}^{n-m+2} \theta_{h_{i+1,m-1}(a_{i+1,m-1}^{0})}.$$  

Then

$$\theta \varphi(e_{ii}) = e_{ii} + t_m, \ i = 1, 2, \ldots, n + 1.$$  

When $m = n + 1$, $\theta \varphi(e_{ii}) = e_{ii}, \ i = 1, 2, \ldots, n + 1.$  

Proof. The process for verifying the result is similar to that of Lemma 3.4. □

LEMMA 3.7. When $n \geq 1$, let $\varphi$ be in $\text{Aut}(\mathfrak{n}_0)$. If $\varphi(e_{ii}) = e_{ii}$, there exists a diagonal automorphism $\lambda_d$ such that $\lambda_d \varphi(e_{i,i+1}) = e_{i,i+1} + t_2$, $i = 1, \ldots, n$.

Proof. By Lemma 3.2 we know that $\varphi(e_{i,i+1}) = b_{i,i+1}^{(i)} e_{i,i+1} + t_2$, $i = 1, \ldots, n$, where $b_{i,i+1}^{(i)} \in R^*$. Let $\lambda_d$ satisfy $e_{i,i+1} \mapsto (b_{i,i+1}^{(i)})^{-1} e_{i,i+1}$, where $d_1 = 1$, $d_i = \prod_{m=2}^{i} b_{i-m+1,i-m+1}^{(i-m+1)}$, $i = 2, \ldots, n + 1$. Applying $\lambda_d \varphi$ to $e_{i,i+1}$ we get the asserted property. □
4. Proofs of main results. Proof of Theorem 1.1. By Lemma 3.1, Lemma 3.4 and Lemmas 3.6–3.8 there are \( \lambda_d^{-1} \), \( \theta^{-1} \) and \( \zeta_e \) such that

\[
\lambda_d^{-1} \theta^{-1} \zeta_e \varphi(e_i) = e_i, \quad i = 1, 2, \ldots, n + 1.
\]

So for \( i = 1, 2, \ldots, n \)

\[
\varphi(e_{i+1}) = \varphi(e_{i+1}) \circ \varphi(e_{i+1}) = \varphi(e_{i+1}) \circ e_{i+1} = e_{i+1}.
\]

In the case \( n = 1 \), \( \varphi(e_2) = e_{12} \).

Proof of Theorem 1.2. By Theorem 1.1 we have \( \text{Aut}(n_0) = G \triangle D \). For any \( x \in n_0 \) we have \( \theta_h \lambda_d(x) = h(xd^{-1})h^{-1} = \lambda_d \theta_d^{-1} hd(x) \), thus \( \theta_h \lambda_d = \lambda_d \theta_d^{-1} hd \). So \( \mathcal{I} \triangle D \). Obviously, \( \mathcal{I} \cap D = 1 \), then \( D = \mathcal{I} \triangle D \). Also we have \( \zeta_0 \theta_h(x) = (e_0(hxh^{-1})e_0)^{\tau} = \theta_{\zeta_0(h^{-1})} \zeta_0(x) \), that is, \( \zeta_0 \theta_h = \theta_{\zeta_0(h^{-1})} \zeta_0 \). From

\[
\theta_{\zeta_0 + (1 - \epsilon) \zeta_0(h^{-1})}(x) = [\epsilon h + (1 - \epsilon) \zeta_0(h^{-1})][\epsilon h + (1 - \epsilon) \zeta_0(h^{-1})]^{-1} \\
= [\epsilon h + (1 - \epsilon) \zeta_0(h^{-1})][\epsilon h^{-1} + (1 - \epsilon)(\zeta_0(h^{-1}))^{-1}] \\
= \epsilon^2 h x h^{-1} + (1 - \epsilon) \zeta_0(h^{-1}) x (\zeta_0(h^{-1}))^{-1} \\
= \epsilon \theta_h(x) + (1 - \epsilon) \theta_{\zeta_0(h^{-1})}(x) \\
= [\epsilon \theta_h + (1 - \epsilon) \zeta_0 \theta_h \zeta_0](x),
\]
we have \( \theta_{e^{h+1-\varepsilon}g_0(h^{-1})} = \varepsilon \theta_h + (1 - \varepsilon) g_0 \theta_h g_0. \) Furthermore,

\[
\zeta \theta_h \zeta = [\varepsilon \zeta_1 + (1 - \varepsilon) g_0] \theta_h [\varepsilon \zeta_1 + (1 - \varepsilon) g_0] \\
= \varepsilon^2 \theta_h + (1 - \varepsilon)^2 g_0 \theta_h g_0 \\
= \theta_{e^{h+1-\varepsilon}g_0(h^{-1})}.
\]

Similarly, \( \zeta \lambda_d \zeta = \lambda_{e^{d+1-\varepsilon}g_0(d^{-1})}. \) Thus \( ID \triangleleft GID. \) Clearly, \( G \cap ID = 1, \) then \( GID = G \ltimes (I \ltimes D), \) that is, \( \text{Aut}(n_0) = G \ltimes (I \ltimes D). \)

**Acknowledgements.** The authors would like to thank the anonymous referees for their useful comments and suggestions. The research was supported partially by National Natural Science Foundation of China (Grant Nos. 10871056 and 10971150) and by Science Research Foundation in Harbin Institute of Technology (Grant No. HITC200708).

**References**